A NOTE ON SOME STUDIES IN FIRST ORDER MATRIX DIFFERENTIAL SYSTEMS

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Abstract:

In this paper, we study linear differential equations of first order whose coefficients are square matrices. The combinatorial method for computing the matrix powers and exponential is adopted. New formulas rep- resenting auxiliary results are obtained. This allows us to prove properties of a large class of linear matrix differential equations of first order.

Keywords: Differential equation, Difference equation and Matrix theory.

1. INTRODUCTION

Linear system. A linear system is a system of differential equations of the form

$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n + f_1,\\
x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n + f_2,\\
&\vdots \qquad \vdots \cdots \vdots \qquad \vdots \\
x_m' &= a_{m1}x_1 + \cdots + a_{mn}x_n + f_m,
\end{aligned} \tag{1}
$$

where $= d/dt$. Given are the functions $a_{ij}(t)$ and $f_j(t)$ on some interval $a < t < b$. The unknowns are the functions $x_1(t), \dots x_n(t)$. The system is called **homogeneous** if all $f_j = 0$, otherwise it is called **non-homogeneous.**

Matrix Notation for Systems. A non - homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$x' = A(t)x + f(t)$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

2. BASIC FIRST ORDER SYSTEM METHODS

Solving 2×2 Systems: It is shown here that any constant linear system

$u' = Au,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be solved by one of the following elementary methods.

(a) The integrating factor method for $y' = p(x)y + q(x)$.
(b) The second order constant coefficient recipe.

Triangular : A. Let's assume $b = 0$, so that A is lower triangular. The upper triangular case is handled similarly. Then $u' = Au$ has the scalar form

$$u_1' = au_1,$$
$$u_2' = cu_1 + du_2$$

The first differential equation is solved by the growth/decay recipe:

$$u_1(t) = u_0 e^{at}$$

Then substitute the answer just found into the second differential equation to give

$$u_2' = du_2 + cu_0 e^{at}$$

This is a linear first order equation of the form $y' = p(x)y + q(x)$, to be solved by the integrating factor method. Therefore, a triangular system can always be solved by the first order integrating factor method.

An illustration: Let us solve $u' = Au$ for the triangular matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

The first equation $u_1' = u_1$ has solution $u_1 = c_1 e^t$. The second equation becomes

$$u_2' = 2c_1 e^t + u_2,$$

which is a first order linear differential equation with solution $u_2 = (2c_1 t + c_2)e^t$. The general solution of $u' = Au$ in scalar form is

$$u_1 = c_1 e^t, \qquad u_2 = 2c_1 t e^t + c_2 e^t$$

The vector form of the general solution is

$$u(t) = c_1 \begin{pmatrix} e^t \\ 2te^t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

The vector basis is the set

$$B = \left\{ \begin{pmatrix} e^t \\ 2te^t \end{pmatrix}, \begin{pmatrix} 0 \\ e^t \end{pmatrix} \right\}$$

Non-Triangular A. In order that A be non-triangular, both $b \neq 0$ and $c \neq 0$ must be satisfied. The scalar form of the system $u' = Au$ is

$$u_1' = au_1 + bu_2,$$
$$u_2' = cu_1 + du_2$$

Theorem: 2.1 (Solving Non-Triangular $u' = Au$)

Solutions u_1, u_2 of $u' = Au$ are linear comninations of the list of atoms obtained from the roots r of the quadratic equation

$$det\,(A - rI) = 0$$

Proof:

The method: differentiate the first equation, then use the equations to eliminate u_2, u_2'. The result us a second order differential equation for u_1. The same differential equation is satisfied also for u_2. The details:

$u_1'' = au_1' + bu_2'$

Differentiate the first equation.

$= au_1' + bcu_1 + bdu_2$ Use equation $u_2' = cu_1 + du_2$.

$= au_1' + bcu_1 + d(u_1' - au_1)$ Use equation $u_1' = au_1 + bu_2$.

$= (a + d)u_1' + (bc - ad)u_1$ Second order equation for u_1 found

The characteristic equation of $u_1'' - (a + d)u_1' + (ad - bc)u_1 = 0$ is

$$r^2 - (a + d)r + (bc - ad) = 0$$

Finally, we show the expansion of $\det(A - rI)$ is the same characteristic polynomial:

$$\det(A - rI) = \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix}$$
$$= (a - r)(d - r) - bc$$
$$= r^2 - (a + d)r + ad - bc$$

The proof is complete.

The reader can verify that the differential equation for u_1 or u_2 is exactly

$$u'' - trace\,(A)u' + \det(A)u = 0$$

Finding u_1. Apply the second order recipe to solve for u_1. This involves writing a list L of atoms corresponding to the two roots of the characteristic equation $r^2 - (a+d)r + ad - bc = 0$, followed by expressing u_1 as a linear combination of the two atoms.

Finding u_2. Isolate u_2 in the first differential equation by division:

$$u_2 = \frac{1}{b}(u_1' - au_1)$$

The two formulas for u_1, u_2 represent the general solution of the system $u' = Au$, when A is 2×2.

An illustration. Let's solve $u' = Au$ when

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The equation $\det(A - rI) = 0$ is $(1-r)^2 - 4 = 0$ with roots $r = -1$ and $r = 3$. The atom list is $L = \{e^{-t}, e^{3t}\}$. Then the linear combination of atoms is $u_1 = c_1 e^{-t} + c_2 e^{3t}$. The first equation $u_1' = u_1 + 2u_2$ implies $u_2 = \frac{1}{2}(u_1' - u_1)$. The general solution of $u' = Au$ is then

$$u_1 = c_1 e^{-t} + c_2 e^{3t}, \qquad u_2 = -c_1 e^{-t} + c_2 e^{3t}$$

In vector form, the general solution is

$$u = c_1 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Triangular Methods: 2.2

Diagonal $n \times n$ matrix $A = diag(a_1, \ldots, a_n)$. Then the system $x' = Ax$ is a set of uncoupled scalar growth/decay equations:
$x_1'(t) = a_1 x_1(t),\ x_2'(t) = a_2 x_2(t), \ldots\ x_n'(t) = a_n x_n(t)$
The solution of the system is given by the formulae
$x_1(t) = c_1 e^{a_1 t}, x_2(t) = c_2 e^{a_2 t}, \ldots\ x_n(t) = c_n e^{a_n t}$
The numbers $c_1, \ldots\ldots, c_n$ are arbitrary constants. Triangular $n \times n$ matrix A. If a linear system $x' = Ax$ has a square triangular matrix A, then the system can be solved by first order scalar methods. To illustrate the ideas, consider the 3×3 linear system

$$x' = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 4 & 4 \end{pmatrix} x$$

The coefficient matrix A is lower triangular. In scalar form, the system is given by the equations
$x_1'(t) = 2x_1(t), x_2'(t) = 3x_1(t) + 3x_2(t),\ x_3'(t) = 4x_1(t) + 4x_2(t) + 4x_3(t)$.

A recursive method.

The system is solved recursively by first order scalar methods only, starting with the first equation $x_1'(t) = 2x_1(t)$. This growth equation has general solution $x_1(t) = c_1 e^{2t}$. The second equation then becomes the first order linear equation

$$x_2'(t) = 3x_1(t) + 3x_2(t) \qquad = 3x_2(t) + 3c_1 e^{2t}$$

The integrating factor method applies to find the general solution $x_2(t) = -3c_1 e^{2t} + c_2 e^{3t}$. The third and last equation becomes the first order linear equation

$$x_3'(t) = 4x_1(t) + 4x_2(t) + 4x_3(t) \ = 4x_3(t) + 4c_1 e^{2t} + 4(-3c_1 e^{2t} + c_2 e^{3t})$$

The integrating factor method is repeated to find the general solution $x_3(t) = 4c_1 e^{2t} - 4c_2 e^{3t} + c_3 e^{4t}$.
In summary, the solution to the system is given by the formulae
$x_1(t) = c_1 e^{2t}, x_2(t) = -3c_1 e^{2t} + c_2 e^{3t}\ x_3(t) = 4c_1 e^{2t} - 4c_2 e^{3t} + c_3 e^{4t}$

Structure of solution.

A system $x' = Ax$ for $n \times n$ traingular A has component solutions $x_1(t), \ldots\ldots x_n(t)$ given as polynomials times exponentials. The exponential factor $e^{a_{11}t}, \ldots\ldots, e^{a_{nn}t}$ are expressed in terms of the diagonal element $a_{11}, \ldots\ldots, a_{nn}$ of the matrix A. Fewer than n distinct exponential factors may appear, due to duplicate diagonal elements. These duplications cause the polynomial factors to appear. The reader is invited to work out the solution to the system below, which has duplicate diagonal entries $a_{11} = a_{22} = a_{33} = 2$.

$x_1'(t) = 2x_1(t),\ x_2'(t) = 3x_1(t) + 2x_2(t), x_3'(t) = 4x_1(t) + 4x_2(t) + 2x_3(t)$.

The solution, given below, has polynomial factors t and t^2, appearing because of the duplicate diagonal entries 2,2,2, and only one exponential factor e^{2t}

$x_1(t) = c_1e^{2t}$, $x_2(t) = 3c_1te^{2t} + c_2e^{2t},\ x_3(t) = 4c_1te^{2t} + 6c_1t^2e^{2t} + 4c_2te^{2t} + c_3e^{2t}$

3. STRUCTURE OF LINEAR SYSTEMS

Theorem: 3.1 (Unique Zero Solution)

Let $A(t)$ be an $m \times n$ matrix with entries continuous on $a < t < b$. Then the initial value problem

$x' = A(t)x,\ x(0) = 0$ has unique solution $x(t) = 0$ on $a < t < b$.

Theorem: 3.2 (Existence-Uniqueness for Constant Linear Systems)

Let $A(t) = A$ be an $m \times n$ matrix with constant entries and let x_0 be any m-vector. Then the initial value problem $x' = Ax,\ x(0) = x_0$ has unique solution $x(t)$ defined for all values of t.

Theorem: 3.3 (Uniqueness and Solution Crossings)

Let $A(t)$ be an $m \times n$ matrix with entries continuous on $a < t < b$ and assume $f(t)$ is also continuous on $a < t < b$. If $x(t)$ and $y(t)$ are solutions of $u' = A(t)u + f(t)$ on $a < t < b$ and $x(t_0) = y(t_0)$ for some $t_0, a < t_0 < b$, then $x(t) = y(t)$ for $a < t < b$.

Superposition. Linear homogeneous systems have linear structure and the solutions to non homogeneous systems obey a principle of superposition.

Theorem: 3.4 (Linear Structure)

Let $x' = A(t)x$ have two solutions $x_1(t), x_2(t)$. If k_1, k_2 are constants, then $x(t) = k_1x_1(t) + k_2x_2(t)$ is also a solution of $x' = A(t)x$. The standard basis $\{w_k\}_{k=1}^{n}$. The Picard-Lindelof theorem applied to initial conditions $x(t_0) = x_0$, with x_0 successively set equal to the columns of the $n \times n$ identity matrix, produces n solutions $w_1, \dots\dots, w_n$ to the equation $x' = A(t)x$, all of which exist on the same interval $a < t < b$. The linear structure theorem implies that for any choice of the constants $c_1, \dots\dots, c_n$, the vector linear combination

$x(t) = c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t)$ (2) is a solution of $x' = A(t)x$.

Conversely, if $c_1, \dots\dots, c_n$ are taken to be the components of a given vector x_0, then the above linear combination must be the unique solution of the initial value problem with $x(t_0) = x_0$. Therefore, all solutions of the equation $x' = A(t)x$ are given by the expression above, where $c_1, \dots\dots, c_n$ are taken to be arbitrary constants.

Theorem: 3.5 (Basis)

The solution set of $x' = A(t)x$ is an n-dimensional subspace of the vector space of all vector-valued functions $x(t)$. Every solution has a unique basis expansion (2).

Theorem: 3.6 (Superposition Principle)

Let $x' = A(t)x + f(t)$ have a particular solution $x_p(t)$. If $x(t)$ is any solution of $x' = A(t)x + f(t)$, then $x(t)$ can be decomposed as homogeneous plus particular:

$x(t) = x_h(t) + x_p(t)$. The term $x_h(t)$ is a certain solution of the homogeneous differential equation $x' = A(t)x$, which means arbitrary constants $c_1, c_2, \dots\dots$ have been assigned certain values. The particular solution $x_p(t)$ can be selected to be free of any unresolved or arbitrary constants.

Theorem: 3.7 (Difference of Solutions)

Let $x' = A(t)x + f(t)$ have two solutions $x = u(t)$ and $x = v(t)$. Define $y(t) = u(t) - v(t)$. Then $y(t)$ satisfies the homogeneous equation $y' = A(t)y$

Solution:

We explain general solution by example. If a formula $x = c_1 e^t + c_2 e^{2t}$ is called a general solution, then it means that all possible solutions of the differential equation are expressed by this formula. In particular, it means that a given solution can be represented by the formula, by specializing values for the constants c_1, c_2. We expect the number of arbitrary constants to be the least possible number. The general solution $x' = A(t)x + f(t)$ is an expression involving arbitrary constants $c_1, c_2, \dots \dots$ and certain functions. The expression is often given in vector notation, although scalar expressions are commonplace and perfectly acceptable.

Required is that the expression represents all solutions of the differential equation, in the following sense:

(a) Every assignment of constants produces a solution of the differential equation.
(b) Every possible solution is uniquely obtained from the expression by specializing the constants.

Due to the superposition principle, the constants in the general solution are identified as multipliers against solutions of the homogeneous differential equation. The general solution has some recognizable structure.

Theorem: 3.8 (General Solution)

Let $A(t)$ be $n \times n$ and $f(t) n \times 1$, both continuous on an interval $a < t < b$. The linear nonhomogeneous system $x' = A(t)x + f(t)$ has general solution x given by the expression

$$x = x_h(t) + x_p(t)$$

The term $y = x_h(t)$ is a general solution of the homogeneous equation $y' = A(t)y$, in which are to be found n arbitrary constants $c_1, \dots \dots, c_n$. The term $x = x_p(t)$ is a particular solution of $x' = A(t)x + f(t)$, in which there are present no unresolved nor arbitrary constants.

Recognition of homogeneous solution terms. An expression x for the general solution of a nonhomogeneous equation $x' = A(t)x + f(t)$ involves arbitrary constants $c_1, \dots \dots, c_n$. It is possible to isolate both terms x_h and x_p by a simple procedure.

To find x_p, set of zero all arbitrary constants $c_1, c_2, \dots.$ the resulting expression is free of unresolved and arbitrary constants.

To find x_h, we find first the vector solutions $y = u_k(t)$ of $y' = A(t)y$, which are multiplied by constants c_k. Then the general solution x_h of the homogeneous equations $y' = A(t)y$ is given by

$$x_h(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_n u_n(t)$$

Use partial derivatives on expression x to find the column vectors

$$u_k(t) = \frac{\partial}{\partial c_k} x$$

The technique isolates the vector components of the homogeneous solution from any form of the general solution, including scalar formulas for the components of x. In any case, the general solution x of the linear system $x' = A(t)x + f(t)$ is represented by the expression

$$x = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_n u_n(t) + x_p(t)$$

In this expression, each assignment of the constants $c_1, \dots \dots, c_n$ produces a solution of the nonhomogeneous system, and conversely, each possible solution of the nonhomogeneous system is obtained by a unique specialization of the constants $c_1, \dots \dots, c_n$.

To illustrate the ideas, consider a 3×3 linear system $x' = A(t)x + f(t)$ with general solution

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

given in scalar form by the expressions

$$x_1 = c_1 e^t + c_2 e^{-t} + t \ , x_2 = (c_1 + c_2)e^t + c_3 e^{2t} \, , x_3 = (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t$$

To find the vector form of the general solutions, we take partial derivatives $u_k = \frac{\partial x}{\partial c_k}$ with respect to the variable names c_1, c_2, c_3:

$$u_1 = \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix}, u_2 = \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix}$$

To find $x_p(t)$, set $c_1 = c_2 = c_3 = 0$

$$x_p(t) = \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}$$

Finally,

$x = c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) + x_p(t)$

$= c_1 \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}$

The expression $x = c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t) + x_p(t)$ satisfies required elements (a) and (b) in the definition of general solution. We will develop now a way to routinely test the uniqueness requirement in (b).

Independence. Constants $c_1, \ldots \ldots, c_n$ in the general solution $x = x_h + x_p$ appear exactly in expression x_h, which has the form

$$x_h = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

A solution x uniquely determines the constants. In particular, the zero solution of the homogeneous equation is uniquely represented, which can be state this way:

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n = 0 \text{ implies } c_1 = c_2 = \cdots c_n = 0$$

This statement is equivalent to the statement that the vector-valued functions $u_1(t), \ldots \ldots, u_n(t)$

It is possible to write down a candidate general solution to some 3×3 linear system $x' = Ax$ via equations like$x_1 = c_1 e^t + c_2 e^t + c_3 e^{2t}$, $x_2 = c_1 e^t + c_2 e^t + 2c_3 e^{2t}$, $x_3 = c_1 e^t + c_2 e^t + 3c_3 e^{2t}$

This example was constructed to contain a classic mistake, for purposes of illustration.

How can we detect a mistake, given only that this expression is supposed to represent the general solution? First of all, we can test that $u_1 = \partial x / \partial c_1, u_2 = \partial x / \partial c_2, u_3 = \partial x / \partial c_3$ are indeed solutions. But to insure the unique representation requirement, the vector functions u_1, u_2, u_3 must be linearly independent. We compute

$$u_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, u_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, u_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

Therefore, $u_1 = u_2$, which implies that the functions u_1, u_2, u_3 fail to be independent. While is possible to test independence by a rudimentary test based upon the definition, we prefer the following test due to Abel.

Theorem: 3.11 (Abel's Formula and the Wronskian)

Let $x_h(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_n u_n(t)$ be a candidate general solution to the equation $x' = A(t)x$. In particular, the vector functions $u_1(t), \ldots \ldots, u_n(t)$ are solutions of $x' = A(t)x$. Define the Wronskian by

$$w(t) = \det(aug\,(u_1(t), \ldots \ldots, u_n(t)))$$

Then Abel's formula holds: $w(t) = e^{\int_{t_0}^{t} trace\,(A(s))ds} w(t_0)$

In particular, $w(t)$ is either everywhere nonzero or everywhere zero, accordingly as $w(t_0) \neq 0$ or $w(t_0) = 0$.

Theorem: 3.12 (Abel's Wronskian Test for independence)

The vector solutions $u_1, \ldots \ldots, u_2$ of $x' = A(t)x$ are independent if and only if the Wronskian $w(t)$ is nonzero for some $t = t_0$.

Clever use of the point t_0 in Abel's Wronskian test can lead to succinct independence tests. For instance, let

$$u_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, u_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, u_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}$$

Then $w(t)$ might appear to be complicate, but $w(0)$ is obviously zero because it has two duplicate columns. Therefore, Abel's Wronskian test detects dependence of u_1, u_2, u_3.

To illustrate Abel's Wronskian test when it detects independence, consider the column vectors

$$u_1 = \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix}, u_2 = \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix}$$

At $t = t_0 = 0$, they become the column vectors

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

Then $w(0) = \det\left(aug\left(u_1(0), u_2(0), u_3(0)\right)\right) = 1$ is non zero, testing independence of u_1, u_2, u_3.

Initial value problems and the rref method. An initial value problem is the problem of solving for x, given

$$x' = A(t)x + f(t), \qquad x(t_0) = x_0$$

If a general solution is known

$$x = c_1 u_1(t) + \cdots + c_n u_n(t) + x_p(t)$$

then the problem of finding x reduces to finding $c_1, \ldots \ldots, c_n$ in the relation

$$c_1u_1(t_0)+\cdots+c_nu_n(t_0)+x_p(t_0)=x_0$$

This is a matrix equation for the unknown constants $c_1, \ldots \ldots, c_n$ of the form $Bc = d$, where

$$B=\text{aug}\big(\, u_1(t_0),\ldots,u_3(t_0)\big),\qquad c=\begin{pmatrix}c_1\\ \vdots\\ c_2\end{pmatrix},\;\; d=x_0-x_p(t_0)$$

The rref-method applies to find c. The method is to perform swap, combination and multiply operations to $C = \text{aug}(B,d)$ until $\text{rref}(C) = \text{aug}(I,c)$.

To illustrate the method, consider the general solution

$x_1 = c_1e^t + c_2e^{-t} + t$, $x_2 = (c_1+c_2)e^t + c_3e^{2t}$, $x_3 = (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t$

We shall solve for c_1, c_2, c_3 given the initial condition $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1$. The above relations evaluated at $t = 0$ give the system

$1 = c_1e^0 + c_2e^0 + 0$,$0 = (c_1+c_2)e^0 + c_3e^0$, $-1 = (2c_2 - c_1)e^0 + (4c_1 - 2c_3)e^0 + 0$

In standard scalar form, this is the 3×3 linear system

$c_1 + c_2 = 1$, $\quad c_1 + c_2 + c_3 = 0$, $\quad 3c_1 + 2c_2 - 2c_3 = -1$,

The augmented matrix C, to be reduced to rref form, is given by

$$C=\begin{pmatrix}1&1&0&1\\1&1&1&0\\3&2&-2&-1\end{pmatrix}$$

After the rref process is completed, we obtain

$$\text{rref}(C)=\begin{pmatrix}1&0&0&-5\\0&1&0&6\\0&0&1&-1\end{pmatrix}$$

From this display, we read off the answer $c_1 = -5, c_2 = 6, c_3 = -1$ and report the final answer $x_1 = -5e^t + 6e^{-t} + t$, $x_2 = e^t - e^{2t}$, $x_3 = 17e^{-t} - 18e^{2t} + 2t$

Equilibria. An equilibrium point x_0 of a linear system $x' = A(t)x$ is a constant solution $x(t) = x_0$ for all t. Mostly, this makes sense when $A(t)$ is constant, although the definition applies to continuous systems. For a solution x to be constant means $x' = 0$, hence all equilibria are determined from the equation

$$A(t)x_0 = 0 \text{ for all } t.$$

This is a homogeneous system of linear algebraic equations to be solved for x_0. It is not allowed for the answer x_0. It is not allowed for the answer x_0 to depend on t (if it does, then it is not an equilibrium). The theory for a constant matrix $A(t) \equiv A$ says that either $x_0 = 0$ is the unique solution or else there are infinitely many answer for x_0(the nullity of A is positive).

4. MATRIX EXPONENTIAL

The problem $x'(t) = Ax(t),\ x(0) = x_0$ has a unique solution, according to the Picard-Londelof theorem. Solve the problem n times, when x_0 equals a column of the identity matrix and write $w_1(t), \ldots . w_n(t)$ for the n solutions so obtained. Define the matrix exponential by packaging these n solutions into a matrix $e^{At} \equiv \text{aug}\big(w_1(t), \ldots . w_n(t)\big)$. By construction, any possible solution $x' = Ax$ can be uniquely expressed in terms of the matrix exponential e^{At} by the formula

$$x(t) = e^{At}x(0)$$

Matrix Exponential Identities:4.1

Announced here and proved below are various formulae and identities for the matrix exponential e^{At}.

$(e^{At})' = Ae^{At}$	Columns satisfy $x' = Ax$.
$e^0 = I$	Where 0 is the zero matrix
$Be^{At} = e^{At}B$	If $AB = BA$.
$e^{At}e^{Bt} = e^{(A+B)t}$	If $AB = BA$.
$e^{At}e^{st} = e^{(A+s)t}$	At and As commute.
$(e^{At})' = e^{-At}$	Equivalently, $e^{At}e^{iAt} = I$.
$e^{At} = r_1(t)P_1 + \cdots + r_n(t)P_n$	Putzer's spectral formula
$e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 \neq \lambda_2$ real.
$e^{At} = e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 = \lambda_2$ real.
$e^{At} = e^{at}\cos bt\, I + \frac{e^{at}\sin nt}{b}(A - aI)$	A is 2×2, $\lambda_1 = \bar{\lambda}_2 = a + ib, b > 0$

$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$ Picard series.

$e^{At} = P^{-1}e^{Jt}P$ Jordan form $J = PAP^{-1}$

Putzer's Spectral Formula: 4.2

The spectral formula of Putzer applies to a system $x' = Ax$ find the general solution, using matrices $P_1, \ldots, P_n$ constructed from A and the eigenvalues $\lambda_1, \ldots, \lambda_2$ of A, matrix multiplication, and the solution $r(t)$ of the first order $n \times n$ initial value problem

$$r'(t) = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & \lambda_2 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \lambda_3 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & \lambda_n \end{pmatrix} r(t), r(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The system is solved by first order scalar method and back-substitution. We will derive the formula separately for the 2×2 case (the one used most often and the $n \times n$ case.

Putzer's 2×2 Spectral Formula:4.3

The general solution of $x' = Ax$ is given by the formula

$$x(t) = (r_1(t)P_1 + r_2(t)P_2)x(0),$$

where r_1, r_2, P_1, P_2 are defined as follows.

The eigenvalues $r = \lambda_1, \lambda_2$ are the two roots of the quadratic equation

$$\det(A - rI) = 0$$

Define 2×2 matrices P_1, P_2 by the formulae

$$P_1 = I, \qquad P_2 = A - \lambda_1 I$$

The functions $r_1(t), r_2(t)$ are defined by the differential system

$$r_1' = \lambda_1 r_1, \qquad r_1(0) = 1,$$
$$r_2' = \lambda_2 r_2 + r_1, r_2(0) = 1$$

Proof:

The Cayley-Hamilton formula $(A - \lambda_1 I)(A - \lambda_2 I) = 0$ is valid for any 2×2 matrix A and the two roots $r = \lambda_1, \lambda_2$ of the determinant equality $\det(A - rI) = 0$. The Cayley-Hamilton formula is the same as $(A - \lambda_2)P_2 = 0$, which implies the identity $AP_2 = \lambda_2 P_2$. Compute as follows.

$$\begin{aligned} x'(t) &= (r_1' P_2 + r_2' P_2)x(0) \\ &= (\lambda_1 r_1(t)P_1 + r_1(t)P_2 + \lambda_2 r_2(t)P_2)x(0) \\ &= (r_1(t)A + \lambda_2 r_2(t)P_2)x(0) \\ &= (r_1(t)A + r_2(t)AP_2)x(0) \\ &= A(r_1(t)I + r_2(t)P_2)x(0) = Ax(t) \end{aligned}$$

This proves that $x(t)$ is a solution. Because $\Phi(t) \equiv r_1(t)P_1 + r_2(t)P_2$ satisfies $\Phi(0) = I$, then any possible solution of $x' = Ax$ can be represented by the given formula. The proof is complete.

4.8 How to Remember Putzer's 2×2 Formula

The expressions $e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I)$,

$r_1 = e^{\lambda_1 t}, r_2 = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}$are enough to generate all three formulae. The fraction $r_2(t)$ is difference quotient with limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \bar{\lambda}_2 = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$r_2(t) = \frac{z - \bar{z}}{w - \bar{w}} = \frac{\sin bt}{b}.$$

Taking real parts to expression (1) then gives the complex case formula for e^{At}.

4.9 Putzer's $n \times n$ Spectral Formula

The general solution of $x' = Ax$ is given by the formula

$$x(t) = (r_1(t)P_1 + r_2(t)P_2 + \cdots + r_n(t)P_n)x(0),$$

where $r_1, r_2, \ldots, r_n, P_1, P_2, \ldots, P_n$ are defined as follows.

The eigenvalues $r = \lambda_1, \ldots, \lambda_n$ are the roots of the polynomial equation

$$\det(A - rI) = 0.$$

Define $n \times n$ matrices $P_1, P_2, \ldots, P_n$ by the formulae

$$P_1 = I, P_k = P_{k-1}(A - \lambda_{k-1}I), \qquad k = 2, \ldots, n.$$

More succinctly, $P_k = \prod_{j=1}^{k-1}(A - \lambda_j I)$. The functions $r_1(t), \ldots, r_n(t)$ are defined by the differential system

$$\begin{aligned} r_1' &= \lambda_1 r_1, & r_1(0) &= 1, \\ r_2' &= \lambda_2 r_2 + r_1, & r_2(0) &= 0, \\ &\vdots \\ r_n' &= \lambda_n r_n + r_{n-1}, & r_n(0) &= 0, \end{aligned}$$

Proof:

The Cayley-Hamilton formula $(A - \lambda_1 I) \ldots (A - \lambda_n I) = 0$ is valid for any $n \times n$ matrix A and the n roots $r = \lambda_1, \ldots, \lambda_n$ of the determinant equality implies $AP_n = \lambda_n P_n$; (2) The definition of P_k implies $\lambda_k P_k + P_{k+1} = AP_k$ for $1 \leq k \leq n - 1$. Compute as follows.

$$\boxed{1}\quad x'(t) = (r_1'(t)P_1 + \cdots + r_n'(t)P_n)x(0)$$

$$\boxed{2}\quad = \left(\sum_{k=1}^{n} \lambda_k r_k(t)P_k + \sum_{k=2}^{n} r_{k-1}P_k\right)x(0)$$

$$\boxed{3}\quad = \left(\sum_{k=1}^{n-1} \lambda_k r_k(t)P_k + r_n(t)\lambda_n P_n + \sum_{k=1}^{n-1} r_k P_{k+1}\right)x(0)$$

$$\boxed{4}\quad = \left(\sum_{k=1}^{n-1} r_k(t)(\lambda_k P_k + P_{k+1}) + r_n(t)\lambda_n P_n\right)x(0)$$

$$\boxed{5}\quad = \left(\sum_{k=1}^{n-1} r_k(t)AP_k + r_n(t)AP_n\right)x(0)$$

$$\boxed{6}\quad = A\left(\sum_{k=1}^{n} r_k(t)P_k\right)x(0)$$

$$\boxed{7}\quad = Ax(t)$$

Details: $\boxed{1}$ Differentiate the formula for $x(t)$. $\boxed{2}$ Use the differential equations for $r_1, \ldots, r_n$. $\boxed{3}$ Split off the last term from the first sum, then re-index the last sum. $\boxed{4}$ Combine the two sums. $\boxed{5}$ Use the recursion for P_k and the Cayley-Hamilton formula $(A - \lambda_n I)P_n = 0$. $\boxed{6}$ Factor out A on the left. $\boxed{7}$ Apply the definition of $x(t)$.

This proves that $x(t)$ is a solution. Because $\Phi(t) \equiv \sum_{k=1}^{n} r_k(t)P_k$ satisfies $\Phi(0) = I$, then any possible solution of $x' = Ax$ can be so represented. The proof is complete.

5. SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

In this section, we will discuss system of first order differential equations. There are many applications that involving find several unknown functions simultaneously. Those unknown functions are related by a set of equations that involving the unknown functions and their first derivatives. For example, in chapter Two, we studied the epidemic of contagious diseases. Now if

- $S(t)$ denotes number of people that is susceptible to the disease but not infected yet.
- $I(t)$ denotes number of people actually infected.
- $R(t)$ denotes the number of people have recovered.

If we assume

- The fraction of the susceptible who becomes infected per unit time is proportional to the number infected, b is the proportional number.
- A fixed faction rS of the infected population recovers per unit time, $0 \leq r \leq 1$.
- A fixed fraction of the recovers g become susceptible and infected, $0 \leq g \leq 1$ proportional function.

The system of differential equations model this phenomena are

$$S' = -bIS + gR$$
$$I' = bIS - rI$$
$$R' = rI - gR$$

The numbers of unknown function in a system of differential equations can be arbitrarily large, but we will concentrate ourselves on 2 to 3 unknown functions.

Principle of superposition:5.1

Let $a_{ij}(t), b_j(t),\ i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n$ be known function, and $x_i t, i = 1,2,\ldots,n$ be unknown functions, the linear first order system of differential equation for $x_i t$ is the following,

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t)$$
$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t)$$
$$x_3'(t) = a_{31}(t)x_1(t) + a_{32}(t)x_2(t) + \cdots + a_{3n}(t)x_n(t) + b_3(t)$$
$$\vdots$$
$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_1(t)$$

Let $x(t)$ be the column vector of unknown functions $x_i t, i = 1,2,\dots,n, A(t) = a_{ij}(t)$ and $b(t)$ be the column vector of known functions $b_i t, i = 1,2,\dots,n$, we can write the first order system of equations as

$$x'(t) = A(t)x(t) + b(t) \tag{1}$$

- When $n = 2$, the linear first order system of equations for two unknown functions in matrix form is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

- When $n = 3$, the linear first order system of equations for three unknown functions in matrix form is,

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13} \\ a_{21}(t) & a_{22}(t) & a_{23} \\ a_{31}(t) & a_{32}(t) & a_{33} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3 t \end{bmatrix} + \begin{bmatrix} b_1(t) \\ b_2(t) \\ b_3(t) \end{bmatrix}$$

A solution of equation (1) on the open interval I is an column vector function $x(t)$ whose derivative (as a vector-values function) equals $A(t)x(t) + b(t)$. The following theorem gives existence and uniqueness of solutions.

Theorem: 5.2

If the vector-valued functions $A(t)$ and $b(t)$ are continuous over an open interval t_0, then the initial value problem

$$\begin{cases} x'(t) = A(t)x(t) + b(t) \\ x(t_0) = x_0 \end{cases}$$

has an unique vector-values solution $x(t)$ that is defined on entire interval I fir any given initial value x_0.

When $b(t) \equiv 0$, the linear first order system of equations becomes

$$x'(t) = A(t)x(t),$$

which is called a homogeneous equation.

As in the case of one equation, we want to find out the general solutions for the linear first order system of equations. To this end, we first have the following results for the homogeneous equations.

Theorem: 5.3

Principle of Superposition Let $x_1(t), bx_2(t), \dots, x_n(t)$ be n solutions of the homogeneous linear equation

$$x'(t) = A(t)x(t)$$

on the open interval I. If $c_1, c_2, \dots, c_n$ are n constants, then the linear combination

$$c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + \cdots + c_nx_n(t)$$

is also a solution on I.

Theorem: 5.4

Let $x_1(t), x_2(t), \dots, bx_n(t)$ be n linearly independent (as vectors) solution of the homogeneous system

$$x'(t) = A(t)x(t)$$

then for any solution $x_c(t)$ there exists n constants $c_1, c_2, \dots, c_n$ such that

$$x_c(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + \cdots + c_nx_n(t)$$

We call $x_c(t)$ the general solution of the homogeneous system.
If $x_p(t)$ is a particular solution of the nonhomogeneous system,

$$x(t) = B(t)x(t) + b(t)$$

and $x_c(t)$ is the general solution to the associate homogeneous system,

$$x(t) = B(t)x(t)$$

then $x(t) = x_c(t) + x_p(t)$ is the general solution.

Homogeneous System:5.5

We will use a powerful method called eigenvalue method to solve the homogeneous system

$$x'(t) = Ax(t)$$

Where A is a matrix with constant entry. We will present this method for A is either a 2×2 or 3×3 cases. The method can be used for A is an $n \times n$ matrix. The idea is to find solutions of form

$$x(t) = ve^{\lambda t}, \tag{3}$$

a straight line that passing origin in the direction v. Now taking derivative on $x(t)$, we have

$$x'(t) = \lambda v e^{\lambda t} \tag{4}$$

put (3) and (2.2) into the homogeneous equation, we get

$$x'(t) = \lambda v e^{\lambda t} = A v e^{\lambda t}$$

So

$$Av = \lambda v$$

which indicates that λ must be an eigenvalue of A and v is an associate eigenvector.
A is 2×2 matrix. suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then the characteristic polynomial $p(\lambda)$ of A is

$$p(\lambda) = |A - \lambda I| = (a_{11} - \lambda) * (a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{22})$$

So $p(\lambda)$ is a quadratic polynomial of λ. From Algebra, we know that $p(\lambda) = 0$ has either 2 distinct real solutions, or a double solution, or 2 conjugate complex solutions. The following theorem summarize the solution to the homogeneous system.

Theorem: 5.6

Let $p(\lambda)$ be the characteristic polynomial of A, for $x'(t) = Ax(t)$,

Case 1:

$p(\lambda) = 0$ has two distinct real solutions λ_1 and λ_2

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ are associate eigen-vector (i.e, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$). Then the general solution is

$$x_c(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

And

$$\Phi(t) = \begin{bmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{bmatrix}$$

is called the fundamental matrix (A fundamental matrix is a square matrix whose columns are linearly independent solutions of the homogeneous system).

Case 2:

$p(\lambda) = 0$ has a double solutions λ_0

In this case $p(\lambda) = (\lambda - \lambda_0)^2$ and λ_0 is a zero of $p(\lambda)$ with multiplicity 2.

(1) λ_0 has two linearly independent eigenvectors:

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ are associate linearly independent eigenvectors. Then the general solution is

$$x_c(t) = (c_1 v_1 + c_2 v_2)e^{\lambda_0 t}$$

And

$$\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

(2) λ_0 has only one associate eigenvector:

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ is the only associate eigenvector and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ is a solution of,

$$(\lambda_0 I - A)v_2 = v_1$$

Then the general solution is,

$$x_c(t) = (c_1 v_1 + c_2(t v_1 + v_2))e^{\lambda_0 t}$$

And

$$\Phi(t) = e^{\lambda_0 t}\begin{bmatrix} v_{11} & (v_{11}t + v_{12}) \\ v_{21} & (v_{21}t + v_{22}) \end{bmatrix}$$

is the fundamental solution matrix.

Case 3:

$p(\lambda) = 0$ has two conjugate complex solutions $a + bi$ and $a - bi$.

Suppose $v = \begin{bmatrix} v_{11} + iv_{12} \\ v_{21} + iv_{22} \end{bmatrix}$ is the associate complex eigen-vector with respect $a + bi$, then the general solution is $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$

$$x_c(t) = [c_1(v_1 \cos(bt) - v_2 \sin(bt))c_2(v_2 \cos(bt) + v_1 \sin(bt))]e^{at}$$

And

$$\Phi(t) = e^{at} \begin{bmatrix} v_{11} \cos(bt) - v_{12} \sin(bt) & v_{12} \cos(bt) + v_{11} \sin(bt) \\ v_{21} \cos(bt) - v_{22} \sin(bt) & v_{22} \cos(bt) + v_{21} \sin(bt) \end{bmatrix}$$

is the fundamental matrix.

Form Theorem 6.6, let $\Phi(t)$ be the fundamental matrix, the general solution is given by $x_c(t) = \Phi(t)c$, with $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and the solution that satisfies a given initial condition $x(t_0) = x_0$ is given by

$$x(t) = \Phi(t)\Phi(t_0)^{-t}x_0$$

A is a 3×3 matrix.:5..7

Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then the characteristic polynomial $p(\lambda)$ of given by

$$p(\lambda) = |A - \lambda I|$$

is a cubic polynomial of λ. From Algebra, we know that $p(\lambda) = 0$ has either 3 distinct real solutions, or 2 distinct solutions and one is a double solution, or one real solution and 2 conjugate complex solutions, or a triple solution. The following theorem summarize the solution to the homogeneous system.

Theorem: 5.8

Let $p(\lambda)$ be the characteristic polynomial of A, for $x'(t) = Ax(t)$,

Case 1:

$p(\lambda) = 0$ has three distinct real solutions λ_1, λ_2, and λ_3

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}, v_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$, and $v_3 = \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$

are associate eigenvector (i.e., $Av_1 = \lambda_1 v_1, Av_2 = \lambda_2 v_2$ and $Av_3 = \lambda_3 v_3$).

Then the general solution is

$$x_c(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + c_3 v_3 e^{\lambda_3 t}$$

And the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} & v_{13}e^{\lambda_3 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & v_{23}e^{\lambda_3 t} \\ v_{31}e^{\lambda_1 t} & v_{32}e^{\lambda_2 t} & v_{33}e^{\lambda_3 t} \end{bmatrix}$$

Case 2:

$p(\lambda) = 0$ has a double solution λ_0.

So $p(\lambda) = (\lambda - \lambda_0)^2(\lambda - \lambda_1)$, and λ_0 has multiplicity 2. Let $v_3 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ is the eigenvector associated with λ_1.

(1) λ_0 has two linearly independent eigenvectors:

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix}$ are associate linearly independent eigenvectors. Then the general solution is

$$x_c(t) = (c_1 v_1 + c_2 v_2)e^{\lambda_0 t} + c_3 v_3 e^{\lambda_1 t}$$

And

$$\Phi(t) = \begin{bmatrix} v_{11}e^{\lambda_0 t} & v_{12}e^{\lambda_0 t} & v_{13}e^{\lambda_1 t} \\ v_{21}e^{\lambda_0 t} & v_{22}e^{\lambda_0 t} & v_{23}e^{\lambda_1 t} \\ v_{31}e^{\lambda_0 t} & v_{32}e^{\lambda_0 t} & v_{33}e^{\lambda_1 t} \end{bmatrix}$$

(2) **λ_0 has one eigenvector:**

Suppose $v_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ is the associated eigenvector with respect to λ_0 and $v_2 = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ is a solution of

$$(\lambda_0 I - A)v_2 = v_1$$

And

$$\Phi(t)=\begin{bmatrix} v_{11}e^{\lambda_0 t} & (v_{11}t+v_{12})e^{\lambda_0 t} & v_{13}e^{\lambda_1 t} \\ v_{21}e^{\lambda_0 t} & (v_{21}t+v_{22})e^{\lambda_0 t} & v_{23}e^{\lambda_1 t} \\ v_{31}e^{\lambda_0 t} & (v_{31}t+v_{32})e^{\lambda_0 t} & v_{33}e^{\lambda_1 t} \end{bmatrix}$$

is the fundamental solution matrix.

Case 3:

$p(\lambda)=0$ has two conjugate complex solutions $a \pm bi$ and a real solution λ_1.

Suppose $v=\begin{bmatrix} v_{11}+iv_{12} \\ v_{21}+iv_{22} \\ v_{31}+iv_{32} \end{bmatrix}$ is the associate complex eigenvector with respect to $a+bi$, then the general solution is, let $v_3=\begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$, are associated eigenvectors with respect to λ_1

$$x_c(t)=[c_1(v_1\cos(bt)-v_2\sin(bt))c_2(v_2\cos(bt)+v_1\sin(bt))]e^{at}+c_3v_3e^{\lambda_3}$$

And

$$\Phi(t)=e^{at}\begin{bmatrix} v_{11}\cos(bt)-v_{12}\sin(bt) & v_{12}\cos(bt)+v_{11}\sin(bt) & v_{13}e^{\lambda_1} \\ v_{21}\cos(bt)-v_{22}\sin(bt) & v_{22}\cos(bt)+v_{21}\sin(bt) & v_{23}e^{\lambda_1} \\ v_{31}\cos(bt)-v_{32}\sin(bt) & v_{32}\cos(bt)+v_{31}\sin(bt) & v_{33}e^{\lambda_1} \end{bmatrix}$$

Case 4:

$p(\lambda)=0$ has solution λ_0 with multiplicity 3.

In this case, $p(\lambda)=(\lambda-\lambda_0)^3$

(1) λ_0 has three linearly independent eigenvectors

Let $v_1=\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}, v_2=\begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$, and $v_3=\begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ be the three linearly independent eigenvectors. Then the general solution is $x_c(t)=(c_1v_1+c_2v_2+c_3v_3)e^{\lambda_0 t}$ and fundamental matrix is

$$\Phi(t)=\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & & \\ v_{32} & & \\ v_{33} & & \end{bmatrix}$$

(2) λ_0 has two linearly independent eigenvectors.

Suppose $v_1=\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}, v_2=\begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ are the linearly independent eigenvectors. Let $v_3=\begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$, then only one of the two equations, $(A-\lambda_0 I)v_3=v_1$ or $(A-\lambda_0 I)v_3=v_2$ can has a solution that is linearly independent with v_1, v_2.

Suppose $(A-\lambda_0 I)v_3=v_2$ generates such a solution. Then the general solution is $x_c(t)=[c_1v_1+c_2v_2+c_3(tv_2+v_3)]e^{\lambda_0 t}$ and fundamental matrix is

$$\Phi(t)=e^{\lambda_0 t}\begin{bmatrix} v_{11} & v_{12} & tv_{12}+v_{13} \\ v_{21} & v_{22} & tv_{22}+V_{23} \\ v_{31} & v_{32} & tv_{32}+V_{33} \end{bmatrix}$$

(3) λ_0 has only one eigenvector.

Let $v_1=\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}$ be the linearly independent eigenvectors. Let $v_2=\begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$ and $v_3=\begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix}$ be two vectors that satisfies

$$(A-\lambda_0 I)v_2=v_1 \ \ and \ \ (A-\lambda_0 I)v_3=v_2$$

Then the general solution

$$x_c(t)=[c_1v_1+c_2(tv_1+v_2)+c_3(t^2v_1+tv_2+v_3)]e^{\lambda_0 t}$$

and fundamental matrix is

$$\Phi(t)=e^{\lambda_0 t}\begin{bmatrix} v_{11} & tv_{11}+v_{12} & t^2v_{11}+tv_{12}+v_{13} \\ v_{21} & tv_{21}+v_{22} & t^2v_{21}+tv_{22}+V_{23} \\ v_{31} & tv_{31}+v_{32} & t^2v_{31}+tv_{32}+V_{33} \end{bmatrix}$$

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