A COMPREHENSIVE REVIEW OF IRREGULAR GRAPH IN PRODUCT GRAPHS

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Abstract:

In this paper we discuss about the irregularity of product graphs and its characterization graph.

Keywords: Graph Theory, regular graph, irregular graph, neighbourly irregular graph and lattice theory.

1 Introduction

Modern business and technology with its vital concern for the efficient utilization of its limited resources provides an excellent source of challenging problems in discrete mathematics. In this paper, we concerned ourselves with NI graphs, namely graphs in which adjacent vertices have distinct degrees. Since not every situation that we will encounter will be this simple, we must be prepared to deal with the graphs with distinct neighbourhoods. When defining highly irregular graphs Yousef Alavi considered the degree of vertices in the neighbourhood set. If we use the closed neighbourhood of vertices instead of using degree, we can arrive a definition of NHI graph. Thus in this paper, we deal with those graphs with any two distinct vertices in the open neighbourhood of any vertex have distinct closed neighbourhoods.

For any vertex v in V, let $N_G(v) = \{ u\epsilon V: uv\epsilon E\}$ be the closed neighbourhood of v, let $N_G[v] = N_G(v)\cup \{v\}$ be the closed neighbourhood of v. A connected graph G is said to be neighbourhood highly irregular (or simply NHI) if for any vertex vϵ V, any two distinct vertices in the open neighbourhood of v have distinct closed neighbourhood sets. Of course, a disconnected graph in which each component is NHI can also be considered as an NHI graph. In this paper we give a necessary and sufficient condition for a graph to be NHI. For any $n\geq 1$, we obtain a sharp lower bound for the order of regular NHI graphs and a sharp lower bound for the order of NHI graphs with clique

2 Results on NHI graphs:

In this section, we characterize the class of NHI graphs. We also find the smallest order of an r- regular NHI graph. In addition, we prove that, the smallest order of an NHI graph with clique number n is n+m where m is the least positive integer such that $n\leq 2^m$. The following results gives a necessary and sufficient condition for a graph to be NHI.

Theorem 2.1

A connected graph G with $n\geq3$ is NHI if and only if $N_G[u] \neq N_G[v]$ for any pair of adjacent vertices u and v in G with d(u) = d(v).

Proof

Let G be a connected graph in which. $N_G[u] \neq N_G[v]$ for any two adjacent vertices u and v of same degree in G. however, obviously, for any two non-adjacent vertices u and v of G and for any two adjacent vertices u and v of distinct degrees,$N_G[u] \neq N_G[v]$ and therefore, G is NHI. Conversely, assume that G is NHI. Let u and v be two adjacent vertices of same degree in V. we claim that $N_G[u] \neq N_G[v]$. If u and v have a common neighbour w, then u and v are in N(w). therefore, since G is NHI, $N_G[u] \neq N_G[v]$. suppose u and v have no common neighbour. In this case, since n≥3 and G is connected, there is a vertex w in N(u) (in N(v)) which is not in N(v) (in N(u)). This forces that $N_G[u] \neq N_G[v]$. Hence the theorem

In the above theorem, if we consider the open neighbourhood instead of the closed neighbourhood then the theorem need not be true. For example, in a complete graph K_n, N[u] ≠N[v] for any pair of adjacent vertices u and v in K_n with d(u) = d(v), but k_n is not NHI, n≥3. Note that K_2 is NHI , in which $N_G[u] = N_G[v]$. In fact, K_n is the only graph in which $N_G[u] = N_G[v]$ for any two vertices u and v. For, clearly in K_n,$N_G[u] = N_G[v]$ for any two vertices u and v. In addition, if G is a graph in which $N_G[u] = N_G[v]$ for any two vertices u and v, then u and v are adjacent in G. this means that, G is complete. For any connected graph G which is not NI, let ℓ_G (or simply ℓ) denote the least positive integer such that G has two adjacent vertices of degree ℓ. Note that, $\ell \geq 2$ whenever $\ell \geq 3$. Recall that for any two vertex disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ the graph $G_1 \cup G_2$ with the vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$ is called the union of G_1 and G_2. The join, $G_1 \vee G_2$, of the graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 by means of an edge. Let K_n^c denote the null graph on n vertices.

Corollary

Let G be a connected graph with n≥ 3. If G is NI or G contains no $K_2 \vee K_{l-1}^c$ as a subgraph, then G is NHI.

Proof

If G is NI, then obviously G is NHI. Assume that G contains no $K_2 \vee K_{l-1}^c$ as a subgraph. If G is not NHI, then by the above theorem, there are two adjacent vertices u and v of same degree m in G such that $N_G[u] = N_G[v]$. Therefore, $|NG[u] = NG[v]|$ = m+1 and hence $K_2 \vee K_{m-1}^c$is a subgraph of G . since $\ell \leq$ m this force that G contains $K_2 \vee K_{l-1}^c$ as a subgraph, a contradiction. Hence G must be NHI. The converse of the above corollary need not be true. For example, the graph shown in Figure 1 in NHI but not NI with $\ell = 2$. In addition, it contains $K_2 \vee K_1^c$ as a subgraph.

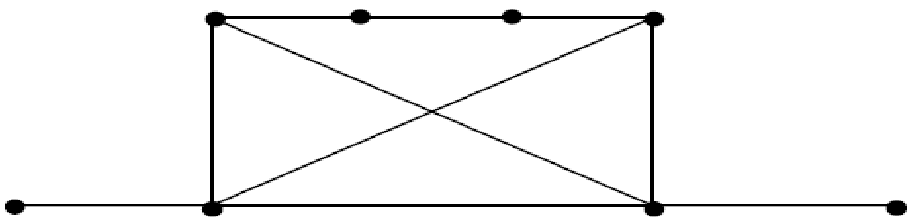

Figure 1

Corollary

Any connected triangle free graph is NHI. Here we present a new proof using the above corollary.

Proof

If n = 1 or 2, the result is obvious, Assume that n≥ 3. If G is NI, then clearly it is NHI. If G is not NI, then $\ell \geq 2$. Since G is triangle free, G contains no $K_2 \vee K_{l-1}^c$ and this follows that G is NHI by corollary. Since any connected bipartite graph is triangle free, we have.

Corollary

Any connected bipartite graph is NHI. Next we establish another characterization for NHI graph.

Theorem 2.2

A connected graph G with n≥ 3 is NHI if and only if $N_{G^c}(u) \neq N_{G^c}(v)$ for any two vertices u and v.

Proof

Let G be an NHI graph. Suppose there are vertices u and v such that $N_{G^c}(u) = N_{G^c}(v)$. Then u and v are not adjacent in G^c and hence adjacent in G and $N_G[u] = N_G[v]$ also. Therefore, u and v have same degree in G such that $N_G[u] = N_G[v]$, which contradicts Theorem 3.2.1. Hence $N_{G^c}(u) \neq N_{G^c}(v)$ for any two vertices u and v. Conversely, suppose G is not NHI. Again, by theorem 2.1, G has two adjacent vertices u and v with same degree such that $N_G[u] = N_G[v]$. This implies that u and v are non-adjacent in G^c with $N_{G^c}(u) = N_{G^c}(v)$. That is, in G^c there are two vertices u and v such that $N_{G^c}(u) = N_{G^c}(v)$. Hence the theorem.

Theorem2.3

For any n ≥ 5, $K_n \setminus H$ is NHI, where H is a Hamiltonian cycle in K_n.

Proof

Let the vertices of K_n be $v_0, v_1, \ldots v_{n-1}$. Through out this proof, the operation + is addition modulo n. Let E(H) = { $e_i = v_i v_{i+1}$: 0 ≤i≤ n-1 }. Now G = $K_n \setminus H$ is an (n-3) -regular graph of order n, in which, for 0 ≤i≤ n- 1, N(v_i) = { $v_{i+2}, v_{i+3}, \ldots . v_{n-2+i}$}. Therefore, if v_i and v_j, 0 ≤i< j ≤ n-1,are adjacent vertices in G, then clearly for (i, j) ∉ { (o, n- 2), (1, n-1)} $v_{i-1} \epsilon$ N[v_j] \ N[v_i] and $v_{j+1} \epsilon$ N[v_i] \ N[v_j] otherwise $v_{i+1} \epsilon$ N[v_j] \ N[v_i] and $v_{j-1} \epsilon$ N[v_i] \ N[v_j]. That is ,N[v_i] ≠ N[v_j]. Hence, by theorem 2.2, G is NHI. The above theorem can be restated as follows:

Corollary

C_n^c is NHI, for all n≥ 5. In a similar way, one can prove that

Theorem 2.4

$P_n^c = K_n \setminus P_n$ **is** NHI, for any n > 3.

For even n≥ 4, let the vertices of $K_n$be $v_1, v_2, \ldots v_n$ and let F = {$v_{2i-1} v_{2i}$, 1≤ i ≤ n\2} be a 1-factor inK_n. Then it has been proved that the regular graph $K_n \setminus$ F is NHI. In fact, more generally, we can prove that complement of an NHI graph G is NHI.

Theorem 2.5

If a graph G is NHI, then its complement G^c is also NHI.

Proof

Let G be an NHI graph. We claim that G^c is also an NHI graph. Let u and v be two adjacent vertices in G^c with $d_{G^c}(u) = d_{G^c}(v)$. Then u and v are non-adjacent in G such that $d_G(u) = d_G(v)$. Since G is NHI, by theorem 2.4 $N_{G^c}(u) = N_{G^c}(v)$. Consequently, $N_{G^c}(u) \neq N_{G^c}(v)$. Hence by theorem 2.1 G^c is NHI.

Theorem 2.6

For r ≥ 2, the smallest order of an r-regular NHI graph is $\begin{cases} r+2, if\ r\ is\ even \\ r+3, if\ r\ is\ odd \end{cases}$. Also the bound is strict.

Proof

Let G be an r- regular NHI graph with vertices. Then p≥ r+1. If p = r+1, then G is complete which is not NHI and hence p≥ r+2. However, when r is even, K_{r+2}\F is an r- regular NHI graph on r+2 vertices. In addition, when r is odd, r+2 is also odd and hence p ≥ r+3. Moreover, K_{r+3}\H where H is a Hamiltonian cycle in K_{r+2}, is an r- regular NHI graph on r+3 vertices. Hence, the smallest order of the r- regular NHI graph is $\begin{cases} r+2, if\ r\ is\ even \\ r+3, if\ r\ is\ odd \end{cases}$.

Theorem 2.7

For any, the smallest order of an NHI graph with clique number n is n+ m where m is the least positive integer such that n $\leq 2^m$. Before proving the theorem, we discuss the following:

For any two positive integers i and k, 1≤ k ≤i, a B(k, i) – graph is a bipartite graph with bipartition (V_1 ,V_2) where $|V_1|$ = $\binom{i}{k}$ and $|V_2|$ = i in which every vertex in V_1 is of degree k and every vertex in V_2 is of degree $\binom{i-1}{k-1}$. For example, the graph shown in Figure 2 is B(2, 4). The existence of such a graph is proved in Lemma 3.2.12. Note that when k = 1, B(1, 1) is a 1- regular graph with 2i vertices.

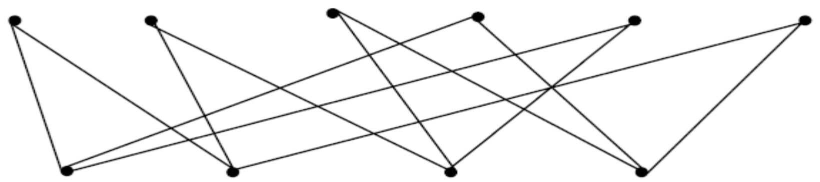

Figure 2

For 1 ≤ k ≤i, a graph is called a B(k, i) -graph if it is a bipartite graph with bipartition (V_1, V_2) where $|V_1|<\binom{i}{k}$ and $|V_2|$ = i in which every vertex in V_1 is of degree k and every vertex in V_2 is of degree less than or equal to $\binom{j-1}{k-1}$. For example, a B(3, 5) – graph is shown in Figure 3.

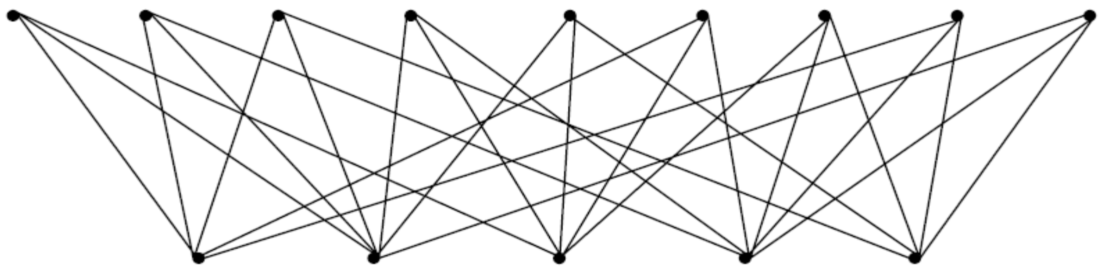

Figure 3

Clearly, all the B (k, i) – graph and the B'(k, i) -graph are NHI. Since they are biparitie.

Lemma 2.8

For any 1≤ k ≤i, B(k, i) -graph exists.

Proof

Let V = $V_1 \cup V_2$ where V_1 contains the vertices $v_{1,} v_2, \dots, v_{\binom{i}{k}}$ and V_2 contains $u_{1,} u_2, \dots, u_i$ and let $u_{1,} u_2, \dots, u_{\binom{i}{k}}$ be the distinct k- subsets (subsets with k elements) of V_2. Join V_j with every element of U_j,for 1 ≤ j ≤$\binom{i}{k}$. Then the resultant graph G is bipartite with bipartition ($V_{1,}$ V_2) in which $|V_1|$ = $\binom{i}{k}$and $|V_2|$ = i. Moreover, every vertex in $V_1$is adjacent to exactly k vertices of V_2 and every vertex in V_2 is adjacent to exactly $\binom{i-1}{k-1}$ vertices of $V_1$is of degree k and each vertex in V_2 is of degree $\binom{i-1}{k-1}$ and hence G is B(k, i)-graph.

Lemma 2.9

For any 1≤ k ≤i, there is a B (k, i) -graph**.**

Proof

Let V= $V_1 \cup V_2$ where V_1 contains the vertices $V_1, V_2,\dots$ such that $|V_1|$<$\binom{i}{k}$ and V_2 = { $U_1, U_2,\dots U_i$} and let $U_1, U_2,\dots U_{\binom{i}{k}}$be the distinct k -subsets (subsets with k elements) of V_2. Join V_j with every element of U_j,for 1≤ j ≤$\binom{i}{k}$. Then the resultant graph G is bipartite with bipartition ($V_{1,}$ V_2) in which $|V_1|$<$\binom{i}{k}$ and $|V_2|$ =i. Moreover, every vertex in $V_1$is of degree k and each vertex in V_2 is of degree less than or equal to $\binom{i-1}{k-1}$ vertices of V_1. Thus each vertex in V_1 is of degree k and each vertex in V_2 is of degree less than or equal to $\binom{i-1}{k-1}$ and hence G is B(k, i) -graph.

Now we prove the main theorem.

Proof of theorem 2.7

For any n≥ 1, we first construct an NHI graph G_n of order n+ m with clique number n.

If n =1 or 2, then K_1 and P_3 are respectively the required graphs. So, assume that n≥ 3.

Let $\{v_1, v_2, \dots v_n{:}u_1, u_2, \dots u_m\}$ be the vertices of G_n. Take V_1 = $\{v_1, v_2, \dots v_n\}$ and w = $\{u_1, u_2, \dots u_m\}$. Suppose U_0 contains the first $\binom{m}{0}$ vertex, that is, v_1 of $V_{1,}U_1$ contains the next $\binom{m}{1}$ vertices of V_1 and so on. In general, $U_k$contains the $\binom{m}{k}$ vertices next to the vertices of U_{k-} in V_1. When n< 2^m, there exists j, 0< j < m, such that $|U_j|$ = $\binom{m}{j}$ and$|V_1 \setminus \cup_{k=0}^{j} U_k|$<$\binom{m}{j+1}$. In this case, take U_{j+1} = $V_1 \setminus \cup_{k=0}^{j} U_k$ and $U_{j+2}, U_{j+3},\dots U_m$ are all empty sets. Note that the set U_{j+1}may also be empty. Now we define the edge set of G_n as follows:

1. Add the edges among the vertices of V_1 such that $\langle V_1 \rangle \cong K_n$.
2. When n = 2^m, for 1≤ k ≤ m, add the edges between the vertices of $U_k$and w such that $\langle U_k, W \rangle$ is a B(k, m) – graph.
3. When n< 2^m,

a. For 1≤ k ≤ j ≤ m, add the edges between the vertices of U_k and W such that $\langle U_k, W\rangle$ is a B(k, m) – graph and

b. If U_{j+1} is nonempty then add the edges between the vertices of U_{j+1} and W such $\langle U_{j+1}, W\rangle$ is a B(j+1, m) -graph.

The resultant graph G_n is an NHI graph of order n+ m with clique number n.

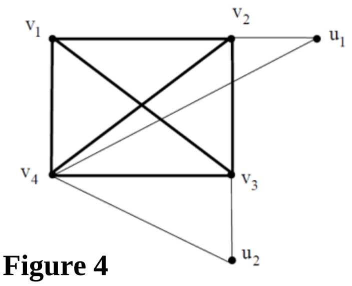

Figure 4

G_4

The graph $G_4$is shown in Figure 4, G_5, G_6,G_8 are shown if Figure 5 $G_9$is shown if Figure 6.

Now, it is enough to show that n+ m is minimum.

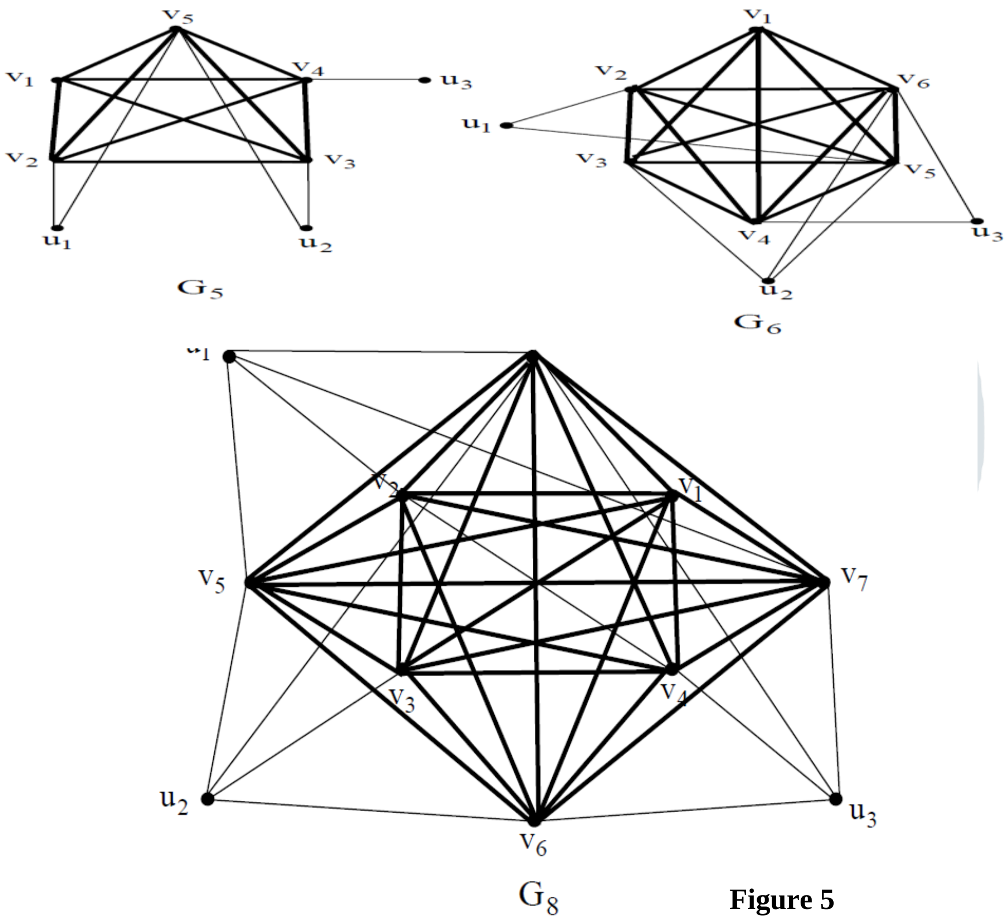

Figure 5

Suppose that there is a graph G with clique number n and order n+ s where s < m. Let W = $\{v_1, v_2, \ldots . v_n\}$be the set of vertices of G which induces K_n in G. Let U = $\{u_1, u_2, \ldots . u_s\}$ be the set of remaining vertices of G. Let W_0 be the set of all vertices of W having no neighbours in U. For 1 ≤ t ≤ s, let W_t⊆ W be the set of all vertices of G with degree t in $\langle W, U\rangle$.

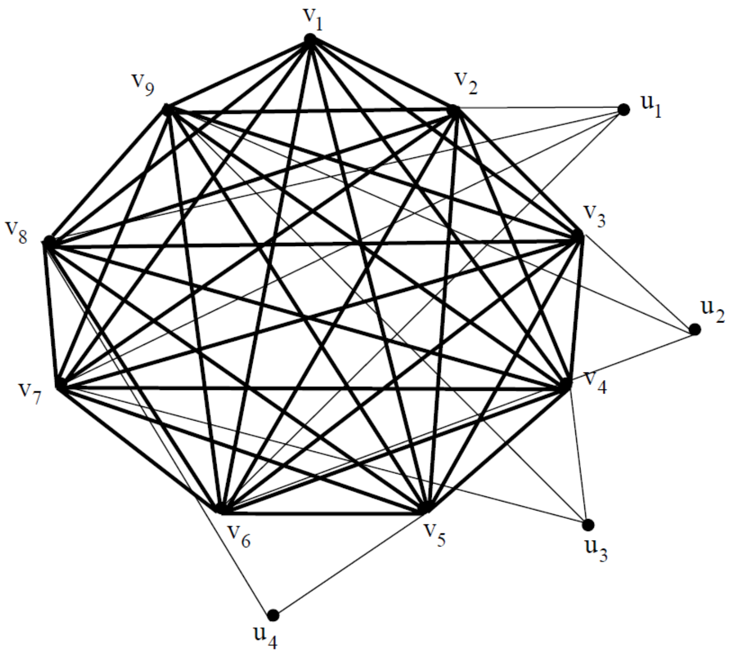

Figure 6

Claim W_t contains at most $\binom{s}{t}$ vertices, o ≤ t ≤ s.

If W_0 contains two vertices u and v, then N[u] = N[v] = W in G . This implies that G is not NHI, which is a contradiction. Therefore W_0 contains at most one vertex, that is, $|W_0| \leq \binom{s}{0}$. Thus the result is true when t=0. When t≥ 1, each vertex in W_t has degree t in $\langle W_t, U\rangle$. But $|U|$ = s. Therefore, for each vertex v in W_t, N(v) in $\langle W_t, U\rangle$is a t-subset (subset with t elements) of U. But the number of distinct t-subsets of U is exactly $\binom{s}{t}$. If W_t contains more than $\binom{s}{t}$vertices, then there are at least two vertices u and v in W_t such that N(u) = N(v) in $\langle W_t, U\rangle$ and hence in G, N[u] = N[v]. this is a contradiction to the fact that G is a NHI. Hence the claim. This forces that,

$$n = |W| = |W_0| + |W_1| + \ldots + |W_s| \leq \binom{s}{0} + \binom{s}{1} + \cdots + \binom{s}{s} = 2^s.$$

Thus n≤ 2^s, where s < m. This is a contradiction, to the choice of m and the proof is complete.

3. IRREGULARITY OF PRODUCT GRAPHS

Theorem 3.1

Let G and H be NI graphs with p(G) and p(H) vertices respectively. Then $G \vee H$ is also NI if and only if $d_{\bar{G}}(u) \neq d_{\bar{H}}(v)$. For any u in G and v in H, that is, $p(G) - d_G(u) \neq p(H) - d_H(v)$.

Proof.

Assume that $G \vee V$ is NI. We claim that for all vertices u in G and v in H, $d_{\bar{G}}(u) \neq d_{\bar{H}}(v)$. Let $u \in G$ and $v \in H$. Then $uv \in E(G \vee H)$. But by our assumption $G \vee H$ is NI and therefore $d_{G \vee H}(u) \neq d_{G \vee H}(v)$. This means that $d_{\overline{G \vee H}}(u) \neq d_{\overline{G \vee H}}(v)$. But $\overline{G \vee H} = \bar{G} \cup \bar{H}$. Thus $d_{\bar{G} \cup \bar{H}}(u) \neq d_{\bar{G} \cup \bar{H}}(v)$ that is, $d_{\bar{G}}(u) \neq d_{\bar{H}}(v)$. Since $u \in \bar{G}$ and $v \in \bar{H}$.

Conversely, suppose $d_{\bar{G}}(u) \neq d_{\bar{H}}(v)$ for all $u \in G$ and $v \in H$. Then we have to prove that $G \vee H$ is NI. Suppose not, then there are two adjacent vertices u and v in $G \vee H$ such that $d_{G \vee H}(u) = d_{G \vee H}(v)$. Since $uv \in E(G \vee H)$, we have either $uv \in E(G)$ or $uv \in E(H)$ or $u \in G$ and $v \in H$. If $uv \in E(G)$, then $u \in G$ and $v \in G$. Also, the degree of u in the join $d_{G \vee H}(u) \neq d_G(u) + p(H)$. Therefore, $d_{G \vee H}(u) = d_{G \vee H}(v)$ implies that $d_G(u) + p(H) =$

$d_G(v) + p(H)$ and this forces that $d_G(u) = d_G(v)$ where $uv \in E(G)$. This is a contradiction. Since G is NI. Thus $uv \notin E(G)$. Suppose $uv \in E(G)$. Then u and v are vertices in the NI graph H. Now $d_{G\vee H}(u) = d_{G\vee H}(v)$ implies that $d_H(u) + p(H) = d_H(v) + p(G)$ and therefore $d_H(u) = d_H(v)$ which is a contradiction. Hence $uv \notin E(H)$. Therefore, the only possibility is that $u \in G$ and $v \in H$ with $uv \in E(G \vee H)$ and $d_{G\vee H}(u) = d_{G\vee H}(v)$, that is $d_{\overline{G\vee H}}(u) = d_{\overline{G\vee H}}(v)$. This means that $d_{\bar{G}\cup\bar{H}}(u) = d_{\bar{G}\cup\bar{H}}(v)$ and hence $d_{\bar{G}}(u) = d_{\bar{H}}(v)$ since $u \in \bar{G}$ and $v \in \bar{H}$. This is a contradiction to the assumption. Hence $G \vee H$ is NI which completes the proof.

Corollary

Let G and H be NI graphs with same order then $G \vee H$ is NI if and only if $d_G(u) \neq d_H(v)$ for any vertex u in G and v in H.

Proof

Let G and H be NI graphs with $p(G) = p(H)$. By the above theorem $G \vee H$ is NI if and only if $d_{\bar{G}}(u) = d_{\bar{H}}(v)$, for all u in G and v in H, that is $p(G) - 1 - d_G(u) \neq p(H) - 1 - d_H(v)$, for all u in G and v in H. This forces that, $d_G(u) \neq d_H(v)$, for any vertex u in G and v in H. This proves the corollary.

Theorem 3.2

G and H are NI graphs if and only if $G \times H$ is NI.

Proof

Let G and H be NI graphs. We claim that $G \times H$ is NI. First we note that for any vertex (u, v) in $G \times H$, $d_{G\times H}(u, v) = d_G(u) + d_H(v)$.

Let (u_1, v_1) and (u_2, v_2) be any two adjacent vertices in $G \times H$. Then, either $u_1 = u_2$and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. Since G and H are NI graphs. We have either $d_G(u_1) = d_G(u_2)$ and $d_H(v_1) \neq d_H(v_2)$, or $d_G(u_1) \neq d_G(u_2)$ and $d_H(v_1) = d_H(v_2)$. In both the cases, $d_G(u_1) + d_H(v_1) \neq d_G(u_2) + d_H(v_2)$ and hence $d_{G\times H}(u_1, v_1) \neq d_{G\times H}(u_2, v_2)$. Consequently, $G \times H$ is NI.

Conversely, suppose $G \times H$ is NI. We will now show that both G and H are NI. Let u_1 and u_2 be any two adjacent vertices in G, and let v be any vertex in H. Now (u_1, v) and (u_2, v) are adjacent vertices in $G \times H$. Since $G \times H$ is NI, $d_{G\times H}(u_1, v) \neq d_{G\times H}(u_2, v)$, that is$d_G(u_1) + d_H(v) \neq d_G(u_2) + d_H(v)$. This forces that, $d_G(u_1) \neq d_G(u_2)$ and hence G is NI. Let u be any vertex in G, and let v_1 and v_2 be any two adjacent vertices in H. Then (u, v_1) and (u, v_2) are adjacent vertices in $G \times H$ which is NI. Therefore, $d_{G\times H}(u, v_1) \neq d_{G\times H}(u, v_2)$. $d_G(u) + d_H(v_1) \neq d_G(u) + d_H(v_2)$. Consequently,$d_H(v_1) \neq d_H(v_2)$ and hence H is NI. Thus the proof follows.

Theorem 3.3

Let G and H be NI graphs. Then $G \otimes H$ is NI if and only if for any two edges $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. $d_G(u_1)d_H(v_1) \neq d_G(u_2)d_H(v_2)$.

Proof

Assume that $G \otimes H$ is NI. We have to prove that $d_G(u_1)d_H(v_1) \neq d_G(u_2)d_H(v_2)$ for any two edges $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. Let u_1u_2 and v_1v_2 be two edges in G and H respectively. Then (u_1, v_1) and (u_2, v_2) are adjacent vertices in $G\otimes H$. But by our assumption $G\otimes H$ is NI, and thus $d_{G\otimes H}(u_1,v_1) \neq d_{G\otimes H}(u_2,v_2)$. Here note that for any vertex (x, y) in $G\otimes H$, $d_{G\otimes H}(x,y) = d_G(x)d_H(y)$. This forces that, $d_G(u_1)d_H(v_1) \neq d_G(u_2)d_H(v_2)$.

Conversely, assume that for any two edges u_1u_2 in G and v_1v_2 in H, $d_G(u_1)d_H(v_1) \neq d_G(u_2)d_H(v_2)$. We will now prove that $G\otimes H$ is NI. Suppose not, then there are adjacent vertices (u_1,v_1) and (u_2,v_2) in $G\otimes H$ such that $d_{G\otimes H}(u_1,v_1) \neq d_{G\otimes H}(u_2,v_2)$. The adjacency between (u_1,v_1) and (u_2,v_2) in $G\otimes H$ means that, $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. Also, $d_{G\otimes H}(u_1,v_1) = d_{G\otimes H}(u_2,v_2)$ results that $d_G(u_1)d_H(v_1) = d_G(u_2)d_H(v_2)$. Thus there are edges $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$ such that $d_G(u_1)d_H(v_1) = d_G(u_2)d_H(v_2)$. Thus, there are edges $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$ such that $d_G(u_1)d_H(v_1) = d_G(u_2)d_H(v_2)$, which is a contradiction. This proves the converse part.

Theorem 3.4

The graphs $G, H_1, H_2, \ldots, H_n$ are NI if and only if $G[H_1 \cup H_2 \cup \ldots \cup H_n]$ is NI.

Proof

Let $= H_1 \cup H_2 \cup \ldots \cup H_n$. Suppose G and H are NI graphs. We have to prove that G[H] is NI. Suppose G[H] is not NI. Then there are adjacent vertices (u_1,v_1) and (u_2,v_2) in $G[H]$ such that $d_{G[H]}(u_1,v_1)$ =$d_{G[H]}(u_2,v_2)$. Therefore, $d_G(u_1)p(H) + d_H(v_1) = d_G(u_2)p(H) + d_H(v_2)$. This means that, $p(H)[d_G(u_1) - d_G(u_2) = d_H(v_2) - d_H(v_1)$ ……….(1). But (u_1,v_1) and (u_2,v_2) are adjacent vertices in G[H], that is either u_1 is adjacent with u_2 in G, or $u_1 = u_2$ and v_1 is adjacent with v_2 in H. If u_1 is adjacent with u_2 in G, then $d_{(G)}(u_1) \neq d_{(G)}(u_2)$, since G is NI. With out loss of generality, we can assume that $d_G(u_1) > d_G(u_2)$. Then, $d_G(u_1) - d_G(u_2) \geq 1$. Therefore by (1), $d_H(v_2) - d_H(v_1) \geq p(H)$, which is impossible and hence G[H] is NI. If $u_1 = u_2$ in G and $v_1v_2 \in E(H)$, then (1) implies that $d_H(v_1) = d_H(v_2)$, which is a contradiction to our assumption that H is the union of NI graphs and therefore G[H] is NI. Conversely, let $G[H_1 \cup H_2 \cup \ldots \cup H_n] = G[H]$ be NI. We claim that all $G, H_1, H_2, \ldots, H_n$ are NI graphs. Let u_1 and u_2 be any two adjacent vertices in G and let v be any vertex in H. Then $(u_1, v)$and (u_2, v) are adjacent vertices in G[H]. Since G[H] is NI, $d_{G[H]}(u_1,v) \neq d_{G[H]}(u_2,v)$. This means that $d_G(u_1)p(H) + d_H(v) \neq d_G(u_2)p(H) + d_H(v)$. So, $d_G(u_1) \neq d_G(u_2)$ and this proves that G is NI. Let u be any vertex in G and let v_{i_1} and v_{i_2} be any two adjacent vertices in H_i. Then $(u, v_{i_1})$and (u, v_{i_2}) are adjacent vertices in G[H]. Since G[H] is NI, $d_{G[H]}(u,v_{i_1}) \neq d_{G[H]}(u,v_{i_2})$. That is $d_G(u)p(H) + d_H(v_{i_1}) \neq d_G(u)p(H) + d_H(v_{i_2})$. This forces that $d_{H_i}(v_{i_1}) \neq d_{H_i}(v_{i_2})$. This implies that H_i is NI which completes the proof.

Theorem 3.5

Tensor product of an NI graph with any regular graph is NI.

Proof

Since$G\otimes H = H\otimes G$. With out loss of generality, we can assume that G is NI andH is regular. We have to prove that $G\otimes H$ is NI. Let (u_1,v_1) and (u_2,v_2) be two adjacent vertices in $G\otimes H$. Then $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$ such that $d_G(u_1) = d_G(u_2)$ and $d_H(v_1) = d_H(v_2)$. Since $d_{G\otimes H}(u,v) = d_G(u)d_H(v)$ for any vertex (u, v) in $G\otimes H$, we have $d_{G\otimes H}(u_1,v_1) = d_G(u_1)d_H(v_1) \neq d_G(u_2)d_H(v_2) = d_{G\otimes H}(u_2,v_2)$.Consequently, $G\otimes H$ is NI. This complies the proof.

4 Product NHI graphs

Some product graphs which are NHI are established in this section.

A vertex v in G is called a full vertex of G if v is adjacent to all the vertices of G except.

Theorem 4.1

Let G and H be NHI graphs. Then $G \vee H$ is NHI if and only if at least one of the graphs G and H has no full vertex.

Proof

If both G and H have full vertex then let $u \in G$ and $v \in H$ be the full vertices of G and H respectively. Therefore, in $G \vee H$. $N[u] = V(G \vee H) = V(G) \cup V(H) = N[v]$. In addition, u and v are adjacent vertices with same degree in $G \vee H$ and thus $G \vee H$ is not NHI. Conversely, suppose that G or H has no full vertex. Without loss of generality, assume that G has no full vertex. We claim that $G \vee H$ is NHI. Let u and v be two adjacent vertices with same degree in $G \vee H$. If both u and v are the vertices of G. Then, since G is NHI. $N_G[u] \neq N_G[v]$ and hence, $N_G[u] \cup V(H) \neq N_G[v] \cup V(H)$. This forces that, $N_{G \vee H}[u] \neq N_{G \vee H}[v]$. Similarly, if both u and v are the vertices of the NHI graph H then $N_H[u] \neq N_H[v]$ and hence $N_H[u] \cup V(G) \neq N_H[v] \cup V(G)$. That is $N_{G \vee H}[u] \neq N_{G \vee H}[v]$. If $u \in G$ and $v \in H$, then as G has no full vertex, there is a vertex w in G such that u and w are non-adjacent in G. This forces that $w \notin N_{G \vee H}$[u] and $w \in N_{G \vee H}$[v] and therefore, $N_{G \vee H}$[u]$\neq$ $N_{G \vee H}$[v].

Theorem 4.2

If G and H are NHI graphs, then $G \times H$ is also NHI.

Proof

Let G be a NHI graph of order m and H be a NHI graph of order n. Let $V(G) = \{u_1, u_2, \dots, u_m\}$ and $V(H) == \{v_1, v_2, \dots, v_n\}$. If $u = (u_1, v_1) \in V(G) \times V(H)$, then the vertices adjacent to u are of the form (u_1, v_i) or (u_j, v_1) where $u_1 u_j \in E(G)$ or $v_1 v_i \in E(H)$. We claim that $G \times H$ is NHI. Let u and v be two adjacent vertices of same degree in $G \times H$. Let $u = (u_1, v_1)$. If $v = (u_1, v_i)$, then v_1 and v_i are adjacent in H, and $d(u_1) + d(v_i) = d(u_1, v_i) = d(v) = d(u) = d(u_1, v_1) = d(u_1) + d(v_1)$ and hence $d(v_1) = d(v_i)$. Also, since H is NHI and v_1 and v_i are adjacent in H, $N_H[v_1] \neq N_H[v_i]$. As $d(v_1) = d(v_i)$, we have $N_H[v_1]$ is not a proper subset of $N_H[v_i]$. Therefore, there exists $w \in N_H[v_1]$ such that $w \notin N_H[v_i]$ and hence in $G \times H$, $(u, w) \in N_{G \times H}[u]$ and $(u, w) \notin N_{G \times H}[v]$. This means that $N_{G \times H}[u] \neq N_{G \times H}[v]$. Similarly, if $v = (u_j, v_1)$, then u_1 and u_j are adjacent in G, and $d(u_j) + d(v_1) = d(u_j, v_1) = d(v) = d(u) = d(u_1, v_1) = d(u_1) + d(v_1)$ and hence $d(u_j) + d(u_1)$. But G is NHI and u_1 and u_j are adjacent in G. Thus $N_G[u_1] \neq N_G[u_j]$. As $d(u_j) = d(u_1)$, we have $N_G[u_1]$ is not a proper subset of $N_G[u_j]$. Therefore, there exits $s \in N_G[u_1]$ such that $s \notin N_G[u_j]$ and hence in $G \times H$, $(s, v) \in N_{G \times H}[u]$ and $(s, v) \neq N_{G \times H}[v]$. Hence $N_{G \times H}[u] \neq N_{G \times H}[v]$. Therefore by Theorem 3.2.1, in both the cases $G \times H$ is NHI. This completes the proof.

Remark

Let G be any NHI graph and let u_i and $u_j$be two adjacent vertices in G. If $d(u_i) \neq d(u_j)$, then $N(u_i) \neq N(u_j)$. If $d(u_i) = d(u_j)$ and if $N(u_i) \neq N(u_j)$, then $N(u_i) \neq N(u_j)$.

Theorem 4.3

Tensor product of an NHI graph with any graph is NHI.

Proof

Let G be an NHI graph and H be any graph. We claim that $G\otimes H$ is NHI. Let (u_1, v_1) and (u_2, v_2) be two adjacent vertices with same degree in $G\otimes H$. Then u_1 is adjacent with u_2 in G and v_1 is adjacent with v_2 in H. We know that in $G\otimes H$, for any vertex $(u.v). d(u.v) = d(u)dv)$ and $N(u, v) = N(u) \times N(v)$.

$N[(u_1, v_1)] = \{(u_1, v_1)\} \cup N(u_1, v_1) = \{(u_1, v_1)\} \cup N(u_1) \times N(v_1)$

$\neq \{(u_2, v_2)\} \cup N(u_2) \times N(v_2)$ (Since G is NHI and $u_1u_2 \in E(G)$) $= N[(u_2, v_2)]$

Theorem 4.4

If G and H are NHI graphs, then $G \circ H$ is also NHI.

Proof

G and H are NHI graphs. We have to prove that $G \circ H$ is NHI. Let (u_1, v_1) and (u_2, v_2) be any two adjacent vertices in $G \circ H$ with same degree. Then (i) $u_1 = u_2$ and v_1 is adjacent with v_2 in H, or (ii) $v_1 = v_2$ and u_1 is adjacent with u_2 in G, or (iii) u_1 is adjacent with u_2 in G and v_1 is adjacent with v_2 in H. If $u_1 = u_2$ and v_1 is adjacent with v_2 in H, then $N[u_1] = N[u_2]$ and $N[v_1] \neq N[v_2]$, since H is NHI. Therefore, $N[u_1] \times N[v_1] \neq N[u_2] \times N[v_2]$ and hence $N[(u_1, v_1)] \neq N[(u_2, v_2)]$. If $v_1 = v_2$ and u_1 is adjacent with u_2 in G, then $N[v_1] = N[v_2]$ and $N[u_1] \neq N[u_2]$, since G is NHI. Therefore, $N[u_1] \times N[v_1] \neq N[u_2] \times N[v_2]$ and hence $N[(u_1, v_1)] \neq N[(u_2, v_2)]$. If u_1 is adjacent with u_2 in G and v_1 is adjacent with v_2 in H, then $N[u_1] \neq N[u_2]$ and $N[v_1] \neq N[v_2]$, since G and H are NHI. Therefore, in this case also, $N[u_1] \times N[v_1] \neq N[u_2] \times N[v_2]$ and $N[(u_1, v_1)] \neq N[(u_2, v_2)]$.

Theorem 4.5

The lexicographic product of two NHI graphs is also NHI.

Proof

Let G and H be NHI graphs. For any vertex (u, v) in G[H], $N[u, v] = \{u\} \times N[v] \cup N(u) \times V(H)$. Let (u_1, v_1) and (u_2, v_2) be two adjacent vertices with same degree in G[H]. Then either (i) u_1 is adjacent with u_2 in G. or (ii) $u_1 = u_2$ and v_1 is adjacent with v_2 in H.

Case (i) Suppose u_1 is adjacent with u_2 in G.

Then since G is NHI. $N[u_1] \neq N[u_2]$. Therefore there is a vertex x distinct from u_1 and u_2 which is not a common neighbor of u_1 and u_2 that is there exists $x \neq u_2$ in $N(u_1)$ such that $x \notin N(u_2)$. or $x \neq u_1$ in $N(u_2)$ such that $x \notin N(u_1)$. Let $x \neq u_2$ in $N(u_1)$ such that $x \notin N(u_2)$.Then, for any vertex $h \in V(H), (x.h) \notin \{u_2\} \times N[v_2]$ and $(x, h) \notin N(u_2) \times V(H)$.

Thus, $(x, h) \notin \{u_2\} \times N(v_2) \cup N(u_2) \times V(H)$. But $(x, h) \in \{u_1\} \times N(v_1) \cup N(u_1) \times V(H)$. $N[(u_1, v_1)] = \{u_1\} \times N[v_1] \cup N(u_1) \times V(H) \neq \{u_2\} \times N[v_2] \cup N(u_2) \times V(H)$ $= [(u_2. v_2)]$ and hence G[H] is NHI. Suppose $x \neq u_1$ in $N(u_2)$ such that $x \notin N(u_1)$. Then, for any vertex $s \in V(H)$. $(x. s) \notin \{u_1\} \times N[v_1]$ and $(x. s) \notin N(u_1) \times V(H)$. Thus, (x, s)$\notin$ $N(u_1) \times V(H)$.

Thus, $(x. s) \notin \{u_1\} \times N[v_1] \cup N(u_1) \times V(H)$. But (x, s) $\in$ $\{u_2\} \times N[v_2] \cup N(u_2) \times V(H)$.

Hence $N[(u_1, v_1)] = \{u_1\} \times N[v_1] \cup N(u_1) \times V(H). \neq \{u_2\} \times N[v_2] \cup N(u_2) \times V(H)$.

$$= N[(u_2, v_2)]$$

Case (ii) Suppose$u_1 = u_2$ and v_1 is adjacent with v_2 in H.

Then $N(v_1) \neq N(v_2)$ since H is NHI. Now, there exists $y \neq v_2$ in $N(v_1)$ such that $y \notin N(v_2)$, or $y \neq v_1$ in $N(v_2)$ such that $y \notin N(v_1)$.If $y \neq v_2$ in N(v_1) such that $y \notin N(v_2)$, then $(u_1, y) \in \{u_1\} \times N[v_1]$. But $(u_1, y) \notin \{u_1\} \times N[v_2]$. Also obviously $(u_1, y) \notin N[u_1] \times V(H)$. Hence $(u_1, y) \notin \{u_1\} \times N[v_2] \cup N[u_1] \times V(H)$ and $(u_1, y) \in \{u_1\} \times N[v_1] \cup N[u_1] \times V(H)$. Thus, $N[(u_1, v_1)] = \{u_1\} \times N[v_1] \cup N(u_1) \times V(H). \neq \{u_1\} \times N[v_2] \cup N(u_1) \times V(H). = N[(u_1, v_2)] = N[(u_2, v_2)]$. Suppose $y \neq v_1$ in $N(v_2)$ such that $y \notin N(v_1)$. Then $(u_1, y) \in \{u_1\} \times N[v_2]$. But $(u_1, y) \notin \{u_1\} \times N[v_1]$. Also, $(u_1, y) \notin N[u_1] \times V(H)$. Hence $(u_1, y) \notin \{u_1\} \times N[v_1] \cup N[u_1] \times V(H)$ and $(u_1, y) \in \{u_1\} \times N[v_2] \cup N[u_1] \times V(H)$. Hence, $N[(u_1, v_1)] = \{u_1\} \times N[v_1] \cup N[u_1] \times V(H) \neq \{u_1\} \times N[v_2] \cup N[u_1] \times V(H) = N[(u_1, v_2)] = N[(u_2, v_2)]$. Hence in both the cases, $N[(u_1, v_1)] = N[(u_2, v_2)]$.

5 Lattice Theoretic Approach To The Study of Irregular Graphs

Both Lattices and Boolean algebra have important applications in the theory and design of computers. There are many other areas such as engineering and science to which Boolean algebra is applied. In this section, the basic concepts in lattice theory and Boolean algebra have been discussed. A set L on which a partial ordering ≤ is defined is called a partially ordered set or a poset and is denoted by (L,≤).

Let (L, ≤) be a poset and let A ⊆ L. Any element x ∈ L is an upper bound for A if for all a ∈ A, a ≤ x. An element x ∈ L is the least upper bound (lub) for A if x is an upper bound for A and x ≤ y, where y is any upper bound for A. Similarly, any element x ∈ L is the greater lower bound (glb) for A if x is a lower bound for A and x ≤ y, where y is any lower bound for A. A lattice is poset L in which every pair of elements has a glb and a lub such that for all a, b, c ∈ L.

a ∨ a = a and a ∧ a = a a ∨ b = b ∨ a and a ∧ b = b ∧ a

a ∨ (b ∨ c) = (a ∨ b) ∨ c and a ∧ (b ∧ c) = (a ∧ b) ∧ c a ∨ (a ∧ b) = a and a ∧ (a ∨ b) = a

Since lattice is an algebraic system with binary operations ∨ and ∧. It is denoted by (L, ∨, ∧).

The glb of a, b ∈ L is denoted by a ∧ b and is also called the meet. The lub of a, b∈ L is denoted by a ∨ b and is also called the join. For example, let A be any set and P(A) be its power set. The posset (P(A),⊆) is a lattice in which the meet and join are respectively the same as the operations intersection ∩ and union ∪ on sets. A lattice (L, ∨, ∧) is called a distributive lattice if for any a, b, c ∈ L. a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c) and a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c)

In other words, in a distributive lattice the operations ∧ and ∨ distribute over each other. For example, the lattice (P(A), ∪, ∩) of the power set of any set A is a distributive lattice. (under ⊆)

A lattice (L, ∨, ∧) is said to be if a ≤ c ⇒ a ∨ (b ∧ c) = (a ∨ b) ∧ c, for any a, b, c ∈ L. A lattice (L, ∨, ∧) which has both a least element, denoted by 0 and a greatest element, denoted by 1 is called a *bounded lattice.* A bounded lattice (L, ∨, ∧) is said to be a complemented lattice if and only if for every element a ∈ L, there exists an element á ∈ L such that a ∧ á = 0 and a ∨ á = 1. The element á is called the complement of the element a. A unary operation': L →L is called an orthocomplementation, if it satisfies the following conditions:

(i) a ∨ á = 1 and a ∧ á = 0 (ii) a ≤ b implies b´ (iii) (a´)´ = a

A posset L together with an orthocomplementation is called an orthoptist. A lattice L together with an orthocomplementation is called an Orth lattice. Let (L, ∨, ∧, ‘) be an orthoclastic. Then L is said to be orthomodular, if it satisfies the orthomodular law “a ≤ b implies a ∨ (á ∧ b) = b.” An Orth lattice, which satisfies the modular law is said to be a modular orthoclastic. A lattice is called complete if each if each of its nonempty subset has a least upper bound and a greatest lower bound. A Boolean algebra is a lattice which contains a least element and a greatest element and which is both complemented and distributive. A Boolean algebra will generally be denoted by (B,’ ∨, ∧, 0, 1) (or) (B,’ ∨, ∧) in which (B, ∨, ∧) is a lattice with two binary operations ∧ and ∨ called the meet and join respectively. The corresponding partially ordered set will be denoted by (B, ≤). The bound of the lattice are denoted by 0 and 1, where 0 is the least element or zero element and 1, the greatest element or unit element of (B, ≤). Since (B, ∨, ∧) is a complemented, distributive lattice, each element of B has unique complement. Unary operation of complementation is denoted by ‘. Thus a Boolean algebra (B,’ ∨, ∧) consists of a set B, a pair of binary operations ∧ (meet) and ∨ (join), and a unary operation’ (complementation). Let (B,’ ∨, ∧) be a Boolean algebra. A non zero element a ∈ B is said to be an atom if for every x ∈ B, x ∧ a = a x ∧ a = 0. Note that in any Boolean algebra, the immediate successors of the zero elements are called atoms.

Theorem 5.1

Lattice of I_n is isomorphic to $B_{\lfloor \frac{n}{2} \rfloor+1}$, Boolean Algebra of $\left\lceil \frac{n}{2} \right\rceil + 1$ atoms and so the graph is I_n is a Boolean graph.

Proof

For the graph I_{2n}, the elements of vertex of $L(I_{2n})$ are given below: $L(I_{2n}) = \{\phi, \{v_n\}, \{v_{n+1}\}, \ldots, \{v_{2n}\}, \{v_2, v_{n+2}\}, \{v_n, v_{n+3}\}, \ldots, \{v_n, v_{2n}\}, \{v_{n+1}, v_{n+2}\}, \{v_{n+1}, v_{2n}\}, \ldots, \{v_{2n-1}, v_{2n}\}, \ldots, \{v_{n-1}, v_n, v_{n+1}\}, \{v_n, v_{n+2}, v_{n+3}\}, \ldots, \{v_{2n-2}, v_{2n-1}, v_{2n}\}, \{v_{n-1}, v_n, v_{n+1}, v_{n+3}\}, \ldots, \{v_{2n-3}, v_{2n-2}, v_{2n-1}, v_{2n}\}, \{v_{n-2}, v_{n-1}, v_{n+1}, v_{n+2}\}, \ldots, \{v_2, v_3 \ldots, v_{2n-2}\}, \ldots, \{v_2, v_3 \ldots, v_{2n-2}, v_{2n}\}, \{v_1, v_2 \ldots, v_{2n-2}, v_{2n-1}\}\}$. The atoms of $L(I_{2n})$ are $\{v_n\}, (v_{n+1}) \ldots, \{v_{2n}\}$. For the graph I_{2n+1}.

$L(I_{2n-1}) = \{\phi, \{v_{n+2}\}, \{v_{n+3}\}, \ldots, \{v_{2n}\}, \{v_{2+1}\}, \{v_n, v_{n+1}\}, \{v_{n+2}, v_{n+3}\}, \ldots, \{v_{n+2}, v_{2+1}\}, \{v_{n+3}, v_{n+4}\}, \{v_{n+3}, v_{2n+1}\}, \ldots, \{v_{2n}, v_{2n+1}\}, \{v_n, v_{n+1}, v_{n+3}\}, \{v_n, v_{n+1}, v_{n+4}\}, \ldots \{v_{2n-1}, v_{2n}, v_{2n+1}\}, \{v_{n-1}, v_n, v_{n+1}, v_{n+2}\}, \ldots \{v_{2n-2}, v_{2n-1}, v_{2n}, v_{2n+1}\}, \{v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{n+4}\}, \ldots \{v_2, v_3, \ldots v_{2n-1}\}, \ldots \{v_2, v_3, \ldots v_{2n-1}, v_{2n+1}\}, \{v_1, v_2, \ldots, v_{2n-1}, v_{2n}\}\}$. The atoms of $L(I_{2n+1})$ are $\{v_{n+2}\}, \{v_{n+3}\}, \ldots \{v_{2n}\}, \{v_{2n+1}\}, \{v_n, v_{n+1}\}$. Thus $L(I_{2n})$ and $L(I_{2n+1})$ contains 2^{n+1}elements and consequently isomorphic to Boolean algebra of n+1 atoms. Thus $L(I_n)$ is isomorphic to $B_{\lfloor \frac{n}{2} \rfloor+1}$, Boolean Algebra of $\left\lceil \frac{n}{2} \right\rceil + 1$ atoms and hence I_n is a Boolean graph.

Theorem 5.2

Highly irregular bipartite graph $H_{n,n}$, $n \geq 2$ is an ortho graph. That is, the lattice of $H_{n,n}$ is an ortho lattice.

Proof

First we prove this result for the cases when n = 2 and 3. In $H_{2,2}$, $N(u_1) = \{v_2\}$, $N(u_2) = \{v_1, v_2\}$, $N(v_1) = \{u_2\}$ and $N(v_2) = \{u_1, u_2\}$. Now $\gamma(\gamma(\{u_1\}) = \gamma(\{v_2\}) = \{u_1,u_2\} \neq \{u_1\}$, and $\gamma(\gamma\{u_2\}) = \gamma(\{v_1, v_2\}) = \{u_2\}$. Hence $\{u_1\} \notin L(H_{2,2})$ and $\{u_2\} \in L(H_{2,2})$.Similarly $\{v_1\} \notin L(H_{2,2})$ and $\{v_2\} \in L(H_{2,2})$. $\gamma(\gamma(\{v_1, v_2\})) = \{v_1, v_2\}$ and $\gamma(\gamma(\{u_1, u_2\})) = \{u_1, u_2\}$.Consequently $\{u_1, u_2\}, \{v_1, v_2\} \in L(H_{2,2})$. Thus $L(H_{2,2}) = \{\phi, V(H_{2,2}), \{v_1, v_2\}, \{u_1, u_2\}, \{v_2\}, \{u_2\}\}$.$L(H_{2,2}) \cong O_6$. If a lattice contains O_6, then it is an ortholattice, consequently $H_{2,2}$ is an ortho graph. For the graph $H_{3,3}$ is an graph.

$L(H_{3,3}) = \{\phi, V(H_{3,3}), \{v_1, v_2, v_3\}, \{u_1, u_2, u_3\}, \{v_2, v_3\}, \{u_2, u_3\}, \{v_3\}, \{u_3\}\}$, since $\gamma(\gamma(\{v_2, v_3\})) = \gamma(\{u_1, u_2, u_3\}) = \{v_3\}$, $\gamma(\gamma(\{v_2, v_3\})) = \gamma(\{u_2, u_3\}) = \{v_2, v_3\}$, $\gamma(\gamma(\{v_1, v_2, v_3\})) = \gamma(\{u_3\}) = \{v_1, v_2, v_3\}$. Similarly $\{u_3\}$, (u_2, u_3), $\{u_1, u_2, u_3\} \in L(H_{3,3})$. $L(H_{3,3})$ contains an isomorphic copy of O_6, given by the elements $\{\phi, \{v_3\}, \{u_3\}, \{v_2, v_3\}, \{u_2, u_3\}, V(H_{3,3})\}$ or $\{\phi, \{v_3\}, \{u_3\}, \{v_1, v_2, v_3\}, \{u_1, u_2, u_3\}, V(H_{3,3})\}$. Consequently $L(H_{3,3})$ is an ortholattice and hence $H_{3,3}$ is an ortho graph. Now we prove the general case. For a graph $H_{n,n}$, $N(v_1) = \{u_n\}$, $N(v_2) = \{u_{n-1}, u_n\}, \ldots N(v_i) = \{u_{n-(i-1)}, \ldots u_{n-1}, u_n\}$, $N(v_n) = \{u_1, u_2, \ldots u_{n-1}, u_n\}$, and $N(u_1) = \{v_n\}$, $N(u_i) = \{v_{n-(i-1)}, \ldots v_{n-1}, v_n\}$ and the lattice of $H_{n,n}$, $L(H_{n,n}) = \{\phi, V(H_{n,n}), \{v_1, v_2, \ldots, v_n\}, \{u_1, u_2, \ldots u_{n-1}, u_n\}, \{v_2, v_3, \ldots, v_n\}, \{v_2, v_3, \ldots v_n\}, \{u_2, u_3, \ldots u_{n-1}, u_n\}, \ldots \{v_n\}, \{u_n\}\}$. The lattice $L(H_{n,n})$ contains a sub lattice which is isomorphic to O_6 and is given bythe element $\{\phi, \{v_n\}, \{u_n\}, \{v_1, v_2, \ldots, v_n\}, \{u_1, u_2, \ldots, u_{n-1}, u_n\}, V(H_{n,n})\}$ and hence $H_{n,n}$ is an ortho graph. Hence the highly irregular bipartite graphs $H_{n,n}$, $n \geq 2$ are ortho graphs.

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