A COMPREHENSIVE REVIEW ON GRAPHS FROM GROUPS

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Abstract:

Graph Theory is one of the important branches of Mathematics which has many applications in other fields of Science. In this paper we discuss about power of a graph and subgroup intersection graph of a group.

Keywords: Group theory, Graph Theory, prefix reversal graph and cayley graph.

1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph. For example, Cayley graph, zero-divisor graph, non-commuting graph, commuting graph, etc., are some of them to mention in this regard.

Even though Cayley graphs were extensively dealt in various literatures, only few authors have worked on domination Cayley graphs. He obtained a necessary and sufficient condition for the existence of an efficient dominating set in a Cayley graph on a class of finite groups, in particular on symmetric groups. As an application, he classified the hyper cubes which admit efficient dominating sets.

There are so many classes of Cayley graphs derived from symmetric groups depending upon various generating sets. Some of the generating sets of symmetric groups S_n and corresponding Cayley graphs are given below: Let $S_n$be the symmetric group of all permutations on the set $< n > = \{1,2,\ldots,n\}$.

$$\Omega_1 = \left\{\frac{1i}{2} \le i \le n\right\}.\ \Omega_2 = \left\{\frac{ii+1}{1} \le i \le n\right\}.\ \Omega_2' = \Omega_2 \cup \{(1\ \mathrm{n})\}.\ \Omega_3 = \left\{\frac{2i-1\ 2i}{1} \le i \le n\right\}.$$

$$\Omega_4 = \{(ij)/1 \le i < j \le n\}.\ \ \Omega_5 = \left\{\frac{3i-2\ 3i-1}{1} \le i \le n\right\} \cup \{(3i-2\ 3i)/1 \le i \le n\}.$$

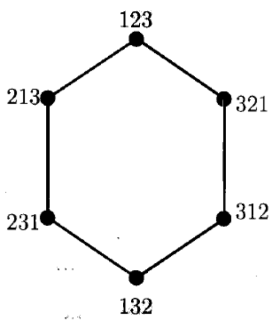
$$\Omega_6 = \{I^{(k)}/2 \le k \le n\}.$$

The Cayley graphs corresponding to S_n and various generating sets are listed below:

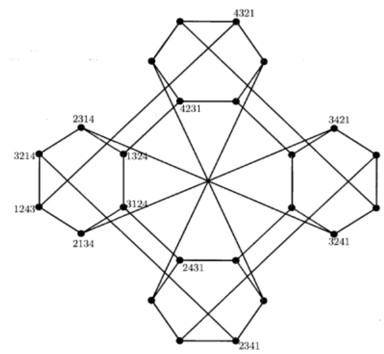
Symbol	**Cayley graph on S_n**	**Generator set**
ST_n	Star graph	Ω_1
BS_n	Bubble-sort graph	Ω_2
MB_n	Modified bubble-sort graph	Ω_2'
BC_n	Binary hypercube	Ω_3
CT_n	Complete-transposition graph	Ω_4
EC_n	Extension of hypercube version 1	Ω_5
PR_n	Prefix-reversal graph	Ω_6

Let S_n denote the set of all permutations over $< n > = \{1,2,\ldots,n\}$. For $,\rho \in S_n$, we take 'o' as $(\sigma \circ \rho)x = \rho(\sigma(x))$. Given $\sigma \in S_n$ and $\sigma = (p_1 p_2 \ldots p_n)$ (image row), define σ^k is obtained from σ by reversing the prefix of length k in σ. Define $\Omega = \{\frac{I^{(k)}}{2} \le k \le n\}$, where I is the identity permutation. $\Omega$is a generating set for S_n and Cayley graph of (S_n, Ω) is called the *prefix-reversal graph* PR_n.

Example



PR₃



PR₄

2. Domination parameters of PR_n

In this section, we determine the values of the domination number $\gamma(PR_n)$, the independent domination number $i(PR_n)$, the perfect domination number $\gamma_p(PR_n)$, the inverse domination number $\gamma^{-1}(PR_n)$, the split domination$\gamma_s(PR_n)$ and we obtain bounds for the total domination number $\gamma_t(PR_n)$ and the connected domination number $\gamma_c(PR_n)$.

Theorem 2.1.

For any positive integer n > 2, $(PR_n) = i(PR_n) = \gamma_p(PR_n) = (n-1)!$.

Proof.

We know that PR_n is a (n-1) regular graph. By Theorem 2.2.70, $\gamma(PR_n) \geq (n-1)!$.

Let D = $\{\sigma \in \frac{V(PR_n)}{\sigma(1)} = 1\}$.

Claim: D is a dominating set of (PR_n).

Let $\sigma \in V(PR_n) - D$ and k = $\sigma(1)$. Clearly $k \neq 1$ and $I^{(k)}(\sigma(1)) = 1$. From this, we have $\sigma \circ I^{(k)} \in D$ and so every element $\sigma \in V(PR_n) - D$ is adjacent to $\sigma \circ I^{(k)} \in D$. Hence D is a dominating set of PR_n. Therefore $\gamma(PR_n) \leq |D| = (n-1)!$. Hence $\gamma(PR_n) = (n-1)!$. Note that, for any k and $\sigma \in D, I^{(k)}(\sigma(1)) \neq 1$.Therefore D is an independent set. Since D is a minimum dominating set, D is a minimum independent dominating set. Hence $i(PR_n) = (n-1)!$. Let $\sigma \in V(PR_n) - D$ and $\sigma \circ I^{(K_1)}, \sigma \circ I^{(K_2)} \in D$ for some $k_1 \neq k_2$. Then $I^{(k_1)}(\sigma(1)) = 1$and $I^{(k_2)}(\sigma(1)) = 1$. This implies that $k_1 = \sigma(1) = k_2$, which is a contradiction. Therefore every vertex in $V(PR_n) - D$ is adjacent to exactly one vertex in D and so D is a minimum perfect dominating set. Hence $\gamma_p(PR_n) = (n-1)!$.

Corollary:

For any integer $n \geq 3$, $d(PR_n) = d_i(PR_n) = d_p(PR_n) = ad(PR_n) = n$.

Proof.

Let $D_i = \left\{\frac{\sigma \in V(PR_n)}{\sigma(i)} = 1\right\}, i = 1,2,\dots,n$. Clearly $V(PR_n) = \cup_{i=1}^{n} D_i$ and each D_i is a minimal dominating set which is also independent, indivisible and perfect. Hence the result follows.

In view of Corollary, we have the following observation.

Corollary:

For any integer $n \geq 3, PR_n$ is dogmatically full.

Theorem 2.2.

Let n > 2 be any integer. Then (i) $\gamma_t(PR_n) = \frac{n!}{n-1}$ if n is even

(ii) $\frac{n!}{n-1} \leq \gamma_t(PR_n) \leq (n+1)(n-2)!$if n is odd.

Proof.

Case 1. Suppose n is even. Define $D_i = \{\sigma \in \frac{V(PR_n)}{\sigma(i)} = 1, \sigma(n-i+1) = n\}$ for i = 1,2,…,n. We claim that $D = \cup_{i=1}^{n} D_i$ is a total dominating set of PR_n. Note that, for $\sigma \in D_i, I^{(n)}(\sigma(i))$ = n and $I^{(n)}\big(\sigma(n-i+1)\big) = 1$ and so $\sigma \circ I^{(n)} \in D_{n-i+1}$. Hence each vertex of D_i is adjacent to exactly one vertex in D_{n-i+1} so that < D >has no isolated vertices. Next we claim that if $\sigma, \rho \in D$, then $N(\sigma) \cap N(\rho) = \emptyset$.If not, there exists an element $\alpha \in N(\sigma) \cap N(\rho)$ for some $\sigma, \rho \in D$. From this, we have $\sigma \circ I^{(k_1)} = \alpha = \rho \circ I^{(k_2)}$for some $k_1 \neq k_2$ and $2 \leq k_1, k_2 \leq n$. (3.1) without loss of generality take $1 < k_1 < k_2 \leq n$.

Sub case 1.1 Suppose$\sigma, \rho \in D_i$ for some i. Then $I^{(k_1)}\big(\sigma(i)\big) = k_1$ and$I^{(k_2)}\big(\rho(i)\big) = k_2$. From 3.1, $k_1 = k_2$, which is a contradiction.

Sub case 1.2 Suppose$\sigma \in D_i$ and $\rho \in D_j$ for some i< j.

Sub case 1.2.1 Suppose $k_2 \neq n$. Then $I^{(k_1)}\big(\sigma(n-i+1)\big) = n$ and $I^{(k_2)}\big(\rho(n-i+1)\big) = n$. From, we get that $n-i+1 = n-j+1$ and hence $i = j$, which is a contradiction.

Sub case 1.2.2 Suppose k_2 = n. From 3.1, $k_1 = I^{(k_1)}\big(\sigma(i)\big) = I^{(k_2)}\big(\rho(i)\big) = I^{(n)}\big(\rho(i)\big)$.This is possible only when $\rho(i) = n - k_1 + 1$. Now, $I^{(k_1)}\big(\sigma(n-i+1)\big) = n$ and $I^{(n)}\big(\rho(j)\big) = n$. Therefore $n - i + 1 = j$ and so $i = n - j + 1$.Therefore $\rho(i) = \rho(n-j+1)$, which implies $n - k_1 + 1 = n$ and hence k_1 = 1, which is a contradiction. Hence $N(\sigma) \cap N(\rho) = \emptyset\ \sigma, \rho\ \in D$. Also $|N(D)| = n! = |V(PR_n)|$ and hence D is a total dominating set. Hence $\gamma_t(PR_n) \leq \frac{n!}{n-1}$. By Lemma, $\gamma_t(PR_n) \geq \frac{n!}{n-1}$ so that $\gamma_t(PR_n) = \frac{n!}{n-1}$ when n is even.

Case 2. Assume that n is odd.

Define $D_i = \left\{\sigma \in \frac{V(PR_n)}{\sigma(i)} = 1, \sigma(n-i+1) = n\right\}, i = 1,2, \ldots, \frac{n-1}{2}, \frac{n+3}{2}, \ldots, n, A_1 = \{\sigma \in \frac{V(PR_n)}{\sigma(1)} = 1,\ \sigma\left(\frac{n+1}{2}\right) = n\}$and $A_2 = \{\sigma \in \frac{V(PR_n)}{\sigma(1)} = n,\ \sigma\left(\frac{n+1}{2}\right) = 1\}$. Let $D = \cup^{n}_{\substack{i=1 \\ i \neq \frac{n+1}{2}}} D_i$and $D_t = D \cup A_1 \cup A_2$. We claim that D_t is a total dominating set of PR_n. N Clearly each vertex of D_i is adjacent to one vertex in D_{n-i+1} and also each vertex of A_1 has exactly one adjacent vertex in A_2 so that < D_t> has no isolated vertices. Repeating the similar argument as in the even case, we obtain $N(\sigma) \cap N(\rho) = \emptyset\ \ \forall_{\sigma,\rho} \in D$. Clearly $N(\sigma) \cap N(\rho) = \emptyset\ \ \forall_{\sigma,\rho} \in A_1$. Let $\sigma \in D$and $\rho \in A_1$.

Claim $N(\sigma) \cap \{N(\rho) - \rho \circ I^{(n)}\} = \emptyset$.

Suppose not, $N(\sigma) \cap \{N(\rho) - \rho \circ I^{(n)}\} \neq \emptyset$. Then there exists $\alpha \in\ N(\sigma) \cap \{N(\rho) - \rho \circ I^{(n)}\}$ and so $\sigma \circ I^{(k_1)} = \alpha = \rho \circ I^{(k_2)}$ for some $k_1 < k_2$ and $2 \leq k_1, k_2 < n$. But $I^{(k_1)}\big(\sigma(n-i+1)\big) = n$ and $I^{(k_2)}\left(\rho\left(\frac{n+1}{2}\right)\right) = n$.From this $n - i + 1 = \frac{n+1}{2}$ and hence $i = \frac{n+1}{2}$ which is a contradiction. Let $X = \{\sigma \circ I^{(n)} :\ \sigma \in A_1\}$. Now $|N(D)| = (n-1)^2\,(n-2)!$and $|N[A_1] - X| = (n-1)(n-2)!$. Hence $|N(D)| \cup \{N[A_1] - X\}| = n! = |V(PR_n)|$. Therefore $D \cup A_1$ is a dominating set and hence D_t is a total dominating set of PR_n. Hence $\gamma_t(PR_n) \leq (n+1)(n-2)!$. On the other hand, $\gamma_t(PR_n) \geq \frac{n!}{n-1}$ and the theorems follows.

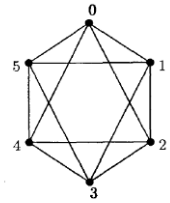
3.Subgroup Complementary Cayley Graph

Definition

Let G be a group and let H be a subgroup of G. The Cayleygraph $Cay(G, \overline{H})$, where $\overline{H} = G - H$ is called the subgroup complementary Cayley graph and is denoted by $SC(G, H)$.

Example

Consider the group $G = \mathbb{Z}_6$ and its subgroup $H = \{0,3\}$. The corresponding subgroup complementary Cayley graph is given below:



$\boldsymbol{SC}(\mathbb{Z}_6, \{\mathbf{0}, \mathbf{3}\})$

Remark

Note that $SC(G, H)$ is a regular graph of degree $|G - H|$. Since G – H is a generating set of G, $SC(G, H)$is a connected graph.

Proposition 3.1

Let G be any group and let H be a subgroup of G. Then diam $\left(SC(G, H)\right) = \begin{cases} 1 & if\ H = \{e\}. \\ 2 & if\ H \neq \{e\}. \end{cases}$

Proof.

If $H = \{e\}$, then $SC(G, H)$ is a regular graph of $|G - H| = |G| - 1$.Therefore$SC(G, H) \cong K_{|G|}$ and hence diam $(SC(G, H)) = 1$. Take $H \neq \{e\}$. Let $a, b \in G - H$. Both a and b are adjacent to e and so $d(a, b) \leq 2$.Since $H \neq \{e\}$, $|H| \geq 2$. Let $a, b \in H$ and $x \in G - H$. If $x^{-1}a \in H$, then $x^{-1}aa^{-1} \in H$ and so $x^{-1} \in H$, a contradiction. From this argument, $x^{-1}a, x^{-1}b \in G - H$ and so x is adjacent to both a and b. Since $a^{-1}b \notin G - H$, a and b are non adjacent. Therefore $d(a, b) = 2$. Hence diam $(SC(G, H)) = 2$.

Proposition 3.2.

Let G be a group and H be a subgroup of G. The maximum independence number $\beta_0\left(SC(G, H)\right) = |H|$.

Proof.

Since H is subgroup of G, H is an independent set in $SC(G, H)$.Suppose there exists an independent set S of $SC(G, H)$ such that |S| > |H|. If $x \in S$, then $\deg(x) < |G - H|$, which is a contradiction to the regularity of $SC(G, H)$. Hence H is a maximum independent set and $\beta_0\left(SC(G, H)\right) = |H|$.

Theorem 3.3.

Let H be a subgroup of a group G. A subset S of G is a maximum independent set of $SC(G, H)$ if and only if S is a left coset of H.

Proof.

Assume that S is a left coset of H and $S = aH$ for some $a \in G$. Let $x, y \in S, x = ah_1$ and $y = ah_2$ for some $h_1, h_2 \in H$. Then $x^{-1}y = h_1^{-1}a^{-1}ah_2 = h_1^{-1}h_2 \in H$ and so $x^{-1}y \notin G - H$. Therefore x and y are non-adjacent in $SC(G, H)$, so S is an independent set of $SC(G, H)$. Since |S| = |H|, by Proposition 3.2, S is a maximum independent set of $SC(G, H)$.

Conversely, assume that S is a maximum independent set of $SC(G, H)$. If S = H, then the result is true. Suppose $S \neq H$. There exists $s \in G - H$ such that $s \in S$.Since$G = \bigcup_{a \in G} aH$, $s \in aH$ for some $a \notin H$.

Claim: $S \subseteq aH$.

If not, $\exists\ y \in S$ such that $y \notin aH$. Let $y \in bH$ for some $b \notin H$. Take $x = ah_1$ and $y = bh_2$, $h_1, h_2 \in H$. Since S is an independent set, $x^{-1}y \in H$, which means $(ah_1)^{-1}(bh_2) \in H$, so that $a^{-1}b \in H$. So $b = ah$ for some $h \in H$. Therefore $y \in bH = (ah)H = aH$ which is a contradiction to the fact $y \notin aH$. Hence $S \subseteq aH$. Since S is a maximum independent set, $|H| = |S| = |aH|$ and hence $S = aH$.

Proposition 3.4

Let H be a proper subgroup of a group G. Then $SC(G, H)$ is Hamiltonian.

Proof.

By Lagrange's theorem, $|G| \geq 2|H|$. Note that $|G-H| \geq |G| - |H| \geq |G| - \frac{|G|}{2} = \frac{|G|}{2}$. Since $\deg(x) = |G - H|$, $\deg(x) \geq \frac{|G|}{2}$ for any $x \in G$. By Dirac's theorem(p.54,[6]) $SC(G,H)$ is Hamiltonian.

Theorem 3.5

Let H be any subgroup of a group G. Then the girth of $SC(G,H)$ is given by

$$gr\big(SC(G,H)\big) = \begin{cases} 4 & if\ |H| = \dfrac{|G|}{2} \\ 3 & otherwise. \end{cases}$$

Proof. Case 1: Suppose $|H| = \frac{|G|}{2}$. $G = H \cup aH$for some $a \in G, a \notin H$. By Proposition 4.2.5 and Theorem 4.2.6, H and aH are two maximum independent sets in $SC(G,H)$. Since $SC(G,H)$ is connected and regular, every element of H is adjacent to every element of aH in $SC(G,H)$ and so $SC(G,H) \cong K_{\frac{|G|}{2},\frac{|G|}{2}}$. This implies that $gr\big(SC(G,H)\big) = 4$.

Case 2: Suppose $|H| \neq \frac{|G|}{2}$ and so $|G - H| > |H|$. From this G – H is not a coset of H. By Theorem 3.3, G – H is not an independent set of $SC(G,H)$. Hence there exist $a, b \in G - H$ such that a is adjacent to b. Since e is adjacent to both a and b, {e,a,b} is a triangle in $SC(G,H)$ and so $gr\big(SC(G,H)\big) = 3$.

Theorem 3.6

Let H be any subgroup of a group G. For $x \in G$, $N(x) = N(x^{-1})$ in $SC(G,H)$ if and only if $x^2 \in H$.

Proof.

Assume that $x^2 \in H$. Let $a \in N(x), a = xx_1$ for some $x_1 \in G - H$. Then $a = x^{-1}x^2x_1$. Since $x^2 \in H, x_1 \in G - H, x^2x_1 \in G - H$. Therefore $a \in N(x^{-1})$ and so $N(x) \subseteq N(x^{-1})$. Similarly one can prove that $N(x^{-1}) \subseteq N(x)$. Hence $N(x) = N(x^{-1})$.Conversely, assume that $N(x) = N(x^{-1})$.Suppose $x^2 \notin H$. Since $x = x^{-1}x^2, x$ and x^{-1} are adjacent in $SC(G,H)$ and so $x \in N(x^{-1})$, whereas $x \notin N(x)$, which is a contradiction to the fact $N(x) = N(x^{-1})$.

Theorem 3.7

Let G be any group. Let $H = \bigcap_{i=1}^{n} H_i$, where each H_i is a subgroup of G. Then $SC(G,H) \cong \bigcup_{i=1}^{n} SC(G,H_i)$.

Proof.

Clearly $V[SC(G,H)] = V[\bigcup_{i=1}^{n} SC(G,H_i)]$.

Then $(x,y) \in E[SC(G,H)] \Leftrightarrow x^{-1}y \notin H$

$\Leftrightarrow x^{-1}y \notin H_i$for some i

$\Leftrightarrow (x,y) \in E[SC(G,H_i)]$for some i

$$\Leftrightarrow (x,y) \in E[\textstyle\bigcup_{i=1}^{n} SC(G,H_i)]$$

Therefore E$[SC(G,H)] = E[\bigcup_{i=1}^{n} SC(G,H_i)]$.

4. Power Graph of a Group

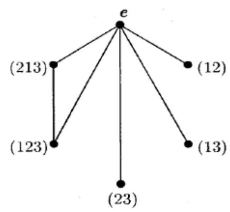
Definition

Let G be a group. The power graph $\Gamma_P(G)$ of G is a graph with $V(\Gamma_P(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_P(G)$ if and only if either $x^i = y$ or $y^j = x$,where $2 \leq i,j \leq n$.

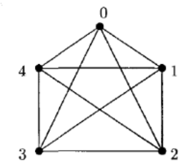
Example

The power graphs of S_3 and $\mathbb{Z}_5$ are given below:

$\boldsymbol{\Gamma_P(S_3)}$



$\boldsymbol{\Gamma_P(\mathbb{Z}_5)}$



The following are some of the vital characterizations of power graphs.

Proposition 4.1

Let G be a group with at least one non-self inverse element. Then $\mathrm{gr}(\Gamma_P(G)) = 3$.

Proof.

Let G be a group with identity e. Let x be a non-self inverse element of G. Note that $< x > = < x^{-1} >$, $e \in < x >$ and so the graph induced by the set $\{e, x, x^{-1}\}$ is K_3in $\Gamma_P(G)$. Hence $gr\left(\Gamma_P(G)\right) = 3$.

Theorem 4.2

Let G be any group with n elements. Then $\Gamma_P(G)$ is a graph with $\frac{\sum_{x \neq e} o(x)}{2}$ edges if and only if every element other than identity of the group G has a prime order.

Proof.

Assume that $\Gamma_P(G)$ is a graph with $\frac{\sum_{x \neq e} o(x)}{2}$ edges. It means that$\deg(x) = o(x) - 1$ for all $e \neq x \in G$ in $\Gamma_P(G)$. Let $x \neq e$ be any element of G. Suppose order of x is not a prime. Without loss of generality, we assume that o(x)=pq, where p,q are distinct primes. Consider the subgroup H = < x > .sincep|o(H), H has an element say y such that o(y) = p. From this deg(y) = o(y)-1 = p-1. Since $x \notin < y >$ and $y \in < x >$, y is adjacent to at least $x, y^2, \dots, y^p = e$.This implies that $\deg(y) > p - 1 = o(y) - 1$, which is a contradiction.

Hence every element other than identity in the group G is of prime order.

Conversely, assume that every element other than identity of the group G is of prime order. We have to prove that $\Gamma_P(G)$ is a graph with $\frac{\sum_{x \neq e} o(x)}{2}$ edges. It is enough to prove that $\deg(x) = o(x) - 1$ for all $e \neq x \in G$in $\Gamma_P(G)$. Suppose $\deg(x) > o(x) - 1$ for some $x \in G - e$.Then there exists $y \notin < x >$ and y is adjacent to x. This implies that $x \in < y >$ and so $< x > \subseteq < y >$. Since $o(x)$ and $o(y)$ are prime, we get that $< x > = < y >$, a contradiction to $y \notin < x >$. Hence $\deg(x) = o(x) - 1$ for all $e \neq x \in G$ in $\Gamma_P(G)$.

Proposition 4.3

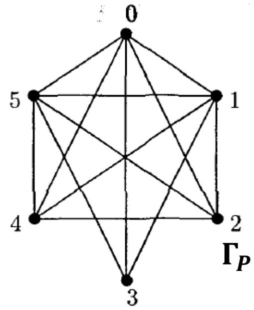
Let G be a group with n elements and Z(G) be its center. If $\deg(x) = n - 1$in $\Gamma_P(G)$, then $x \in Z(G)$.

Proof.

Let $x \in G$ be a vertex with $\deg(x) = n - 1$in $\Gamma_P(G)$. Let $H = < x >$. Since $\deg(x) = n - 1$in $\Gamma_P(G)$, $x \in < y >$ for all $y \in G - H$. Hence x commutes with all elements in G and so $x \in Z(G)$.

Remark

The converse of Proposition 4.3 is not true. For example, consider the group $(\mathbb{Z}_6, +_6)$ and the graph $\Gamma_P(\mathbb{Z}_6)$ (given below. Here $3 \in Z(\mathbb{Z}_6)$, whereas $\deg(3) = 3 \neq 5 = n - 1$.



$\Gamma_P(\mathbb{Z}_6)$

Theorem 4.4

Let G be a group with n elements. Then the following are equivalent:

(i) $\Gamma_P(G) \cong K_{1,n-1}$ (ii)$\Gamma_P(G)$ is a tree (iii) Every element of G is its own inverse.

Proof. $(i) \Longrightarrow (ii)$ Proof is trivial

$(ii) \Longrightarrow (iii)$

Assume that $\Gamma_P(G)$ is a tree. Suppose that there exist an element $a \in G$ such that $a \neq a^{-1}$. Then the graph induced by $\{e, a, a^{-1}\}$ is a triangle in $\Gamma_P(G)$, which is a contradiction.

$(iii) \Longrightarrow (i)$

Assume that every element of G has self inverse. We have $\Gamma_P(G) \cong K_{1,n-1}$ and hence it is a tree. From this $< x >= \{e, x\}$ for al $x \in G - e$. Then $\Gamma_P(G) \cong K_{1,n-1}$.

Theorem 4.5

Let G be a group of order pq, where $p < q$, p, q are two distinct primes and $\emptyset$ is the Euler function. Then

(i) G is cyclic if and only if $\Gamma_P(G) \cong \left(K_{p-1} \cup K_{q-1}\right) + K_{\emptyset(pq)+1}$
(ii) G is non-cyclic if and only if $\Gamma_P(G) \cong K_1 + \left(qK_{p-1} \cup K_{q-1}\right)$.

Proof.

(i): Let G be a cyclic group of order pq. Then G has a unique p-Sylow subgroup namely H_1 and a unique q-Sylow subgroup namely H_2. We have, $\Gamma_P(H_1) \cong K_p$ and $\Gamma_P(H_2) \cong K_q$. Note that all elements in $G - (H_1 \cup H_2)$ are generators of G and so $\left|G - \left(H_p \cup H_q\right)\right| = \emptyset(pq)$. Since the generators and the identity element e of G have full degree in $\Gamma_P(G)$ and every non identity element in H_1 is not adjacent to every non identity element in H_2, $\Gamma_P(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\emptyset(pq)+1}$.

Conversely, assume that $\Gamma_P(G) \cong (K_{p-1} \cup K_{q-1}) + K_{\emptyset(pq)+1}$.If G is non-cyclic, then every non identity element of G has order either p or q. From this, identity is the only vertex of full degree in $\Gamma_P(G)$, which is a contradiction. Hence G is cyclic.

(ii): Let G be a non-cyclic group. Then the number of p-Sylow subgroups of G is q and G has a unique q-Sylow subgroup. Also the identity element of G has full degree in $\Gamma_P(G)$. Hence $\Gamma_P(G) \cong K_1 + \left(qK_{p-1} \cup K_{q-1}\right)$. Conversely, assume that $\Gamma_P(G) \cong K_1 + \left(qK_{p-1} \cup K_{q-1}\right)$. If G is cyclic, then G has $\emptyset(pq)$ generators and so $\Gamma_P(G)$ has $\emptyset(pq)$ +1 full degree vertices, which is a contradiction. Hence G is non cyclic.

Theorem 4.6

Let G be a finite group. Then $\Gamma_P(G)$ is Eulerian if and only if o(G) is odd.

Proof.

Assume that o(G) is odd. Clearly $\deg(e)$ is even in $\Gamma_P(G)$.For any $e \neq x \in G$, clearly o(x) is odd and so o(x)-1 is even. If $\deg(x) = o(x) - 1$in $\Gamma_P(G)$, then $\deg(x)$ is even. If $\deg(x) > o(x) - 1$, then there exists an element $y \in G$ such that $y \notin < x >$ and $x \in < y >$. Since $< y >=< y^{-1} >$, $x \in < y^{-1} >$. From this x is adjacent to $\{e, x^2, \ldots, x^{o(x)-1}, y_1, y_1^{-1}, \ldots, y_k y_k^{-1}\}$ for some $k \geq 1$. Since o(G) is odd, G has no self inverse element and so $\deg(x)$ is even in $\Gamma_P(G)$. Hence $\Gamma_P(G)$ is Eulerian.

Conversely, assume that $\Gamma_P(G)$ is Eulerian. If o(G) is even, then $\deg(e)$ is odd, which is a contradiction to $\Gamma_P(G)$ is Eulerian. Hence o(G) is odd.

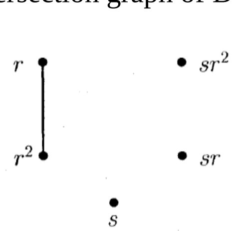
5. Sub Group Intersection Graph of a Group

Definition

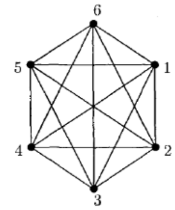
Let G be a group. The subgroup intersection graph Γ_{SI}(G) of G is a graph with V(Γ_{SI}(G) = G - e and two distinct vertices x and y are adjacent in Γ_{SI}(G) if $|< x > \cap < y >|$>1, where < x > is the subgroup generated by x ϵG.

Example

The subgroup intersection graph of $D_6 = \{e, r, r^2, s, sr, sr^2\}$ and $\mathbb{Z}_7$ are below:



Γ_{SI}(D$_6$))



(Γ_{SI}($\mathbb{Z}_7$))

Proposition 5.1

Let G be a finite group with identify e. For any x ϵG – e, $deg_{\Gamma_{SI}(G)}(x) \geq 0(x) - 2$.

Proof.

Let x ϵG – e. since $<x^i> \cap <x> = <x^i>$ for all $2 \leq i \leq o(x) - 1$, x is adjacent to $x^2, x^3, \ldots, x^{0(x)-1}$ and so $deg_{\Gamma_{SI}(G)}(x) \geq o(x) - 2$ for all x ϵ G-e.

Proposition 5.2

Let G be a finite group. Isolated vertices of Γ_{SI}(G) are self-inverse elements in G.

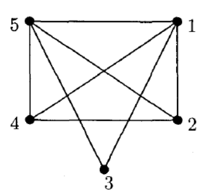
Proof.

If a is not a self-inverse element in G, then 0(a) >2 and so

$|< a > \cap < a^{-1} >| > 1$. Hence a, a^{-1} are adjacent in Γ_{SI}(G) and so a is not an isolated vertex of Γ_{SI}(G).

Remark

The converse of proposition 6.2.4 is not true. For, let G = ($\mathbb{Z}_6$, $+_6$) and a = 3. Here 3 is a self-inverse element in G, but deg(3) =2≠0 as can be seen from the graph given below.



$\Gamma_{SI}(\mathbb{Z}_6)$

Proposition 5.3

Let G be a finite group and q be number the of edges Γ_{SI}(G).

Then $q \geq \frac{\sum_{x \in G-e} 0(x) - 2}{2}$. Moreover, this bound is sharp.

Proof.

By proposition 6.2.3, for all x∈G-e, $deg_{r_{SI}(G)}(x) \geq 0(x)$ -2. From this, $\sum_{x\in G-e} deg_{\Gamma_{SI}(G)}(x) \geq \sum_{x\in G-e} 0(x) - 2$. Hence q $\geq \frac{\sum_{x\in G-e} 0(x)-2}{2}$. Consider the group $\mathbb{Z}_3 \times \mathbb{Z}_3$, order in Γ_{SI}(G) equals $\frac{\sum_{x\in G-e} 3-2}{2} = 4$. We now characterize the groups G for which the associated graph Γ_{SI}(G) attains the bound specified above.

Theorem 5.4.

Let g be a finite group and q be number the of edges in Γ_{SI}(G). Then

q = $\frac{\sum_{x\in G-e} 0(x)-2}{2}$ if and only if every element other than identity of the group G is of prime order.

Proof.

Assume that Γ_{SI}(G) is a graph with $\frac{\sum_{x\in G-e} 0(x)-2}{2}$ edges.

In view of proposition 5.3, we get that $deg_{\Gamma_{SI}}(x) = 0(x)$ -2 for all vertices x∈G-e = Γ_{SI}(G). for x∈G-e. suppose 0(x) is not prime without loss of generality, we can take 0(x) = pq, where p is a prime and q is a positive integer. Consider the subgraph H =<x>. since p\o(H) and by Cauchy's Theorem, H has an element say y such that o(y) = p. By assumption, $deg_{\Gamma_{SI}}(y)$ = o(y)-2 = p-2. On the other hand, note that x∉<y> and y∈<x>, y is adjacent to at least x,y^2,…, y^{n-1} which every element other that identity of the group G has a prime order.

Conversely, assume that every element other than identity of the group g is of prime order. Suppose there exists an element x∈G -e such that deg(x) = 0(x) -2. Then there exists an element y∈ G- {e, x}, y ∉<x> and y is adjacent to x. this implies that $|< x > \cap < y >| > 1$. Since o(x) and o(y) are prime, o(x) =o(y). in such a case <x> = <y>, a contradiction to y ∉<x>. Hence deg(x) = =o(x) - 2 for all x∈G-e in Γ_{SI}(G).

Theorem 5.5.

Let G be a finite group. Then Γ_{SI}(G) is a complete graph if and only if G has a unique subgroup of order p and o(G) = p^m for some prime number p and positive integer m.

Proof.

Let G be a finite group of order n. Assume that Γ_{SI}(G) is a complete graph. If n is not a prime power, then there exist two prime divisors p and q of n. By Cauchy's theorem, G has two elements a and b such that o(a) = p and o(b) =q. Clearly |<a>∩<b> | =1 so that a and b are non-adjacent, which is a contradiction to the assumption that Γ_{SI}(G) is complete. Hence o(G) = P^m for a prime number p,

Suppose G has two different subgroups of order p. then there exist two non-identity elements a,b∈G such that o(a) = o(b) =p and |<a>∩<b> | =1. From this a and are non-adjacent in Γ_{SI}(G), which is again a contradiction. Hence G has a unique subgroup of order p.

Conversely, assume that o(G) =p^m for some prime p ad G has a unique subgroup H of order p. since p is prime, there exist a∈G such that H=< a >.

Since o(G) =p^m, for any b∈G – e, o(b) =P^k where k is an integer with 1≤k≤m. Since H=<a> is a unique subgroup of order p, $< a > \subseteq < b >$ for all b∈G – e. Therefore $|< x > \cap < y >| \geq p > 1$ for all x, y ∈G – e = V(Γ_{SI}(G)) i.e., x and y are adjacent and hence Γ_{SI}(G) is complete.

Theorem 5.6.

For any finite group G, Γ_{SI}(G) is a tree if and only if G is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Proof.

By Theorem 5.5, $\Gamma_{SI}(\mathbb{Z}_2) = K_1$,$\Gamma_{SI}(\mathbb{Z}_3) = K_2$ and hence $\Gamma_{SI}(G)$ is a tree when G is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Conversely, assume that $\Gamma_{SI}(G)$ is a tree.

If p|o(G) for some prime number p ≥ 5, by Cauchy's Theorem G has an element of order p. By Theorem 5.5, K_{p-1}(p ≥ 5) is a subgroup of $\Gamma_{SI}(G)$ and so $\Gamma_{SI}(G)$ is not a tree, a contradiction. From this G must be of order 2^n or 3^n for some n≥ 1.

Suppose G is an elementary abelian group of order 2^n or 3^n for some n>1. Then $\Gamma_{SI}(G)$ is either $(2^n - 1)$ K_1 or $\frac{3n-1}{2}K_2$ respectively and so $\Gamma_{SI}(G)$ is disconnected, a contradiction.

Suppose every element of G is of order 2 or 3. Then $\Gamma_{SI}(G)$ is disconnected, a contradiction to the assumption. Hence G contains an element a such that o(a) ≥ 4. Now, the subgraph induced by <a> contains $K_3 \subseteq \Gamma_{SI}(G)$, again a contradiction to the assumption. From this G is isomorphic to one of the groups $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

Theorem 5.7.

Let G be a finite group of order n = $P_1^{\alpha_1}P_2^{\alpha_2},\ldots.P_k^{\alpha_k}$, where $p_1,p_2,,\ldots,p_k$ are distinct prime numbers and $\alpha_1,\alpha_2,\ldots,\alpha_k$ are positive integers. If G has unique subgroups of orders $p_1,p_2,,\ldots,p_k$,then $\Gamma_{SI}(G)$ is connected and diam($\Gamma_{SI}(G)$)≤4.

Proof:

Since each p_i divides o(G), by Cauchy's Theorem, G contain elements $a_i$such that, $o(a_i) = p_i$, for 1≤i≤ k. consider a_i, a_j∈ G with $o(a_i) = p_i$ and $o(a_j) = p_j$ for some 1≤i, j≤k and j≠j. H_i =<a_i> and H_j = <a_j> be subgroups of G. By assumption, H_i and $H_j$are unique subgroups of orders p_i and $p_j$respectively and so H_i and $H_j$are normal subgroups of G. Therefore H_iH_j is also a normal subgroup of G and so $o(H_iH_j) = p_ip_j$. since H_i and H_j are cyclic subgroups, the subgroup H_iH_j is cyclic and so contains an element b of order p_ip_j. By assumption, <a_i> ∩ < b> = <a_i> and <a_j>∩ < b> = <a_j>. Therefore, a_iba_j is a path in $\Gamma_{SI}(G)$. Let x, y ∈ G. Then there exists p_i and p_j for some I, j with 1≤i, j≤k such that p_i\o(x) and p_j\o(y). Note that xa_ibz_jy is a path between x and y in $\Gamma_{SI}(G)$. Hence $\Gamma_{SI}(G)$ is connected and diam ($\Gamma_{SI}(G)$) ≤4.

Theorem 5.8.

Let G be a finite group of order n = $P_1^{\alpha_1}P_2^{\alpha_2},\ldots.P_k^{\alpha_k}$, where $p_1,p_2,,\ldots,p_k$ are distinct primes and $\alpha_1,\alpha_2,\ldots,\alpha_k$ are positive integers. Then the independence number β_0 ($\Gamma_{SI}(G)$)≥k.

Proof.

Since each p_i divides o(G), by Cauchy's Theorem G contain elements a_i such that $o(a_i) = p_i$ for 1≤i≤k. Note that <a_j>∩ <a_j> = { e} for all i ≠ j. From this $\{a_1, a_2,\ldots,a_n\}$ is an independent set of $\Gamma_{SI}(G)$and hence the result follows.

REFERENCE(S):

[1]. Abdollahi, A., Akbari, S., and Maimani, H.R., Non-commuting graph of a group of a group, J. Algebra, 298 (2006), 468-492.

[2]. Acharya, B.D., Walikar, H.B., and Sampathkumar. E., Recent development in the theory of domination in graphs, ,Mehta Research Institute, Allahabad, MRI Lecture notes in math. (1) (1979).

[3]. Akbari, S., Mohammadian, A., On the zero-divisor graph of a commutative ring, J. Algebra, 274(2)(2004), 847-855.

[4]. Arumugam, S., and Kala, R., Domination parameters of star graph, Ars Combin., 44(1996), 93-96.

[5]. Biggs, N.L., Algebraic graph theory, Cambridge University Press, Cambridge, (1974).

[6]. Bondy, J.A., and Murty, U.S.R., Graph theory with applications Elsevier, (1997).

[7]. Cameron, P.J., and Ghosh, S., T he powe graph of afinite group, diiscrte math, 311(13)(2011), 1220-1222.

[8]. Chakrabarty, I., Ghosh, S., and Sen, M.K., Undirected power graph of semigroups, Semigroup Forum, 78 (2009), 410-426.

[9]. Chartrand, G., and Zhang, P., Introduction to graph theory, Tata McGraw-Hill, (2006).

[10]. Dejter, I.J., Perfect domination in regular grid graphs, Aust. J. Math., 42(2008), 99-114.Algebra, 4(22)(2010), 1051-1056.