Development of Mathematics in Ancient India

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Abstract

An amazing feature of all ancient Indian mathematical literature, beginning with the Sulbasutras, is that they are composed entirely in verses – an incredible feat! This tradition of composing terse sutras, which could be easily memorized, ensured that, inspite of the paucity and perishability of writing materials, some of the core knowledge got orally transmitted to successive generations. India gave to the world a priceless gift – the decimal system. This profound anonymous Indian innovation is unsurpassed for sheer brilliance of abstract thought and utility as a practical invention. The decimal notation derives its power mainly from two key strokes of genius: the concept of place-value and the notion of zero as a digit. Apart from developing the subject of algebra proper, Indians also began a process of algebrisation and consequent simplification of other areas of mathematics. For instance, they developed trigonometry in a systematic manner, resembling its modern form, and imparted to it its modern algebraic character. The algebraisation of the study of infinitesimal changes led to the discovery of key principles of calculus by the time of Bhaskaracharya.

Keyword: - mathematical literature, algebraisation, decimal notation, Trigonometry.

Introduction

Mathematics, in its early stages, developed mainly along two broad overlapping traditions: (i) the geometric and (ii) the arithmetical and algebraic. Among the pre-Greek ancient civilizations, it is in India that we see a strong emphasis on both these great streams of mathematics. Other ancient civilizations like the Egyptian and the Babylonian had progressed essentially along the computational tradition. A Seidenberg, an eminent algebraist and historian of mathematics, traced the origin of sophisticated mathematics to the originators of the Rig Vedic rituals.

The oldest known mathematics texts in existence are the Sulba-sutras of Baudhayana, Apastamba and Katyayana which form part of the literature of the Sutra period of the later Vedic age. The Sulbasutras had been estimated to have been composed around 800 BC. But the mathematical knowledge recorded in these sutras are much more ancient; for the Sulba authors emphasise that they were merely stating facts already known to the composers of the Brahmanas and Samhitas of the early Vedic age.

The Sulbasutras give a compilation of the results in mathematics that had been used for the designing and constructions of the various elegant Vedic fire-altars right from the dawn of civilization. The altars had rich symbolic significance and had to be constructed with accuracy. The designs of several of these brick-altars are quite involved – for instance, there are constructions depicting a falcon in flight with curved wings, a chariot-wheel complete with spokes or a tortoise with extended head and legs! Constructions of the fire-altars are described in an enormously developed form in the Sata-patha Brahmana; some of them are mentioned in the earlier Taittiriya Samhita, but the sacrificial fire-altars are referred – without explicit construction – in the even earlier Rig Vedic Samhitas, the oldest strata of the extant Vedic literature. The descriptions of the fire-altars from the Taittiriya Samhita onwards are exactly the same as those found in the later Sulbasutras.

Plane geometry stands on two important pillars having applications throughout history; (i) the result popularly known as the 'Pythagoras theorem' and (ii) the properties of similar figures. In the Sulbasutras, we see an explicit statement of the Pythagoras theorem and its applications in various geometric constructions such as construction of a square equal to the sum, or difference, of two given squares, or to a rectangle, or to the sum of *n* squares. These constructions implicitly involve application of algebraic identities such as $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $a^2 - b^2 = (a + b)(a - b)$, $ab = ((a + b)/2)^2 - ((a - b)/2)^2$ and $na^2 = ((n + 1)/2)^2a^2 - ((n - 1)/2)^2a^2$. They reflect a blending of geometric and subtle algebraic thinking and insight which we associate with Euclid. In fact, the Sulba construction of a square equal in area to a given rectangle is exactly the same as given by Euclid several centuries later! There are geometric solutions to what are algebraic and number-theoretic problems.

Pythagoras theorem was known in other ancient civilizations like the Babylonian, but the emphasis there was on the numerical and not so much on the proper geometric aspect while in the Sulbasutras one sees depth in both aspects – especially the geometric. This is a subtle point analyzed in detail by Seidenberg. From certain diagrams described in the Sulbasutras, several historians and mathematicians like Burk, Hankel, Schopenhauer, Seidenberg and Van der Waerden have concluded that the Sulba authors possessed proofs of geometrical results are analyzed in the pioneering work of Datta. One of proofs of the Pythagoras theorem, easily deducible from the Sulba verses, is later described more explicitly by Bhaskara – II.

Apart from the knowledge, skill and ingenuity in geometry and geometric algebra, the Vedic civilization was strong in the computational aspects of mathematics as well – they handled the arithmetic of fractions as well as surds with ease, found good rational approximations to irrational numbers like the square root of 2, and, of course, used several significant results on mensuration.

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The Decimal Notation and Arithmetic:–

India gave to the world a priceless gift – the decimal system. This profound anonymous Indian innovation is unsurpassed for sheer brilliance of abstract thought and utility as a practical invention. The decimal notation derives its power mainly from two key strokes of genius: the concept of place-value and the notion of zero as a digit.

The decimal system has a deceptive simplicity as a result of which children all over the world learn it even at a tender age. It has an economy in the number of symbols used as well as the space occupied by a written number, an ability to effortlessly express arbitrarily large numbers and, above all, computational facility. Thus the twelve-digit Roman number (DCCCLXXXVIII) is simply 888 in the decimal system!

Most of the standard results in basic arithmetic are of Indian origin. This includes neat, systematic and straight-forward techniques of the fundamental arithmetic operations: addition, subtraction, multiplication, division, taking squares and cubes, and extracting square and cube roots; the rules of operations with fractions and surds; various rules on ratio and proportion like the rule of three; and several commercial and related problems like income and expenditure, profit and loss, simple and compound interest, discount, partnership, computations of the average impurities of gold, speeds and distances, and the mixture and cistern problems similar to those found in modern texts.

The excellence and skill attained by the Indians in the foundations of arithmetic was primarily due to the advantage of the early discovery of the decimal notation – the key to all principal ideas in modern arithmetic. For instance, the modern methods for extracting square and cube roots, described by Aryabhata in the 5th century AD, cleverly use the ideas of place value and zero and the algebraic expansions of $(a + b)^2$ and $(a + b)^3$.

The concept of zero existed by the time of Pingala (dated 200 BC). The idea of place-value had been implicit in ancient Sanskrit terminology – as a result, Indians could effortlessly handle large numbers right from the Vedic Age. There is terminology for all multiples of ten up to 10^{18} in early Vedic literature, the Ramayana has terms all the way up to 10^{55}. The structure of the Sanskrit numeral system and the Indian love for large numbers must have triggered the creation of the decimal system.

As well shall see later, even the smallest positive integral solution of the equation $x^2 - Dy^2 = 1$ could be very large; in fact, for D = 61, it is (1766319049, 226153980).

The decimal system stimulated and accelerated trade and commerce as well as astronomy and mathematics. It is no coincidence that the mathematical and scientific renaissance began in Europe only after the Indian notation was adopted. Indeed the decimal notation is the very pillar of all modern civilization.

ALGEBRA:–

Algebra was only implicit in the mathematics of several ancient civilizations till it came out in the open with the introduction of literal or symbolic algebra in India. By the time of Aryabhata (488 AD) and Brahmagupta (628 AD), symbolic algebra had evolved in India into a distinct branch of mathematics and became one of its central pillars. After evolution through several stages, algebra has now come to play a key role in modern mathematics both as an independent area in its own right as well as an indispensable tool in other fields. In fact, the 20th century witnessed a vigorous phase of 'algebraisation of mathematics' Albebra provides elegance, simplicity, precision, clarity and technical power in the hands of the mathematicians. It is remarkable how early the Indians had realized the signifiance of algebra and how strongly the leading Indian mathematicians like Brahmagupta (628 AD) and Bhaskara II (1150 AD) asserted and established the importance of their newly-founded discipline as we shall see in subsequent issues.

Indians began a systematic use of symbols to denote unknown quantities and arithmetic operations. The four arithmetic operations were denoted by "yu", "ksh" "gu" and "bha" which are the first letters of the corresponding Sanskrit words yuta (addition), ksaya (subtraction), guna (multiplication) and bhaga (division; similarly "ka" was used for karani (root), while the first letters of the names of different colours were used to denote different unknown variables.

This introduction of symbolic representation was an important step in the rapid advancement of mathematics. While a rudimentary use of symbols can also be seen in the Greek texts of Diophantus, it is in India that algebraic formalism achieved full development.

The Indians classified and made a detailed study of equations (which were called sami-karana), introduced negative numbers together with the rules for arithmetic operations involving zero and negative numbers, discovered results on surds, described solutions of linear and quadratic equations, gave formulae for arithmetic and geometric progression as well as identities involving summation of finite series, and applied several useful results on permutation and combinations including the formulae for ^nP_r and ^nC_r. The enlargement of the number system to include negative numbers was a momentous step in the development of mathematics. Thanks to the early recognition of the existence of negative numbers, the Indians could give a unified treatment of the various forms of quadratic equations (with positive co-efficient), i.e., $ax^2 + bx = c$, $ax^2 + c = bx$, $bx + c = ax^2$. The Indians were the first to recognize that a quadratic equation has two roots. Sridharacharya (750 AD) gave the well-known method of solving a quadratic equation by completing the square – an idea with far – reaching consequences in mathematics. The Pascal's triangle for quick computation of ^nC_r is described by Halayudha in the 10th century AD as Meru-Prastara 700 years before it was stated by Pascal; and Halayudha's Meru-Prastara was only a clarification of a rule invented by Pingala more than 1200 years earlier.

<u>Trigonometry and Calculus</u>:–

Apart from developing the subject of algebra proper, Indians also began a process of algebrisation and consequent simplification of other areas of mathematics. For instance, they developed trigonometry in a systematic manner, resembling its modern form, and imparted to it its modern algebraic character. The algebraisation of the study of infinitesimal changes led to the discovery of key principles of calculus by the time of Bhaskaracharya.

Indians invented the sine and cosine functions, discovered most of the standard formulae and identities, including the basic formula for sin $(A \pm B)$, and constructed fairly accurate sine tables. Bramagupta and Govindaswami gave interpolation formulae for calculating the sines of intermediate angles from sine tables – these are special cases of the Newton – Stirling and Newton – Gauss formulae for second-order difference. Remarkable approximations for π are given in Indian texts including 3.1416 of Aryabhata, 3.14159265359 of Mad-have and 355/113 of Nilakanta. An anonymous work Karanapaddhati gives the value 3.1415926358979324 which is correct up to seventeen decimal places.

The Greeks had investigated the relationship between a chord of a circle and the angle it subtends at the centre – but their system is quite cumbersome in practice. The Indians realized the significance of the connection between a half-chord and half of the angle subtended by the full chord. In the case of a unit circle, this is precisely the sine function.

The Sanskrit word for half-chord "ardha-jya", later abbreviated as "jya", was written by the Arabs as "jyb" Curiously, there is a similar-sounding Arab word "jaib" which means "heart, bosom, fold, bay or curve". When the Arab works were being translated into Latin, the apparently meaningless word "jyb" was mistaken for the word "jaib" and translated as "sinus" which has several meanings in Latin including "heart, bosom, fold, bay or curve"! This word became "sine" in the English version. Aryabhata's "kotijya" became cosine.

The tradition of excellence and originality in Indian trigonometry reached a high peak in the outstanding results of Madhavacharya on the power series expansions of trigonometric functions. Three centuries before Gregory, Madhava had described the series

$$\theta = \tan\theta - (1/3)(\tan\theta)^3 + (1/5)(\tan\theta)^5$$

$$- \quad (1/7)(\tan\theta)^7 + \qquad (|\tan\theta| \leq 1).$$

Proof, as presented in Yuktibhasa, involves the idea of integration as the limit of a summation and corresponds to the modern method of expansion and term-by-term integration. A crucial step is the use of the result.

$$\frac{\lim}{n \to \infty}(1^P + 2^P + \qquad + (n-1)^P)/n^{P+1} = 1/(P+1).$$

The explicit statement that (| tan θ| ≤ 1) reveals the level of sophistication in the understanding of infinite series including an awareness of convergence. Madhava also discovered the beautiful formula.

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 +$$

Obtained by putting $\theta = \pi/4$ in the Madhava – Gregory series. This series was rediscovered three centuries later by Leibniz. As one of the first applications of his newly invented calculus, Leibniz was thrilled at the discovery of this series which was the first of the results giving a connection between π and unit fractions. Madhava also described the series.

$$\pi/\sqrt{12} = 1 - 1/3.3 + 1/5.3^2 - 1/7.3^3 +$$

First given in Europe by A Sharp (1717). Again, three hundred years before Newton (1676 AD), Madhava had described the well-known power series expansions.

$$\sin x = x - x^3 / 3! + x^5 / 5! -$$

$$\text{and } \cos x = 1 - x^2 /2! + x^4/4! -$$

These series were used to construct accurate sine and cosine tables for calculations in astronomy.

Conclusion

Among ancient mathematicians whose texts have been found, special mention may be made of Aryabhata, Brahmagupta and Bhaskaracharya. All of them were eminent astronomers as well. The Indian contributions in arithmetic, algebra and trigonometry were transmitted by the Arabs and Persians to Europe. Unfortunately, the original texts of several outstanding mathematicians like sridhara, Padmanabha, Jayadeva and Madhava have not been found yet – it is only through the occasional reference to some of their results in subsequent commentaries that we get a glimpse of their work. An amazing feature of all ancient Indian mathematical literature, beginning with the Sulbasutras, is that they are composed entirely in verses – an incredible feat! This tradition of composing terse sutras, which could be easily memorized, ensured that, inspite of the paucity and perishability of writing materials, some of the core knowledge got orally transmitted to successive generations.

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