FIXED POINT THEOREM IN CONE METRIC SPACE

Harikrishna Patel[1]

[1]Department of Mathematics,
Dr. C. V. Raman University,
Kota, Bilaspur, Chhattisgarh, 495113, India.

ABSTRACT:
In this paper, I have prove as unique fixed point there ever in cone metric spaces with applying c-distance. This result generalize and extend the recent results of Dubey, A.K. et al. [2015] in the sense that applying c-distances in contractive conditions, which extends the further scope of our results.

1 . INTRODUCTION

Metric Space

Let X be a nonempty set .A metric on X is a real function "d" of order pair of element Of X , which satisfies the following three conditions :

d(x,y)≥0 and d(x,y) =0 ⇔ x=y

d(x,y)=d(x,y) (symmetry)

d(x,y)≤ d(x,z) +d(z,y) (The Triangle inequalities)

A metric space consists of two objects :a non-empty set X and a metric 'd' on X. The element Of X are called the point of the metric space (X,d) .A metric space is called complete if all Cauchy Sequences converge .Every incomplete space is isometric ally embedded into its completion .

Every compact metric space is complete ,the real line is non-compact but complete ,and the Open interval (0,1) is incomplete .Every Euclidean space is also complete space .

Cone metric space :

Let X be a non –empty set .Suppose the mapping d:X x X⟶ E satisfies (E is always be a real Banach).

(1)0<d(x,y) for all x,y ϵ X and d(x,y) =0 if and only if x=y;

(2)d(x.y) =d(y,x) for all x,y ϵ X;

(3)d(x,y)≤ d(x,z)+d(y,z) for all x,y,z ϵX

Then d is called cone metric on X , and (X,d) is called cone metric space .

2 MAIN RESULTS.

We prove the following theorem

Theorem : 2. 1 Let (X,d)be cone metric spaces, P be a solid cone and q be a c-distance on$X$. Suppose that $T: X \to X$ be continuous and satisfies the contractive condition;

$$\boldsymbol{q}(Tx,Ty) \le a_1 q(x,y) + a_2 q(x,Tx) + a_3 q(y,Ty) + a_4[(q(x,Tx) + q(y,Ty)] + a_5[q(y,Tx) + q(x,Ty)] \quad \text{……………………………………….. (2.1)}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are non negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$. Then T has a fixed point $x^* \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point. If $u = Tu$. Then $(u,u) = \theta$. The fixed point is unique.

Proof: Choose $x_0 \in X$. Set $x_1 = Tx_{0,}$, $x_2 = Tx_1 = T^2 \ldots\ldots\ldots\ldots. x_{n+1} = Tx_n = T^n x_0$

Then we have,

$$q(x_n, x_{n+1}) \leq q(Tx_{n-1,}Tx_n) \quad \ldots\ldots \text{(2.2)}$$

$$\leq a_1 q(x_{n-1}, x_n) + a_2 q(x_{n-1}, Tx_{n-1}) + a_3 q(x_n, Tx_n)$$

$$+ a_4[q(x_{n-1}, Tx_{n-1}) + q(x_n, Tx_n)] + a_5[q(x_n, Tx_{n-1}) + q(x_{n-1}, Tx_n)$$

$$= a_1 q(x_{n-1}, x_n) + a_2 q(x_{n-1}, x_n) + a_3 q(x_n, x_{n+1})$$

$$+ a_4[q(x_{n-1}, x_n) + q(x_n, x_{n+1})] + a_5[q(x_n, x_n) + q(x_{n-1}, x_{n+1})$$

$$q(x_n, x_{n+1}) \leq (a_1 + a_2 + a_3 + a_4 + a_5) q(x_{n-1}, x_n) + (a_3 + a_4 + a_5) q(x_n, x_{n+1})$$

So, $q(x_n, x_{n+1}) \leq \frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)} q(x_{n-1}, x_n)$

$= hq(x_{n-1}, x_n)$, where $h = \frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)} < 1$. ....(2.3)

Let $m > n \geq 1$. Then it follows that

$$q(x_{n,} x_m) \leq q(x_{n,}x_{n+1}) + q(x_{n+1,}x_{n+2}) + \ldots\ldots + q(x_{n-1,}x_n)$$

$$\leq (h^n + h^{n+1} + \cdots \ldots\ldots\ldots\ldots + h^{n-1}) q(x_{0,}x_1)$$

$$\leq \frac{h^n}{1-h} q(x_{0,}x_1) \to \infty, h \to \infty. \quad \ldots\ldots \text{(2.4)}$$

Thus, Lemma 2.10 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. Since T is continuous, then $x^* = \lim_{x_{n+1}} = limT(x_n) = T(limx_n) = T(x^*)$. Therefore, x^* is a fixed point of T.Suppose that $u = Tu$.

Then we have

$$q(u, u) \leq q(Tu, Tu)$$

$$\leq a_1 q(u, u) + a_2 q(u, Tu) + a_3 q(u, Tu) + a_4[q(u, Tu) + q(u, Tu)]$$

$$+ a_5[q(u, Tu) + q(u, Tu)$$

$$= (a_1 + a_2 + a_3 + 2a_4 + 2a_5) q(u, u). \quad \ldots\ldots \text{(2.5)}$$

Since $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$, Lemma 2.5 shows that $q(u, u) = \theta$. Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point ofy^* of T, then we have

$$q(x^* y^*) \leq q(Tx^*, Ty^*)$$

$$\leq a_1 q(x^*, y^*) + a_2 q(x^*, Tx^*) + a_3 q(y^*, Ty^*) + a_4[q(x^*, Tx^*) + q(y^*, Ty^*)]$$

$$+ a_5[q(y^*, Tx^*) + q(x^*, Ty^*)$$

$$= (a_1 + 2a_5) q(x^*, y^*).$$

$$\leq (a_1 + a_2 + a_3 + 2a_4 + 2a_5)\, q(x^*, y^*). \quad \ldots\ldots \text{(i)}$$

Since $(a_1 + a_2 + a_3 + 2a_4 + 2a_5) < 1$, then by Lemma 2.5 we have$q(x^*, y^*) = \theta$and also we have$(x^*, x^*) = \theta$. Hence by Lemma 2.10(1), $x^* = y^*$.Therefore the fixed point is unique.

Remark 2.2

(1). Put $a_4 = 0$ and $a_4 = a_5$ in theorem 2.1, we get the result of theorem 2.1 of Dubey, A. K.et al.[2015].

(2). If we put $a_4 = 0$ and $a_5 = 0$ in theorem 2.1, we get the result of theorem 3.3 of Fadail,et al. [9].

(3). If we put $a_1 = a_2 = a_3 = a_5 = 0$ and $a_2 = a_4$in theorem 2.1, we get the result of Corollory3.4 of Fadail, et al. [2012].

Theorem 2.3: Let (X,d)be cone metric spaces, P be a solid cone and q be a c-distance on$X$. Suppose that $T:X\to X$ be continuous and satisfies the contractive condition;

$$\boldsymbol{q}(Tx,Ty)\le a_1q(x,y)+a_2[q(x,Tx)+q(y,Ty)]+a_3[q(x,Ty)+q(y,Tx)]$$
$$+a_4[q(x,Tx)+q(x,y)]+a_5[q(y,Ty)+q(x,y)]\ldots\ldots\ldots\ldots\ldots\ldots (2.6)$$

for all $x,y\in X$, where a_1,a_2,a_3,a_4,a_5 are non negative real numbers such that $a_1+2a_2+2a_3+2a_4+2a_5<1$. Then T has a fixed point $x^*\in X$, iterative sequence $\{T^nx\}$ converges to the fixed point. If $u=Tu$. Then $(u,u)=\theta$. The fixed point is unique.

Proof: Choose $x_0\in X$. Set $x_1=Tx_0,\ x_2=Tx_1=T^2\ldots\ldots\ldots\ldots x_{n+1}=Tx_n=T^nx_0$

Then we have,

$$q(x_n,x_{n+1})\le q(Tx_{n-1},Tx_n)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$
$$\le a_1q(x_{n-1},x_n)+a_2[q(x_{n-1},Tx_{n-1})+q(x_n,Tx_n)]$$
$$+a_3[q(x_{n-1},Tx_n)+q(x_n,Tx_{n-1})]+a_4[q(x_{n-1},Tx_{n-1})+q(x_{n-1},x_n)]$$
$$+a_5[q(x_n,Tx_n)+q(x_n,x_n)]$$
$$=a_1q(x_{n-1},x_n)+a_2[q(x_{n-1},x_n)+q(x_n,x_{n+1})]$$
$$+a_3[q(x_{n-1},x_{n+1})+q(x_n,x_n)]+a_4[q(x_{n-1},x_n)+q(x_{n-1},x_n)]$$
$$+a_5[q(x_n,x_{n+1})+q(x_{n-1},x_n)]$$
$$q(x_n,x_{n+1})\le(a_1+a_2+a_3+2a_4+a_5)q(x_{n-1},x_n)+(a_2+a_3+a_5)q(x_n,x_{n+1})$$

So,$q(x_n,x_{n+1})\le\frac{(a_1+a_2+a_3+2a_4+a_5)}{1-(a_3+a_4+a_5)}q(x_{n-1},x_n)$

$$=hq(x_{n-1},x_n)\text{ , where }h=\frac{(a_1+a_2+a_3+a_4+a_5)}{1-(a_3+a_4+a_5)}<1.\qquad\ldots\ldots\ldots\ldots(2.8)$$

Let $m>n\ge 1$. Then it follows that

$$q(x_n,x_m)\le q(x_n,x_{n+1})+q(x_{n+1},x_{n+2})+\ldots\ldots\ldots+q(x_{n-1},x_n)$$
$$\le(h^n+h^{n+1}+\cdots\ldots\ldots\ldots\ldots\ldots+h^{n-1})q(x_0,x_1)$$
$$\le\frac{h^n}{1-h}q(x_0,x_1)\to\infty,h\to\infty.\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .(2.9)$$

Thus, Lemma 2.10 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x^*\in X$ such that $x_n\to x^*$. Since T is continuous, then $x^*=\lim\limits_{x_{n+1}}=limT(x_n)=T(limx_n)=T(x^*)$. Therefore, x^* is a fixed point of T.Suppose that $u=Tu$.

Then we have

$$q(u,u)\le q(Tu,Tu)$$
$$\le a_1q(u,u)+a_2[q(u,Tu)+q(u,Tu)]+a_3[q(u,Tu)+q(u,Tu)]$$
$$+a_4[q(u,Tu)+q(u,u)]+a_5[q(u,Tu)+q(u,u)]$$
$$=[(a_1+2a_2+2a_3+2a_4+2a_5)]q(u,u)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.10)$$

Since $a_1+2a_2+2a_3+2a_4+2a_5<1$, Lemma 2.5 shows that $q(u,u)=\theta$. Next we prove that the uniqueness of the fixed point. Suppose that, there is another fixed point ofy^* of T, then we have

$$q(x^*y^*)\le q(Tx^*,Ty^*)$$
$$\le a_1q(x^*,y^*)+a_2[q(x^*,Tx^*)+q(y^*,Ty^*)]+a_3[q(x^*,Ty^*)+q(y^*,Tx^*)]$$
$$+a_4[q(x^*,Tx^*)+q(x^*,y^*)]+a_5[q(y^*,Ty^*)+q(x^*,y^*)]$$
$$=(a_1+2a_3+a_4+a_5)q(x^*,y^*).$$
$$\le(a_1+a_2+a_3+2a_4+2a_5)\,q(x^*,y^*).\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

Since $(a_1 + a_2 + a_3 + 2a_4 + 2a_5) < 1$, then by Lemma 2.5 we have $q(x^*, y^*) = \theta$ and also we have $(x^*, x^*) = \theta$. Hence by Lemma 2.10(1), $x^* = y^*$.Therefore the fixed point is unique.

Remark3.2

(1). Put $a_4 = 0$ and $a_4 = a_5$ in theorem 2.2, we get the result of theorem 2.2of Dubey, A. K.et al.[2015].

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