Bi-domination in Graphs

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Abstract: Let G = (V, E) be a simple graph. A set $S \subseteq V(G)$ is a bi-dominating set if S is a dominating set of G and every vertex in S dominates exactly two vertices in V-S. The bi-domination number $\gamma_{bi}(G)$ of a graph G is the minimum cardinality of the minimal bi-dominating set. In this paper, bi-domination number for some standard graphs are determined. Bounds for bi-domination number are obtained.

Key Words: Domination number, Bi-Domination number and Upper bi-domination number.

1.Introduction: Let G(V,E) be a simple, connected graph where V is its vertex set and E is its edge set. The degree of any vertex v in G is the number of edges incident with v and is denoted by deg v, the minimum degree of a graph is denoted by δ(G) and the maximum degree of a graph G is denoted by Δ (G). A vertex of degree 0 is called an isolated vertex and a vertex of degree 1 is called a pendent vertex. A subdivision of an edge *uv* is obtained by replacing the edge uv with the edges *uw* and *vw* with a new vertex *w*. A subset S of V(G) is called a dominating set of G if every vertex in V − S is adjacent to at least one vertex in S [2-3]. The domination number γ(G) of G is the minimum cardinality of all dominating sets in G. In this paper, bi-domination number for some standard graphs are determined. Bounds for bi-domination number are obtained.For graph theoretic notations, Harary [1] is referred to.

Definition 1.1: Let G = (V, E) be a simple graph. Let $|V(G)| = n$. A vertex v is called a full degree vertex if $\deg v = n - 1$.

Definition 1.2: The graph $B_{n,n}$, $n \geq 2$ is a bistar obtained from two disjoint copies of $K_{1,n}$ by joining the centre vertices by an edge. It have 2n + 2 vertices and 2n + 1 edges.

Definition 1.3: A spider is a tree on 2n + 1 vertices obtained by subdividing each edge of a star. One or more (but not all) of the edges from this subdivision exempted results a wounded spider.[5]

2. Main Results:

Definition 2.1: A set $S \subseteq V(G)$ is a bi-dominating set if S is a dominating set of G and every vertex in S dominates exactly two vertices in V-S.

Remark 2.2: The bi-domination number $\gamma_{bi}(G)$ of a graph G is the minimum cardinality of all minimal bi-dominating sets. The maximum cardinality of a bi-dominating set of G is called the upper bi-domination number of G and it is denoted by Γ_{bi}(G).

Example 2.3: Consider the following graph G in figure 2.1

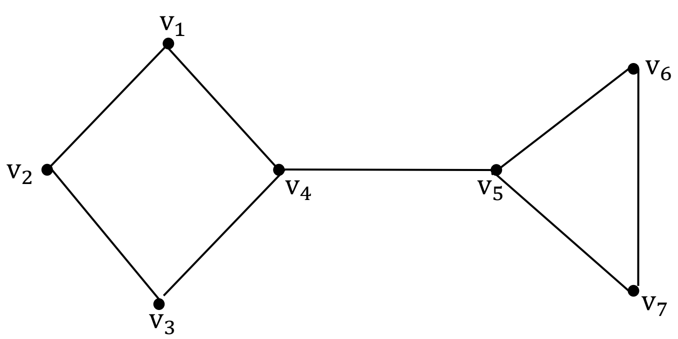

Figure 2.1

Let $S_1 = \{v_1, v_3, v_6\}$, $S_2 = \{v_1, v_3, v_7\}$, $S_3 = \{v_2, v_4, v_5\}$, every vertex of the set S_i, $1 \le i \le 3$ dominate exactly two vertices of V - S_i. Hence S_i, $1 \le i \le 3$ are bi-dominating sets of G. Therefore $\gamma_{bi}(G) \le 3$. $\{v_2, v_5\}$ is the unique minimum dominating set, $\gamma(G) = 2$. It is not a bi-dominating set, since v_5 dominates three vertices v_4, v_6 and v_7. Therefore $\gamma_{bi}(G) \ge 3$. Hence $\gamma_{bi}(G) = 3$.

Example 2.4: Consider the following graph G in figure 2.2

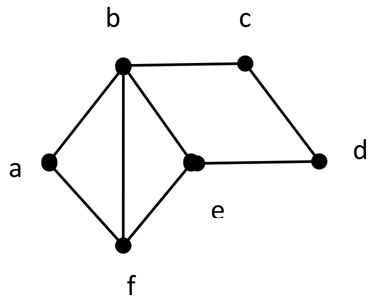

Figure 2.2

Let $S_1 = \{a, d\}$, $S_2 = \{e, c, f\}$. Every vertex of the set S_i $1 \le i \le 2$ dominates exactly two vertices in V – S_i. Hence S_i $1 \le i \le 2$ are bi-dominating sets of G. Hence $\gamma_{bi}(G) \le 2$, Since there is no full degree vertex in G, $\gamma(G) \ge 2$. Therefore $\gamma_{bi}(G) \ge 2$. Also it is verified that no set with four vertices is a bi-dominating set. Hence S_1 is the minimum bi-dominating set and S_2 and S_3 are maximum bi-dominating sets . Hence $\gamma_{bi}(G) = 2$ and $\Gamma_{bi}(G) = 3$.

Remark 2.5: Any bi-dominating set does not contain pendent vertices.

Example 2.6: Graph G without bi-dominating set is given in the following figure 2.3

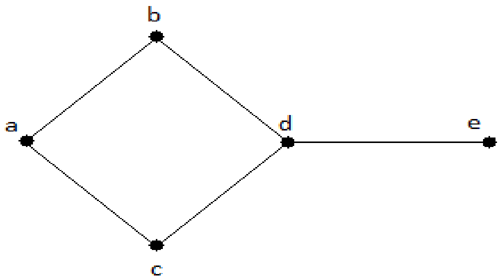

Figure 2.3

Suppose G has bi-dominating set S. Since there is no full degree vertex in G, $\gamma(G) \ge 2$. Hence $\gamma_{bi}(G) \ge 2$. Since every vertex in S dominates exactly two vertices in V-S, $\gamma_{bi}(G) \le 3$. Since e does not belong to S, d must belong to S. Let $S_1 = \{a, d\}$, $S_2 = \{c, d\}$ and $S_3 = \{b, d\}$. In S_1, d dominates three vertices b, c and e of V - S_1 of G. Hence S_1 is not a bi-dominating set of G. In S_2, c dominates only one vertex a of V – S_2 of G. Hence S_2 is not a bi-dominating set of G.
In S_3, b dominates only one vertex a of V – S_3 of G. Hence S_3 is not a bi-dominating set of G. Therefore no set with two elements is a bi-dominating set. In the set $S_4 = \{a, b, d\}$, the vertices a and b do not dominate two elements of V – S_4 of G. In the set $S_5 = \{a, c, d\}$, the vertices a and c do not dominate two elements of V – S_5 of G. In the set $S_6 = \{c, b, d\}$, no element dominate two vertices of V – S_6 of G. Therefore no sets with three elements is a bi-dominating set of G. Hence bi-dominating set does not exist.

Observation 2.6: For any connected graph G with p vertices, $\gamma_{bi}(G) = 1$ if and only if
$G \cong P_3$ or K_3.

Remark 2.7: $\gamma(G) \le \gamma_{bi}(G)$, since every bi-dominating set is a dominating set.

Example 2.8: Consider a graph G given in example 2. $\gamma(G) = 2$ and $\gamma_{bi}(G) = 3$. Therefore $\gamma(G) < \gamma_{bi}(G)$.

Example 2.9: For the Graph G given in figure 2.4, $\gamma_{bi}(G) = \gamma(G)$

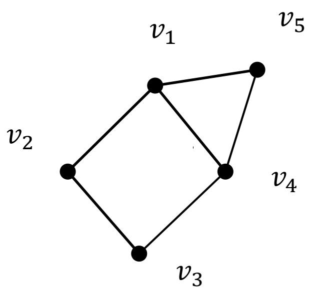

Figure 2.4

There is no full degree vertex in G. Therefore γ(G) ≥ 2. Let S_1 = {v_1, v_4}, S_2 ={v_2, v_5} and S_3 ={v_3, v_5}, Every vertex of the set S_i, 1 ≤ i ≤ 3 dominates exactly two vertices of V - S_i. Hence S_i, 1 ≤ i ≤ 3 are bi-dominating sets of G. Therefore γ(G) ≤ 2 and $\gamma_{bi}(G)$ ≤ 2. Hence γ(G) = 2. Since G is not either P_3 or K_3, $\gamma_{bi}(G)$ = 2. Hence $\gamma_{bi}(G)$ = 2 = γ(G)

Theorem 2.10: Let P be a path of length n, then $\gamma_{bi}(P_n) = \lceil\frac{n}{3}\rceil$, n ≥ 3, n ≠ 2, 4.

Proof: case (i): Let n = 3m, m ≥ 1. {v_2, v_8, …, v_{3m-1}} is the unique bi-dominating set of P_{3m}. Therefore $\gamma_{bi}(P_n) = \frac{n}{3}$.

Case (ii): Let n = 3m + 1, m ≥ 2. {v_2, v_5, v_8, …,v_{3m-4}, v_{3m-2}, v_{3m}} is a bi-dominating set of P_{3m+1}. Therefore $\gamma_{bi}(P_n) = m + 1 = \lceil\frac{n}{3}\rceil$.

Case (iii): Let n = 3m + 2, m ≥ 1. {v_2, v_4, v_7, v_{10}, …, v_{3m+1}} are some bi-dominating set of P_{3m+2}. Therefore $\gamma_{bi}(P_n) = m + 1 = \lceil\frac{n}{3}\rceil$.

Case (iv): when n = 2. {v_1} and {v_2} are dominating sets and it dominates exactly one vertex. Hence it is not a bi-dominating set.

Case (v): S_1= {v_1, v_4}, S_2= {v_1, v_3}, S_3= { v_2, v_3} and S_4= { v_2, v_4} are dominating sets of P_4. No vertices in S_1 and S_3 dominates exactly two vertices. V- S_1 and V – S_3 respectively v_1 in S_2 and v_4 in S_4 do not dominate exactly two vertices in V- S_2 and V – S_4.

Theorem 2.11: Let C be a cycle of length n, then $\gamma_{bi}(C_n) = \lceil\frac{n}{3}\rceil$, n ≥ 3.

Proof: Case (i): Let n = 3m, m ≥ 1. {v_2, v_5, v_8, …, v_{3m-1}} is the bi-dominating set of C_{3m}. Therefore $\gamma_{bi}(C_n) = \frac{n}{3}$.

Case (ii): Let n = 3m + 1, m ≥ 2. {v_2, v_5, v_8, …,v_{3m-4}, v_{3m-2}, v_{3m}} is a bi-dominating set of C_{3m+1}. Therefore $\gamma_{bi}(C_n) = m + 1 = \lceil\frac{n}{3}\rceil$.

Case (iii): Let n = 3m + 2, m ≥ 1. {v_2, v_4, v_7, v_{10}, …, v_{3m+1}} are some bi-dominating set of C_{3m+2}. Therefore $\gamma_{bi}(C_n) = m + 1 = \lceil\frac{n}{3}\rceil$.

Case (iv): when n = 4. The antipodal vertices {v_1, v_2} or {v_3, v_4} is a bi-dominating set of C_4.

Theorem 2.12: Let G be a connected graph, If $G = K_n$ then $\gamma_{bi}(G)$ = n - 2.

Proof: Let $G = K_n$ be a connected graph $\{v_1, v_2, \dots, v_n\}$ be the vertex set of G. Since G is complete and degree of every vertex in G is n – 1, clearly by definition the vertices $v_1, v_2, \dots, v_{n-2}$ dominate exactly two vertices. Hence $\gamma_{bi}(G)$ = n - 2.

Theorem 2.13: Let G be a complete bi-partite graph $K_{m,n}$ and $m,n \geq 3$.Then $\gamma_{bi}\left(G = K_{m,n}\right) = m + n - 4$.

Proof: Let U = {u_1, u_2, … ,u_m} and W = {w_1, w_2, … ,w_n}be a bi-partition of G. Every u_i∈ U dominates all the vertices in W. As each vertex can dominate exactly two vertices except two w_i's all other w_i's are in S, the bi dominating set. Similarly all u_i's except two are in S. Hence γ_{bi}(G) = m-2 + n-2 = m + n – 4.

Theorem 2.14: For a star G = $K_{1,k}$, k ≠ 2 dominating set exist,$\gamma(G)$ = 1 but bi-dominating set does not exits.

Proof: Let $K_{1,k}$ be a star with k pendant vertices. Let v be a vertex with maximum degree and $v_1, v_2, \dots, v_k$ be the pendant vertices of$K_{1,k}$. The vertex v can dominate $v_1, v_2, \dots, v_k$ vertices and the vertices $v_1, v_2, \dots, v_k$ dominate exactly one vertex v. Hence {v} is the minimum dominating set and $\gamma(G)$ = 1. The vertex v dominates more than two vertices but not exactly two. Clearly {v} is a dominating set but not bi-dominating. Therefore bi-dominating set does not exist for $K_{1,k}$.

Note 2.15: If G = $K_{1,2}$then $\gamma(G) = \gamma_{bi}(G)$ =1.

Proof: Let G = $K_{1,2}$. Let v be a vertex with maximum degree 2 and v_1 *and* v_2 be the pendant vertices of$K_{1,2}$.Since degree of v is 2 and it dominate other vertices v_1 *and* v_2. Hence {v} is the minimum dominating set and it dominate exactly two vertices and it also a bi-dominating set. Therefore $\gamma(G) = \gamma_{bi}(G)$ =1.

Remark 2.16: For any connected graph G with n vertices, $1 \leq \gamma_{bi}(G) \leq n-2$.

Theorem 2.18: A bi-dominating set does not exist for a graph G in which a vertex adjacent with more than two pendent vertices.

Proof: Let $u_1, u_2, \ldots, u_m$, $m \geq 3$ be the pendent vertices adjacent with a vertex v in G. Suppose S is a bi-dominating set of G. Since pendent vertices do not belong to S, the vertex v must belong to S to dominate $u_1, u_2, \ldots, u_m$, $m \geq 3$. Then v dominate more than two vertices of V – S, a contradiction. Hence bi-dominating set does not exist in G.

Observation2.19: For a bistar, $B_{m,,n}$, m and n ≠ 2 bi-dominating set does not exist.

Theorem 2.20: For $B_{2,2}$ a bi-star,$\gamma(G) = \gamma_{bi}(G) = 2$

Proof: Let u and v be the central vertices of $B_{2,2}$. Let u_1, u_2 and v_1, v_2 be the vertices adjacent with u and v respectively. Since {u, v} is the unique minimum dominating set of $B_{m,n}$and $\gamma(G) = 2$. Clearly u dominate exactly two vertices u_1 and $u_2$and v dominate exactly two verticesv_1 and v_2. Hence {u, v} is unique bi-dominating set and$\gamma_{bi}(G) = 2$. Therefore $\gamma(G) = \gamma_{bi}(G) = 2$.

Theorem2.21: For a wounded spider with at least one leg, bi-dominating set does not exist.

Proof: Consider the following wounded spider $K^*_{1,t}$, where k edges are subdivided and $k<t$.

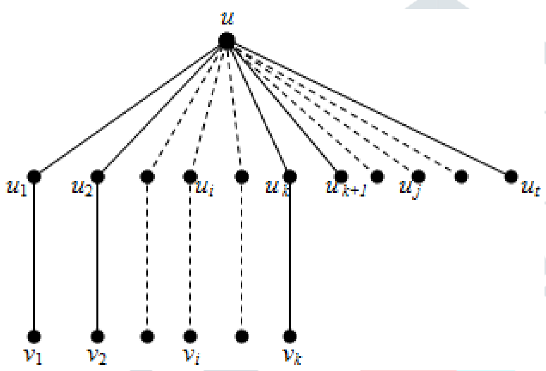

Figure 2.5

Let u be the central vertex. Take D = {u_i}, $1 \leq i \leq t$ is a unique minimum dominating set, since the vertices $u_{k+1}, \ldots, u_t$ has only one neighbourhood u. Hence the set D is dominating set but not a bi-dominating set. Therefore bi-dominating set does not exist for wounded spider.

Remark 2.22: Complement of S a bi-dominating set of a graph G need not be a bi-dominating set . Consider the following graph P_5

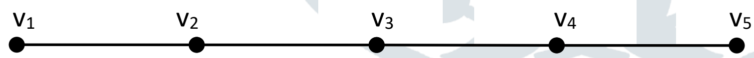

Figure 2.6

Here {v_2, v_4} is the unique bi-dominating set of P_5. V – S is a dominating set but not a bi-dominating set of P_5

Theorem 2.23: Let G be a connected graph with n edges. Then $\gamma_{bi}\big(S(G)\big) = n$.

Proof: Let G be a simple graph with n edges. Let H = S(G) and |E(H)| = 2n. let the edges of G subdivided by the new vertices u_1, $u_2, \ldots, u_n$. {$u_1, u_2, \ldots, u_n$ } is the unique bi-dominating set of H. Therefore $\gamma_{bi}(H) = n$.

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