CONTRACTION MAPPING IN B-METRIC SPACE

Savita Patel
M.Phil. Scholar
Department of Mathematics
Dr. C.V. Raman University, Kota, Bilaspur.

Abstract:

In this paper, we prove a fixed point theorem of contraction mapping in B-metric spaces.

Introduction:

Some problems, particurarly the problem of the convergence of measurable functions with respects to measure lead to a generalization of notion of metric. Using this idea we shall present generalization of a fixed point theorems of Banach type.

DEFINITION (1.1) Let (x,d) be a metric space then a mapping $T : X \to X$ is called a contraction mapping on X if there exists q$\in$ (0,1) such that

$d(Tx, Ty) \leq q\ d(x,y)$ for all x, y is in X.

if q = 1 then a mapping contractive mapping such that it is also knopwn as a nopn expansive mapping because non expansive mappings are more general than contractive mappings.

b-Metric Space:

The idea of b-metric was initiated from the works of Bourbaki [1974] and Bakhtin [1989]. Czerwik [1993] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [1998] discussed some kind of relaxation in triangular inequality and called this new distance measure as non-linear elastic mathing (NEM). All these applications intrigued and pushed us to introduce the concept of extended b-metric space. So that the results obtained for such rich spaces become more viable in different directions of applications.

Lex X be a space and let R_+ denotes the set of all nonnegative numbers. A function

d : X × X → R+ is said to be an b-metric iff for all x, y, z ∈ X and all r > 0 the following conditions are satisfacted:

d{x, y) = 0 iff x = y (1)

d{x, y) = d{y, x) (2)

d{x, y) < r and d{x, z) < r imply d{y, z) < 2r. (3)

A pair {X, d) is called an b-metric space.

Definition 1. Let X be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric (Bakhtin [1989], Czrerwik [1993]) if it satisfies the following properties for each x, y, z ∈ X. (b1): $d(x, y) = 0 \Leftrightarrow x = y$; (b2): $d(x, y) = d(y, x)$; (b3): $d(x, z) \leq s[d(x, y) + d(y, z)]$. The pair (X, d) is called a b-metric space.

2. Main Results:

The following results, which we will generalize and extend the results of B – Metric Space:-

Theorem Let (X, d) be a complete b-metric space with constant $s \geq 1$ and suppose that $T : X \to X$ satisfies

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \ldots\ldots(2.1)$$

for all x, y ∈ X, where $\varphi: [0, \infty) \to [0, \infty)$ is increasing an

for each $t \geq 0$. Then T has a unique fixed point x* ∈ X and limn→∞ T n(x) = x* for each x ∈ X.

In define upon theorem I recall the notion of b-metric spaces, the statement of Theorem 2.1 in [Kajanto, S. 2018] and I present extend in theorem.

Theorem : Let (x, d) be a complete b-matric space with the confficient s≥ 1. Sappose that the mapping $T : X \to X$ satisfies the Condition :

$$d\,(Tx, Ty) \leq \propto d\,(x,y) + \beta \frac{[1+d\,(x,Tx)]d\,(y,Ty)}{[1+d\,(x,y)]}$$

$$+ Y[d(x,Tx) + d\,(y,Ty)] + \delta\,[d(x,Ty) + d\,(y,Tx)] \quad \ldots\ldots\ldots\ldots(2.2)$$

for all $x, y \in X$, when α, β, r, δ are nonnegative reals with $\alpha + \beta + 2r + 2s\delta < 1.$ *then T* has a unique fixed point in X.

Proof. Choose $x_o \in X$. we construct the iteratine sequence $\{x_n\}$, where $x_n = Tx_{n-1}, n \geq 1$, that in $x_{n+1} = Tx_n = T^{n+1} x_o$ from (1.1) me have $d\,(x_n, x_{n+1}) = (Tx_{n-1}, Tx_n)$

$$\leq \alpha\, d(x_{n-1}, x_n) + \beta \frac{[1+d(x_{n-1}, Tx_{n-1})]\, d\,(x_n, Tx_n)}{[1+d\,(x_{n-1}, x_n)]}$$

$+ r\,[d(x_{n-1}, Tx_{n-1}) + d\,(x_n, Tx_n)]$

$+ \delta\,[d(x_{n-1}, Tx_n) + d\,(x_n, Tx_{n-1})]$

$= \alpha\, d\,(x_{n-1}, x_n) + \beta\, \frac{[1 + d\,(x_{n-1}, x_n)]d\,(x_n, x_{n+1})}{[1 + d\,(x_{n-1}, x_n)]}$

$+ r\,[d\,(x_{n-1}, x_n) + d\,(x_n, x_{n+1})]$

$+ \delta\,[d\,(x_{n-1}, x_{n+1}) + d\,(x_n, x_n)]$

$\leq \alpha\, d\,(x_{n-1}, x_n) + \beta\, d\,(x_n, x_{n-1})$

$+ s\,\delta\,[d\,(x_{n-1}, x_n) + d\,(x_n, x_{n+1})]$

$= (\alpha + \gamma + s\,\delta)d(x_{n-1}, x_n) + (\beta + \gamma + s\,\delta)d(\,x_n, x_{n+1})$

Which implies that

$d\,(\,x_n, x_{n+1}) \leq \left(\frac{\alpha + \gamma + s\,\delta}{1 - \beta - \gamma - s\,\delta}\right) d(x_{n-1}, x_n)$

$= \mu d(x_{n-1}, x_n)$………………….. (2.3)

Where

$\mu = \left(\frac{\alpha+\gamma+s\,\delta}{1-\beta-\gamma-s\,\delta}\right)$

Since $\alpha + \beta + 2\gamma + 2s\,\delta < 1$, it follows that $0 < \mu < 1$. By induction, we have

$d(x_{n+1}, x_n) \leq \mu d(x_n, x_{n-1}) \leq \mu^2 d(x_{n-1}, x_{n-2}) \leq \mu^n d(x_1, x_0)$ ………………….. (2.4)

Let, $An = \mu^n A_o$ ………………………… (2.5)

Hence, for any $m, n \geq 1$ and $m > n$, we have

$d(x_n, x_m) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)]$

$= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m)$

$\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)]$

$= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m)$

$\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots \ldots \ldots .. + s^{n+m-1} d(x_{n+m-1}, x_m)$

$\leq [s\mu^n + s^2\mu^{n+1} + s^3\mu^{n+2} + \cdots \ldots \ldots . + s^m \mu^{n+m-1}]A_o$

$= s\mu^n[1 + s\mu + s^2\mu^2 + s^2\mu^2 + \cdots \ldots \ldots . + (s\mu)^{m-1}]A_o$

$\leq \left[\frac{s\mu^n}{1 - s\mu}\right] A_o$

Since 0< $s\mu < 1$, therefore taking limit m, n → ∞, we have

$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$

Hence $\{x_n\}$ is a Cauchy sequence in complete b-matric space X, is complete. So there exists $q\epsilon X$ such that $\lim_{m,n\to\infty} x_n$=q. Now, we have to show that q is a fixed point of T. we have

$d(x_{n+1}, T_q) = d(T\, x_n, T_q)$

$\leq \alpha d(x_n, q) + \beta \frac{[1+d(x_n, Tx_n)]d(q,\ Tq)}{[1+d(x_n, q)]}$

$+\gamma[d(x_n, Tx_n) + d(q, Tq)]$

$+\delta[d(x_n, Tq) + d(q, Tx_n)]$

$= \alpha d(x_n, q) + \beta \frac{[1+d(x_n, x_{n+1})]d(q,\ Tq)}{[1+d(x_n, q)]}$

$+\gamma[d(x_n, x_{n+1}) + d(q, Tq)]$

$+\delta[d(x_n, Tq) + d(q, x_{n+1})]$

Taking limit $n \to \infty$, we have

$d(q, Tq) \leq (\beta + \gamma + \delta)d(q, Tq).$

The above inequality in true only if *d (q, Tq) =0* and so *Tq = q*. Thus q is a fixed point of T.

Now, we show the uniqueness of T. for this, let q, be another fixed point of T, that is $Tq = q_1$, such that $q \neq q_1$, from (1.1), we have

$d(q, q_1) = d(Tq, Tq_1).$

$\leq \alpha d(q, q_1) + \beta \frac{[1+d(q,\ Tq)]d(q_1,\ Tq_1)}{[1+d(q,q_1)]}$

$+\gamma[d(q, Tq) + d(q_1, Tq_1)]$

$+\delta[d(q, Tq_1) + d(q_1, Tq)]$

$= \ (\alpha, 2\delta)d(q, q_1).$

The above inequality is possible only if $d(q, q_1) = 0$ and so q = q_1. Thus q is a unique fixed point of T. This completes the proof.

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