A STUDY ON P-WEAKLY REGULAR NEAR-RING

[1]D. RADHA, [2]C. RAJA LAKSHMI

[1]Assistant Professor of Mathematics,
A.P.C. Mahalaxmi College for Women, Thoothukudi, Tamil Nadu, India
[2]M. Phil Scholar,
A.P.C. Mahalaxmi College for Women, Thoothukudi, Tamil Nadu, India.

ABSTRACT:

The notion of P-regular rings was introduced by V. A. Andrunakievich (1990) which is the generalization of regularity in rings. In 1991, S.J. Choi extended the P-regularity in rings to P-regularity in near-rings. In this paper, P-weak regularity in near-ring was defined. It is proved that a near-ring N is P-left weakly regular near-ring if and only if every two-sided ideal and every quotient near-ring is P-left weakly regular near-ring. Also, in a reduced P-left w-weakly regular near-ring, N and for any proper ideal P of N TFAE (i) P is prime (ii) P is completely prime (iii) P is maximal. It is proved that a reduced near-ring N is P-left w-weakly regular if and only if N/P is a simple domain for every prime ideal P of N.

KEYWORDS:

P-left weakly regular near-ring, P-left w-weakly regular near-ring, quotient near-ring, simple domain, reduced.

I. INTRODUCTION:

In a ring (N, +, .), if we ignore commutativity of + and one distributive law, then (N, +, .) is a near-ring. If we do not stipulate the left distributive law, (N, +, .) is a right near-ring. The notion of P-regular rings was introduced by V. A. Andrunakievich (1990) which is the generalization of regularity in rings. In 1991, S.J. Choi extended the P-regularity in rings to P-regularity in near-rings. P. Dheena and C. Rajeswari introduced the notion of w-weakly (weak weakly) regular near-rings and give several characterization of w-weakly regular near-ring. P.Dheena and C. Jenila introduced the notion of P-Strongly regular near-rings in 2012. In this paper, some results on P-Weakly regular near-ring and P- weak weakly regular near-ring are discussed.

II. PRELIMINARIES:

DEFINITION: II.1 [7]

A *right near-ring* is a non-empty set N together with the two binary operators '+' and '.' such that (i) (N, +) is a group (need not be abelian)

(ii) (N, .) is a semi-group

(iii) $(a+b).c = a.c + b.c$ hold for all a, b, c in N

Instead of (iii), if N satisfies the left distributive law, then (N, +, .) is called a left near-ring.

DEFINITION: II.2 [7]

A near-ring N is called *regular near-ring*, if for every element a in N, there exists an element x in N such that $a = axa$

DEFINITION: II.3 [7]

An element a in the near-ring N is called *nilpotent* if $a^k = 0$ for some positive integer k.

DEFINITION: II.4 [7]

A near-ring N is said to be *reduced* if it has no nonzero nilpotent element in N.

DEFINITION: II.5 [2]

Let P be any ideal of the near-ring, N. Then the near-ring N is said to be a *P-regular near-ring* if for each a in N, there exists an element x in N such that $a = axa + p$ for some p in P.

If $P = 0$, then a P-regular near-ring is a regular near-ring.

DEFINITION: II.6

A near-ring N is said to be *left weakly regular near-ring* if $x \in (Nx) * (Nx)$ for all x in N.

NOTATION: II.7

(1) For any subsets A, B of N, $A * B$ denotes the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A, b_k \in B$
(2) For any x in N, $< x >$ stands for the principal ideal of N generated by x

DEFINITION: II.8

A near-ring N is said to be *left w-weakly (weak-weakly) regular* if for any $x \in N, x = ux$ for some $u \in < x >$

III. A STUDY ON P-WEAKLY REGULAR NEAR-RINGS

Hereafter, N stands for right near-ring.

DEFINITON: III.1

A near-ring N is said to be *P-left weakly regular near-ring* if $x \in (Nx) * (Nx) + P$ for all x in N and for some ideal P of N

DEFINITION: III.2

A near-ring N is said to be *P-left w-weakly regular near-ring* if for any x in N, $x = ux + p$ for some $u \in < x >$ & $p \in P$, the ideal of N

In a near-ring, P-left weakly regular near-ring always implies P-left w-weakly regular near-ring.

LEMMA: III.3

If a near-ring N is P-left weakly regular near-ring then $P = P^2$ for every ideal P of N

Proof:

The proof clearly follows from the definition of P-left weakly regular near-ring.

LEMMA: III.4

Every two sided ideal and every quotient near-ring of a P-left weakly regular near-ring is P-left weakly regular. On the other hand, if a near-ring N has a two-sided ideal P such that P and N/P are both P-left weakly regular then N is P-left weakly regular near-ring.

Proof:

The proof follows from lemma: III.3. On the other hand, Suppose that N has a two-sided ideal P which is P-left weakly regular and that the quotient near-ring N/P is also P-left weakly regular near-ring. Let $x \in N$. Since N/P is P-left weakly regular near-ring, $x + P = (n_1 x n_2 x + n_3 x n_4 x + \cdots \ldots n_k x n_{k+1} x) + P$. Then $x - x' \in P$

where $x' = n_1xn_2x + n_3xn_4x + \cdots \ldots n_kxn_{k+1}x$. Since P is P-left weakly regular, $x - x' \in [P(x-x') * P(x-x')] + P$. Now claim that $[P(x-x') * P(x-x')] + P \subset (Nx * Nx) + P$. For, let $y \in [P(x-x') * P(x-x')] + P$. Then $y = [z_1(x-x')z_2(x-x') + \cdots \ldots z_k(x-x')z_{k+1}(x-x')] + P$. Now, $z_j(x-x')z_{j+1}(x-x') + P = z_j(x - n_1xn_2x + n_3xn_4x + \cdots \ldots n_kxn_{k+1}x)z_{j+1}(x - n_1xn_2x + n_3xn_4x + \cdots \ldots n_kxn_{k+1}x) + P = z_j[1 - (n_1xn_2 + n_3xn_4 + \cdots \ldots n_kxn_{k+1})]xz_{j+1}[1 - (n_1xn_2 + n_3xn_4 + \cdots \ldots n_kxn_{k+1})]x + P \in [Nx * Nx] + P$. Thus $y \in (Nx * Nx) + P$. Hence, $x - x' \in (Nx * Nx) + P$. Clearly, $x' \in (Nx * Nx)$. Therefore, $x \in (Nx * Nx) + P$. Thus the required is proved.

LEMMA: III.5

Every ideal of a reduced P-left w-weakly regular near-ring N is completely semi-prime.

Proof:

Suppose P is an ideal of N and $a^2 \in P$. By hypothesis we have, $a^2 = ua^2 + p$ for some u in $< a^2 >$ & p in P. This gives, $(a - ua)a \in P$. Also, since P is an ideal, $u(a - ua) \in P$. Now, $(a - ua)^2 = a(a - ua) - ua(a - ua) \in P$. Hence $(a - ua)^2 + P \subseteq P$. Since N is reduced, $a - ua \in P$. Therefore, $a = ua + p \in P$.

COROLLARY: III.6

Every ideal of a reduced P-left weakly regular near-ring is completely semiprime.

Proof:

Since P-left weakly regular near-ring always implies P-left w-weakly regular near-ring, the proof follows from lemma: III.5

THEOREM: III.7

Let N be a reduced near-ring. N is P-left weakly regular if and only if

(i) Every ideal is completely semiprime
(ii) N/P is P-left weakly regular for all prime ideals P of N

Proof:

Let us assume that N is P-left weakly regular near-ring.

(i) follows from corollary: III.6 and (ii) follows from lemma: III.4.

Conversely, assume that the conditions (i) and (ii) holds. Now, to prove that N is P-left weakly regular. If not, then there is an element x in N such that $x \notin (Nx * Nx) + P$. Let S={completely semiprime ideals I of N / $x \notin (Nx * Nx) + P$ } Clearly, S$\neq \emptyset$. By Zorn's lemma, S has a maximal element say P such that $x \notin (Nx * Nx) + P$. By (ii), P is not a prime ideal. So, there exists ideals A, B of N such that $P \subset A, P \subset B$ but $AB \subseteq P$. Let K={$n \in N/nB \subseteq P$} and L={$n \in N/nK \subseteq P$}. Clearly, K and L are ideals. And also, $A \subseteq K$ &$BK \subseteq P$. This gives, $B \subseteq L$. So we get, $P \subset K$ & $P \subset L$ but $K \cap L = P$. By maximality of P, $x \in (Nx * Nx) + K$ and $x \in (Nx * Nx) + L$ so $x - e_1x \in K$ & $x - e_2x \in L$ for some $e_1, e_2 \in (Nx * N)$. Let $e = e_1(1 - e_2) + e_2$. Then $x - ex = x - (e_1(1 - e_2) + e_2)x = x - e_2x - e_1(x - e_2x) \in L$. Since K is completely semiprime, $x(1 - e_1) \in K$. Consider, $[(1 - e_1)(x - e_2x)]^2 = (1 - e_1)[x(1 - e_1) - e_2x(1 - e_1)](x - e_2x) \in K$. Since K is completely semiprime, $(1 - e_1)(x - e_2x) \in K$, which gives, $x - ex \in K$. Thus, $x - ex \in K \cap L = P$ which is a contradiction to $x \notin (Nx * Nx) + P$. Hence the required is proved.

THEOREM: III.8

If N is a reduced P-left w-weakly regular near-ring then for any proper ideal P of N, TFAE:

(i) *P is prime*
(ii) *P is completely prime*
(iii) *P is maximal*

Proof:

(i)⟹(ii): Suppose $ab \in P$. Then clearly, $< a >< b >\subseteq P$. Since P is prime, $< a >\subseteq P$ or $< b >\subseteq P$. This gives, $a \in P$ or $b \in P$. (ii) ⟹ (i) follows clearly. Now to prove, (i) ⟹(iii). Let P be a proper prime ideal. Suppose P is properly contained in an ideal M. Let $x \in M/P$. Since N is P-left w-weakly regular, and for any n in N, $nx = nyx + np$. This gives, $< n - ny >< x >\subseteq P$. Since P is completely prime, $n - ny \in P \subset M$. This implies, $ny \in M$ which in turn, $n \in M$. Hence N = M. (iii) ⟹(i) is obvious.

THEOREM: III.9

A reduced near-ring N is P-left w-weakly regular iff N/P is a simple domain for every prime ideal P of N.

Proof:

Let N be a P-left w-weakly regular near-ring. Then by theorem:III.8, N/P is a simple domain for every prime ideal P of N. Conversely, let N/P be a simple domain for every prime ideal P of N. Let $0 \neq a \in N$ and $\mathcal{N} = N/A(a)$ be reduced and $\bar{a} \in N/A(a)$ be not a zero-divisor. Let M be the multiplicative semigroup generated by all the elements of the form $\bar{a} - \bar{x}\bar{a}$ where $x \in < a >$. Now, claim that $\bar{0} \in M$. If not, then there exists a completely semiprime ideal $\mathcal{P}$ with $\mathcal{P} \cap M = \emptyset$. Suppose $< \bar{a} >\subseteq \mathcal{P}$. Then for any $x \in < a >, \bar{a} - \bar{x}\bar{a} \in \mathcal{P}$ which is a contradiction to $\mathcal{P} \cap M = \emptyset$. Suppose $< \bar{a} >\not\subset \mathcal{P}$. Since $\mathcal{P}$ is maximal, $\mathcal{P} + < \bar{a} > = \mathcal{N}$. This gives, $1 - \bar{x} = \bar{a} \in \mathcal{P}$. This implies, $\bar{a} - \bar{x}\bar{a} \in \mathcal{P}$ which is a contradiction to $\mathcal{P} \cap M = \emptyset$. Hence $\bar{0} \in M$. Now, $\bar{0} = (\bar{a} - \overline{x_1}\bar{a})(\bar{a} - \overline{x_2}\bar{a}) \dots \dots (\bar{a} - \overline{x_n}\bar{a})$ where $x_i \in < a >$. Since N is reduced and $\bar{a}$ is not a zero-divisor, $(\bar{1} - \overline{x_1})(\bar{1} - \overline{x_2}) \dots \dots (\bar{1} - \overline{x_n}) = \bar{0}$. Now, claim that, $1 + P = x + P$ for some $x \in < a >$. Let $n = 2$. Then $(\bar{1} - \overline{x_1})(\bar{1} - \overline{x_2}) = \bar{0}$. This gives, $\bar{1} = [\overline{x_1}(\bar{1} - \overline{x_2}) + \overline{x_2}] + P$, which in turn gives, $1 + P = [x_1(1 - x_2) + x_2] + P$. Since $x_1, x_2 \in < a >, x_1(1 - x_2) + x_2 \in < a >$. Let $x_1(1 - x_2) + x_2 = x$. Then $1 + P = x + P$. Hence, $1 - x \in P$. This yields, $a = xa + p$ for some $x \in < a >$ & $p \in P$. Thus, N is a P-left w-weakly regular near-ring.

IV CONCLUSION:

In this paper, it is proved that every ideal in a reduced P-left w-weakly regular near is completely semi-prime. Also, in a reduced near-ring N, it is proved that N is P-left weakly regular if and only if every ideal is completely semiprime and N/P is P-left weakly regular for all prime ideals P of N. It is proved that a reduced near-ring N is P-left w-weakly regular iff N/P is a simple domain for every prime ideal P of N.

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