Simplex algorithm

Vinit Kumar

Mathematics

ABSTRACT :-

The linear programming problems discussed so far are concerned with two variables ,the solution of which can be found out easily by the graphical method. But most real life problems when formulated as an LP model involve more than two variables and manay constraints. Thus, there is a need for a method other than the graphical method. The most popular non-graphical method of solving an LPP is called the simplex method. This method developed by George B. Dantzig in 1947, is applicable to any problem that can be formulated in terms of linear objective function subject to a set of linear constraints. There are no theoretical restrictions placed on the number of decision variables or constraints. The development of computers has further made it easy for the simplex method to solve large scale LP problem very quickly.

The concept of simplex method is similar to the graphical method. For LP problems with several variables, the optimal solution lies at a corner point of the many-faced, multi-dimensional figure, called an n-dimensional polyhedron. The simplex method examines the corner points in a systematic manner. It is a computational routine of repeating the same set of steps over and over until an optimal solution is reached. For this reason, it is known as an iterative method. As we move from one iteration to the other, the method improves the value o the objective function and achieves optimal solution in a finite number of iterations.

INTRODUCTION :-

SOME USEFUL DEFINITIONS

(i) Slack Variable - A variable added to the left hand side of a less than or equal to constraint to convert the constraint into an equality is called a slack variable.

For example, to convert the constraint $3x + 2y \leq 18$ into an equation, we add a slack variable s to the left-hand side thereby getting the equality $3x + 2y + s = 18$……..(1)

Clearly, s must be non-negative, since $s = 18-(3x + 2y) \geq 0$ [by (1)]

In economic terminology, a slack variable represents unused resource in the form of money, labour hours, time on a machine etc.

(ii) Surplus Variable. A variable subtracted from the left-hand side of a greater thanor equal to constraint to convert the constraint into an equality is called a surplus variable.

For example, to convert the constraint $2x + 3y \geq 30$ into an equation, we subtract a surplus variable s from the left-hand side thereby getting the equality $2x+ 3y-s = 30$

Clearly, s must be non-negative, since $s =2x+ 3y- 30 \geq 0$

A surplus variable represents the surplus of left-hand side over the right-hand side. It is also called a negative slack variable.

(iii) Basic Solution. For a system of m simultaneous linear equations in n variables (n> m), a solution obtained by setting (n m) variables equal to zero and solving for the remaining m variables for a unique solution is called a basic solution. The (n - m) variables set equal to zero in any solution are called non-basic variables. The other m variables are called basic variables.

(iv) Basic Feasible Solution. A basic solution which happens to be feasible (i.e., a solution in which each basic variable is non-negative) is called a basic feasible solution.

(v) Degenerate and Non-degenerate Solution. If one or more of the basic variables

in the basic feasible solution are zero, then it is called a degenerate solution. If all the

variables in the basic feasible solution are positive, then it is called a non-degenerate solution.

STANDARD FORM OF LPP

The standard form of an LPP should have the following characteristics

(i) Objective function should be of maximization type.

(ii) All constraints should be expressed as equations by adding slack or surplus variables, one for each constraint.

(iii) The right-hand side of each constraint should be non-negative. If it is negative, then to make it positive, we multiply both sides of the constraint by (-1), changing ≤ to ≥ and vice versa.

(iv) All variables are non-negative

Thus, the standard form of an LPP with n variables and m constraints is:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots\cdots + c_nx_n + 0s_1 + 0s_2 + \ldots\ldots + 0s_m$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n + s_2 &= b_2 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

and $$x_1, x_2 \ldots\ldots, x_n, s_1, s_2, \ldots\ldots, s_m \geq 0$$

WORKING PROCEDURE OF SIMPLEX METHOD

Assuming the existence of an initial basic feasible solution, the optimal solution to a LPP by simplex method is obtained as follows:

Step 1: To express the LPP in the Standard Form

(i) Formulate the mathematical model of the given LPP.

(ii) If the objective function is to be minimized, then convert it into a maximization problem by using Min. Z =-Max (-Z)

(iii) The right-hand side of each constraint should be non-negative.

(iv) Express all constraints as equations by introducing slack/surplus variables, in foreach constraint.

(v) Restate the given LPP in standard form:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0s_1 + 0s_2 + \cdots + 0s_m$$

$$\text{subject to the constraints} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m = b_m$$

$$\text{and} \quad x_1, x_2, \ldots\ldots, x_n, s_1, s_2, \ldots\ldots, s_m \geq 0$$

Step 2: To set up the Initial Basic Feasible Solution

Take the m slack/surplus variables s as the basic variables so that the n given variables $x_1,x_2,\ldots\ldots x_n$ are non-basic variables. As such $x_1=x_2=\ldots\ldots=x_n=0$ and $s_1= b_1$, $s_2= b_2, \ldots\ldots\ s_m= b_m$

Since each b_i is non-negative (see Step I (iii), the basic solution is feasible. This basic feasible solution is the starting point of the iterative process. The simplex method then proceeds to solve the LPP by designing and redesigning successively better basic feasible solutions until an optimal solution is obtained

Step 3: To set up the Initial Simplex Table

The above information is conveniently expressed in the tabular form as shown below

		$c_j \rightarrow$	c_1	c_2	$\ldots\ldots c_n$	0	0	 0	Ratio
c_B	Basic Variables B	Solution $b(=x_B)$	x_1	x_2	Variables $\ldots\ldots x_n$	s_1	s_2	$\ldots\ldots s_m$	x_B/x_b
$(c_{B1}=)\ 0$	s_1	$(x_{B1}=)b_1$	a_{11}	a_{12}	$\ldots\ldots a_{1n}$	1	0	$\ldots\ldots 0$	
$(c_{B2}=)\ 0$	s_2	$(x_{B2}=)b_2$	a_{21}	a_{22}	$\ldots\ldots a_{2n}$	0	1	$\ldots\ldots 0$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$(c_{Bm}=)\ 0$	s_m	$(x_{Bm}=)b_m$	a_{m1}	a_{m2}	$\ldots\ldots a_{mn}$	0	0	$\ldots\ldots 1$	
$Z = \Sigma c_{Bi}\, x_{Bi}$		$Z_j = \Sigma c_{Bi} a_{ij}$	0	0	$\ldots\ldots 0$	0	0	$\ldots\ldots 0$	
		$C_j = c_j - Z_j$	$c_1 - Z_1$	$c_1 - Z_2$	$\ldots\ldots c_n - Z_n$	0	0	$\ldots\ldots 0$	

Step 4: Test for Optimality. Examine the entries in C_j -row. If all entries in this row are negative or zero, i.e., if C_i ‘s ≤ 0, then the basic feasible solution is optimal. Any positive entry in the row indicates that an improvement in the value of objective function Z is possible and, hence, we proceed to the next step.

Step 5: To Identify the Incoming and Outgoing Variables. If there is a positive entry in the C-row, then simplex method shifts from the current basic feasible solution to a better basic feasible solution. For this, we have to replace one current basic variable (called the outgoing or departing variable) by a new non-basic variable (called the incoming or entering variable)

Determination of Incoming Variable. The column with the largest positive entry in

the C_j-row is called the key (or pivot) column (which is shown marked with an arrow T). Thenon-basic variable which will replace a basic variable is the one lying in the key column. Thusthe incoming variable is located If more than one variable has the same positive largest entry in the C_j-row, then any ofthese variables may be selected arbitrarily as the incoming variable.

Determination of Outgoing Variable. Divide each entry of the solution column

(i.e., X_B-column) by the corresponding positive entry in the key column. These quotients are written in the last column labelled 'Ratio'. The row which corresponds to the smallest non negative quotient is called the key (or pivot) row (which is shown marked with an arrow ↑)The departing variable is the corresponding basic variable in this row. The element at the intersection of the key row and key column is called the key (or pivot) element. We place a circle around this element. If all these ratios are negative or zero, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an unbounded solution and no further iteration is required.

Step 6: To set up the new simplex table from the current one.

Step 7: Test for Optimality. If there is no positive entry in C-row, we have an optimal solution. Otherwise, go to Step 4 and repeat the procedure until all entries in C-row are either negative or zero.

References

Murty, Katta G. (1983). Linear programming. New York: John Wiley & Sons, Inc. pp. xix+482. ISBN 978-0-471-09725-9. MR 0720547.