CONSTRUCTIONS OF NEW SEMITOPOLOGICAL LATTICE ORDERED GROUPS FROM OLD

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Abstract : K. Parhi and P. Kumari constructed lattice ordered group [3]. Many interesting results concerning the separation axioms, connectedness, etc. requires continuity of group operations in both variables together as well as the continuity of inversion. Therefore, such results are postponed until the further study. However, a few results will show that subgroups, quotient groups and product group of semitopological lattice ordered groups as well.

Key words : Semitopological lattice ordered group, Homeomorphisms, Tychonoff theorem. [1, 224]

Introduction

Let G be a semitopological lattice ordered group and H a lattice ordered subgroup of G. Then H, endowed with the topology induced from G, is called a semitopological lattice ordered subgroup. The fact that the group multiplication is separately continuous in H is easy to see. Moreover, if G is a semitopological lattice ordered group and if H is an lattice ordered invariant subgroup of G, then G/H, the collection of all distinct cosets $\{xH\}$, $x \in G$, forms a group called the lattice ordered quotient of G. Let ϕ denote canonical mapping. A usual, we define a quotient topology on G/H as follows: A set W in G/H is open if, and only if, $x^{-1}(W)$ is an open subset of G. It is easy to see that with this topology G/H is a semitopological lattice ordered group.

Furthermore, if $G_\alpha(\alpha \in A)$ is a family of semitopological lattice ordered groups, then the Cartesian product $G = \prod_{\alpha \in A} G_\alpha$ is the set of all functions $x : A \to G$ such that $x(\alpha) = x_\alpha$ for each $\alpha \in A$. G is also a lattice ordered group with co-ordinate wise multiplication or addition as the case may be.

We can define lattice ordered product topology on $\prod_{\alpha \in A} G_\alpha$ is the one having as its subbase the family $\left\{p_\alpha^{-1}(U_\alpha)\right\}$, where p_α is the α^{th} projection mapping : $G \to G_\alpha$ and U_α runs over open sets of G_α for each α.

Proposition 1. If, for each $\alpha \in A$, G_α is a semitopological lattice ordered group, so is the direct product $G = \prod_{\alpha \in A} G_\alpha$ endowed with the product topology.

Proof. We have to show that the mapping : $(x, y) \to xy$ of $G \times G$ onto G is continuous in each variable separately. Let W be a neighbourhood of $xy \in G$ [2]. Then there exists a member U of the base of the product topology such that $xy \in U \subset W$. But $U = \prod_{\alpha \in A} U_\alpha$, where $U_\alpha = G_\alpha$ for all $x \in A$ except for a finite subset B of A and for $\beta \in B$, U_β is an open set containing $x_\beta y_\beta = p_\beta(xy)$, x_α, y_α G_α for all α.

Now, since each G_α is a semitopological lattice ordered group, for each $\beta \in B$, there exits a neighbourhood V_β of y_β such that $x_\beta V_\beta = U_\beta$.

But then $V = \prod_{\alpha \in A} V_\alpha$, where $V_\alpha = G_\alpha$ for each $\alpha \in A \sim B$ and $V_\alpha = V_\beta$, $\alpha \in B$ is a neighbourhood of y in G.

Moreover, $p_\alpha(xy) = x_\alpha\, y_\beta \in x_\alpha V_\alpha = x_\alpha p_\alpha(V) \subset U_\alpha$ for each $\alpha \in A$ shows that $xV \subset U \subset W$.

This proves the continuity of $(x, y) \to xy$ then x is kept fixed.

Similarly, we prove the continuity in y.

Proposition 2. For each x, p_α is continuous and open homomorphism of $G = \prod_{\alpha \in A} G_\alpha$ onto the semitopological lattice ordered group G_α.

Proof. For $x, y \in G$, $p_\alpha(xy) = x_\alpha\, y_\alpha \in p_\alpha(x)\, p_\alpha(y)$. Therefore p_α is a homomorphism. Since for each $x_\alpha = G_\alpha$, $x = (x_\beta)$, where $x_\beta = p_\alpha$, $\beta \neq \alpha$, $x_\beta = x_\alpha$, $\beta = \alpha$ is an element of G and $p_\alpha(x) = x_\alpha$, it shows that p_α is onto. Therefore each projection mapping p_α is continuous and open.

Theorem 1. If $G_\alpha(x \in A)$ is a compact semitopological lattice ordered group for each α, so is $G = \prod_{\alpha \in A} G_\alpha$.

Proof. By proposition 1, G is a semitopological lattice ordered group. It is compact owing to Tychonoff's theorem [1, p. 224]

Reference

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