FRACTIONAL CALCULUS OF K4 FUNCTION USING MARICHEV–SAIGO- MAEDA OPERATORS

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Abstract – The object of this paper is to derive fractional integral and derivative formulas associated with K_4 function by using Marichev – Saigo- Maeda (M.S.M.) fractional calculus operators and we will also launch Caputo type M.S.M. fractional Differential operators on K_4 function. The established results are in terms of generalized wright function. We also consider some corollary and relevant special cases of our main results.

Keywords – Special function, K_4 Function , Marichev – Saigo- Maeda operator , Caputo fractional Differential operator , Generalized Wright function , Erde'lyi – kober fractional differential and integral operator.

1. Introduction –

During the last century K_4 function play a vital role in solving the problem of many fractional order equation. Due to this important function , many researcher {see [2] , [9] , [23] }have studied in depth the Application , Properties and different generalization of K_4 function on fractional differential and integral operators.

Now we start this paper with the K_4 function [24] defined in the form of following power series

$$K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}(a_1 \dots\dots a_p\ ;\ b_1 \dots . b_q\ ; x) = K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}(x)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_{n \dots} (a_p)_n}{(b_1)_{n \dots} (b_q)_n} \frac{u^n\,(\gamma)_n\,(x-v)^{(n+\gamma)\alpha-\beta-1}}{n!\ \Gamma((n+\gamma)\alpha-\beta)} \qquad (1.1)$$

Where $\alpha, \beta, \gamma, x \in C$, $R(\alpha\gamma-\beta) > 0$; $(a_i)_n\ (i = 1,2 \dots p)$ *and* $(b_j)_n\ (j = 1,2 \dots q)$ are the Pochhammer symbols which as

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \qquad (1.2)$$

Relation between K4 function and other special function –

(i) Put $\beta = \alpha - \beta$, $\gamma = 1$, $u = 1$ *and* $v = 0$ in equation (1.1) then K_4 function reduce to Generalized M – Series { [8], [25] , [26]} defined as

$$K_4^{(\alpha,\alpha-\beta,1),(1,0),(p,q)}(x) = x^{\beta-1}\ {}_pM_q^{\alpha,\beta}(x)$$

$$= x^{\beta-1} \sum_{n=0}^{\infty} \frac{(a_1)_{n \dots} (a_p)_n}{(b_1)_{n \dots} (b_q)_n} \frac{x^{n\alpha}}{\Gamma(\alpha n+\beta)} \qquad (1.3)$$

(ii) If there is no upper and lower parameters in equation (1.1) , then K_4 function convert G- function introduced by Lorenzo and Hartley { [10] , [22] }and defined as

$$K_4^{(\alpha,\beta,\gamma),(u,v),(0,0)}(x) = G_{\alpha,\beta,\gamma}(u,v,x) = \sum_{n=0}^{\infty} \frac{u^n\,(\gamma)_n\,(x-v)^{(n+\gamma)\alpha-\beta-1}}{\Gamma((n+\gamma)\alpha-\beta)} \qquad (1.4)$$

(iii) If we set $\gamma = 1$ in equation (1.4) then K_4 function convert R- function [11] defined as

$$K_4^{(\alpha,\beta,1),(u,v),(0,0)}(x) = R_{\alpha,\beta}(u,v,x) = \sum_{n=0}^{\infty} \frac{u^n\,(x-v)^{(n+1)\alpha-\beta-1}}{\Gamma((n+1)\alpha-\beta)}. \qquad (1.5)$$

(iv) If we put $v = \beta = 0$ in equation (1.5) then K_4 function reduce to F function [3] defined as $K_4^{(\alpha,0,1),(u,0),(0,0)}(x) =$

$$F_\alpha(u,0,x) = \sum_{n=0}^{\infty} \frac{u^n\,(x)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha}. \qquad (1.6)$$

(v) If we put $\alpha = u = 1$ in equation (1.6) then K_4 function convert Mittag – Leffler function $E_1(x)$ or Exponential function e^x [19] .defined as

$$K_4^{(1,0,1),(1,0),(0,0)}(x) = E_1(x) = e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{\Gamma((n+1)}. \qquad (1.7)$$

(vi) K_4 function reduce to Miller and Ross function [16] if we set $\alpha = 1$, $\beta = -\beta, \gamma = 1$ *and* $v = 0$ in equation (1.4) i.e.

$$K_4^{(1,-\beta,1),(u,0),(0,0)}(x) = E_x(\beta,u) = \sum_{n=0}^{\infty} \frac{u^n\ (x)^{n+\beta}}{\Gamma((n+1)+\beta)} \quad (1.8)$$

K_4 function reduce to some more special function like Agarwal function [1] , Mittag – Leffler function { [14] , [15] } , Generalized Mittag – Leffler function {[15],[18]}, H- function [13] and wright function [27] etc.
Saigo and Maeda [21] introduced the generalized fractional integral operator with Appell function $F_3(.)$ as the kernel , using the idea of Marichev [12] and saigo [20], this operator is known as Marichev-Saigo-Maeda operator (MSM) which is defined as

$$\left(I_{0+}^{\mu,\mu',\eta,\eta',\xi} f\right)(x) = \frac{x^{-\mu}}{\Gamma(\xi)} \int_0^x (x-t)^{\xi-1}\, t^{-\mu'}\, F_3\left(\mu,\mu',\eta,\eta':\xi; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \quad (1.9)$$

And $\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi} f\right)(x) = \frac{x^{-\mu'}}{\Gamma(\xi)} \int_x^\infty (x-t)^{\xi-1}\, t^{-\mu}\, F_3\left(\mu,\mu',\eta,\eta':\xi; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt \quad (1.10)$

Where $\mu,\mu',\eta,\eta',\xi \in C$ and $R(\xi) > 0$, $x > 0$.

The corresponding generalized fractional differential operator are defined as follows $\left(D_{0+}^{\mu,\mu',\eta,\eta',\xi} f\right)(x) =$

$$\left(I_{0+}^{-\mu',-\mu,-\eta',-\eta,-\xi} f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{x^{-\mu'}}{\Gamma(n-\xi)} \int_0^x (x-t)^{n-\xi-1}\, t^{-\mu}\, F_3\left(-\mu',-\mu,n-\eta,-\eta,n-\xi\,;1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt$$

(1.11)

And

$$\left(D_{0-}^{\mu,\mu',\eta,\eta',\xi} f\right)(x) = \left(I_{0-}^{-\mu',-\mu,-\eta',-\eta,-\xi} f\right)(x) = \left(-\frac{d}{dx}\right)^n \frac{x^{-\mu'}}{\Gamma(n-\xi)} \times$$

$$\int_x^\infty (x-t)^{n-\xi-1} t^{-\mu'}\, F_3\left(-\mu',-\mu,n-\eta,-\eta,n-\xi\,;1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt \quad (1.12)$$

Where $n = [R(\xi)] + 1$.

The corresponding Caputo- type generalized M.S.M. fractional differential operators [5] are defined as $({}^cD_{0+}^{\mu,\mu',\eta,\eta',\xi} f)(x) =$

$$\left(I_{0+}^{-\mu',-\mu,-\eta'+m,-\eta,-\xi+m} f^m\right)(x) \quad (1.13)$$

And $({}^cD_{0-}^{\mu,\mu',\eta,\eta',\xi} f)(x) = (-1)^m \left(I_{0-}^{-\mu',-\mu,-\eta',-\eta+m,-\xi+m} f^m\right)(x) \quad (1.14)$

Next we required the definition of Fox-wright hypergeometric function [27] given by

$$ {}_p\psi_q = {}_p\psi_q \left\{ \begin{matrix} (a_1,A_1),\ldots\ldots(a_p,A_p); \\ (b_1,B_1),\ldots\ldots(b_q,B_q); \end{matrix} x \right\} = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i+nA_i)}{\prod_{j=1}^q \Gamma(b_j+nB_j)} \frac{x^n}{n!} \quad (1.15)$$

where $a_i, b_j \in C$, $A_i, B_j \in R-\{0\}$ and $\sum_{i=1}^p A_i - \sum_{j=1}^q B_j \le 1$

2. The MSM Fractional integration of K_4 function

In this section, we have established the MSM fractional integration formula of K_4 function in terms of generalized Wright function so for this purpose , we required following Lemma [21]-

Lemma 1 Let $\mu,\mu',\eta,\eta',\xi \in C$ and $R(\xi) > 0$, $R(\rho) > \max\{0, R(\mu+\mu'+\eta-\xi), R(\mu'-\eta')\}$ then

$$\left(I_{0+}^{\mu,\mu',\eta,\eta',\xi} t^{\rho-1}\right)(x) = x^{\rho-\mu-\mu'+\xi-1} \frac{\Gamma(\rho)\Gamma(\rho+\xi-\mu-\mu'-\eta)\Gamma(\rho+\eta'-\mu')}{\Gamma(\rho+\xi-\mu-\mu')\,\Gamma(\rho+\xi-\mu'-\eta)\Gamma(\rho+\eta')} \quad (2.1)$$

Lemma 2 Let $\mu,\mu',\eta,\eta',\xi \in C$ and $R(\xi) > 0$, $R(\rho) < 1 + \min\{R(-\eta), R(\mu+\mu'-\xi), R(\mu+\eta'-\xi)\}$ then

$$\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi} t^{\rho-1}\right)(x) = x^{\rho-\mu-\mu'+\xi-1} \frac{\Gamma(1-\rho-\eta)\Gamma(1-\rho-\xi+\mu+\mu')\Gamma(1-\rho+\eta'+\mu-\xi)}{\Gamma(1-\rho+\eta'-\xi+\mu+\mu')\,\Gamma(1-\rho+\mu-\eta)\Gamma(1-\rho)} \quad (2.2)$$

Theorem 1. Let $\mu,\mu',\eta,\eta',\xi,\rho,\alpha,\beta,\gamma \in C$ be such that $R(\xi) > 0$, $R(\rho+\gamma\alpha-\beta) > \max.\{0,(\mu'-\eta'), R(\mu+\mu'+\eta-\xi)\}$ and $R(\alpha\gamma-\beta) > 0$ then for $x > 0$, we obtain following result –

$$\left(I_{0+}^{\mu,\mu',\eta,\eta',\xi} (t-v)^{\rho} \, K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)} \right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\;(x-v)^{\gamma\alpha+\rho+\xi-\beta-\mu-\mu'-1}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho+\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha), \\ (\rho+\gamma\alpha-\beta+\eta'-\mu',\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta+\eta',\alpha), (\rho+\gamma\alpha+\xi-\beta-\mu-\mu',\alpha), \\ (\rho+\gamma\alpha+\xi-\beta-\mu'-\eta,\alpha) \end{matrix}\right] \quad (2.3)$$

Proof : Firstly we denote L.H.S. of above equation is I_1 and apply equation (1.1) we get,

$$I_1 = \left(I_{0+}^{\mu,\mu',\eta,\eta',\xi} (t-v)^{\rho} \, K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)} \right)(x)$$

$$= I_{0+}^{\mu,\mu',\eta,\eta',\xi} \left\{ (t-v)^{\rho} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{u^n (\gamma)_n \; (t-v)^{(n+\gamma)\alpha-\beta-1}}{n! \; \Gamma((n+\gamma)\alpha-\beta)} \right\} (x)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{u^n (\gamma)_n}{n! \; \Gamma((n+\gamma)\alpha-\beta)} \; I_{0+}^{\mu,\mu',\eta,\eta',\xi} \{ (t-v)^{\rho+(n+\gamma)\alpha-\beta-1} \} (x)$$

Apply equation (2.1) with $\rho = \rho + (n+\gamma)\alpha - \beta$, we get

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{u^n (\gamma)_n}{n! \; \Gamma((n+\gamma)\alpha-\beta)} (x-v)^{\rho+\gamma\alpha+n\alpha-\beta-\mu-\mu'+\xi-1} \times \frac{\Gamma(\rho+n\alpha+\gamma\alpha-\beta)\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu-\mu'-\eta)\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\eta'-\mu')}{\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\eta')\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu-\mu')\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu'-\eta)}$$

If we use equation (1.2) $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$ then

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\dots\Gamma(a_p+n)}{\Gamma(b_1+n)\dots\Gamma(b_q+n)} \frac{\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\dots\Gamma(a_p)} \frac{\Gamma(\gamma+n)\,u^n}{n! \; \Gamma(\gamma)\,\Gamma((n+\gamma)\alpha-\beta)} (x-v)^{\rho+\gamma\alpha+n\alpha-\beta-\mu-\mu'+\xi-1}$$

$$\times \frac{\Gamma(\rho+n\alpha+\gamma\alpha-\beta)\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu-\mu'-\eta)\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\eta'-\mu')}{\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\eta')\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu-\mu')\,\Gamma(\rho+n\alpha+\gamma\alpha-\beta+\xi-\mu'-\eta)}$$

$$= (x-v)^{\rho+\gamma\alpha-\beta-\mu-\mu'+\xi-1} \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} \frac{1}{\Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho+\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha), \\ (\rho+\gamma\alpha-\beta+\eta'-\mu',\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta+\eta',\alpha), (\rho+\gamma\alpha+\xi-\beta-\mu-\mu',\alpha), \\ (\rho+\gamma\alpha+\xi-\beta-\mu'-\eta,\alpha) \end{matrix}\right]$$

which is required proof of theorem (1).

Theorem 2. Let $\mu, \mu', \eta, \eta', \xi, \rho, \alpha, \beta, \gamma \in C$ be such that $R(\xi) > 0$, $R(1+\gamma\alpha-\rho-\beta) < 1+ \min\{ R(-\eta), R(\mu+\mu'-\xi), R(\mu+\eta'-\xi)\}$ and $R(\alpha\gamma-\beta) > 0$ then for $x > 0$, we obtain following result –

$$\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi} (t-v)^{-\rho+1} \, K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)} \right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\;(x-v)^{\gamma\alpha+\xi-\rho-\beta-\mu-\mu'}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta-\eta,-\alpha), (\rho-\gamma\alpha-\xi+\beta+\mu+\eta',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta,-\alpha), (\rho-\gamma\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta+\mu,-\alpha) \end{matrix}\right] \quad (2.4)$$

Proof : We denote L.H.S. of above equation is I_2 and apply equation (1.1) we get,

$$I_2 = \left(I_{0-}^{\mu,\mu',\eta,\eta',\xi} (t-v)^{-\rho+1} K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)} \right)(x)$$

$$= I_{0-}^{\mu,\mu',\eta,\eta',\xi} \left\{ (t-v)^{-\rho+1} \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{u^n (\gamma)_n (t-v)^{(n+\gamma)\alpha-\beta-1}}{n!\, \Gamma((n+\gamma)\alpha-\beta)} \right\} (x)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{u^n (\gamma)_n}{n!\, \Gamma((n+\gamma)\alpha-\beta)} I_{0-}^{\mu,\mu',\eta,\eta',\xi} \{ (t-v)^{n\alpha+\gamma\alpha-\beta-\rho} \} (x)$$

Apply equation (2.2) with $\rho = n\alpha + \gamma\alpha - \beta - \rho + 1$ and using this relation (1.2) $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$,we get

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n) \dots \Gamma(a_p+n)}{\Gamma(b_1+n) \dots \Gamma(b_q+n)} \frac{\Gamma(b_1) \dots \Gamma(b_q)}{\Gamma(a_1) \dots \Gamma(a_p)} \frac{\Gamma(\gamma+n)\, u^n}{n!\, \Gamma(\gamma)\, \Gamma((n+\gamma)\alpha-\beta)} (x-v)^{\gamma\alpha+n\alpha-\rho-\beta-\mu-\mu'+\xi}$$

$$\times \frac{\Gamma(\rho-n\alpha-\gamma\alpha-\eta+\beta)\, \Gamma(\rho-n\alpha-\gamma\alpha+\mu+\mu'+\beta-\xi-\eta)\, \Gamma(\rho-n\alpha-\gamma\alpha-\xi+\eta'+\beta+\mu)}{\Gamma(\rho-n\alpha-\gamma\alpha+\beta)\, \Gamma(\rho-n\alpha-\gamma\alpha+\beta-\xi+\mu+\mu'+\eta')\, \Gamma(\rho-n\alpha-\gamma\alpha+\beta+\mu-\eta)}$$

$$= \frac{\prod_{j=1}^{q} \Gamma(b_j)\, (x-v)^{\gamma\alpha+\xi-\rho-\beta-\mu-\mu'}}{\prod_{i=1}^{p} \Gamma(a_i)\, \Gamma(\gamma)} \times$$

$${}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta-\eta,-\alpha), (\rho-\gamma\alpha-\xi+\beta+\mu+\eta',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta,-\alpha), (\rho-\gamma\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta+\mu,-\alpha) \end{matrix} \right]$$

which is the required proof of theorem (2).

3. **Special Cases –** In this section we have established some known and new M.S.M. fractional integral formula , which are related to theorem (1) and (2).

(i) If we derive MSM fractional integral formula of generalized M series then we set $\beta = \alpha - \beta, \gamma = 1, u = 1$ *and* $v = 0$ in equation (2.3) and (2.4), we have

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi} \{ t^{\rho}\, {}_pM_q^{\alpha,\beta} (a_1 \dots a_p ; b_1 \dots b_q) \} (x) = \frac{\{ \prod_{j=1}^{q} \Gamma(b_j) \} x^{\rho+\xi-\mu-\mu'}}{\{ \prod_{i=1}^{p} \Gamma(a_i) \}} \times$$

$${}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (1,1), (\rho+\beta,\alpha), (\rho+\xi+\beta-\mu-\mu'-\eta,\alpha), \\ (\rho+\beta+\eta'-\mu',\alpha) \\ ; u(x)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\beta,\alpha), (\rho+\beta+\eta',\alpha), (\rho+\xi+\beta-\mu-\mu',\alpha), \\ (\rho+\xi+\beta-\mu'-\eta) \end{matrix} \right] \quad (3.1)$$

And

$$I_{0-}^{\mu,\mu',\eta,\eta',\xi} \{ t^{-\rho+1}\, {}_pM_q^{\alpha,\beta} (a_1 \dots a_p ; b_1 \dots b_q) \} (x) = \frac{\{ \prod_{j=1}^{q} \Gamma(b_j) \} x^{1-\rho+\xi-\mu-\mu'}}{\{ \prod_{i=1}^{p} \Gamma(a_i) \}} \times$$

$${}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (1,1), (\rho-\beta-\eta,-\alpha), (\rho-\xi-\beta+\mu+\eta',-\alpha), \\ (\rho-\beta-\eta-\xi+\mu+\mu',-\alpha) \\ ; u(x)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\beta,\alpha), (\rho-\beta,-\alpha), (\rho-\xi-\beta+\mu+\mu'+\eta',-\alpha), \\ (\rho-\beta-\eta+\mu,-\alpha) \end{matrix} \right] \quad (3.2)$$

(ii) If we put p = 1 = q and $a = b$ in equation (2.3) and (2.4) then we obtain the following MSM fractional integral formula of G function

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi}\{(t-v)^{\rho}\,G_{\alpha,\beta,\gamma}(u,v,x)\} = \frac{(x-v)^{\gamma\alpha+\rho+\xi-\beta-\mu-\mu'-1}}{\Gamma(\gamma)} \times$$

$${}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho+\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha),\\(\rho+\gamma\alpha-\beta+\eta'-\mu',\alpha)\\ ;u(x-v)^{\alpha}\\(\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha-\beta+\eta',\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu',\alpha),\\(\rho+\gamma\alpha+\xi-\beta-\mu'-\eta,\alpha)\end{matrix}\right] \quad (3.3)$$

And

$$\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}\{(t-v)^{-\rho+1}\,G_{\alpha,\beta,\gamma}(u,v,x)\}\right)(x) = \frac{(x-v)^{\gamma\alpha+\xi-\rho-\beta-\mu-\mu'}}{\Gamma(\gamma)} \times$$

$${}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho-\gamma\alpha+\beta-\eta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha)\\ ;u(x-v)^{\alpha}\\(\gamma\alpha-\beta,\alpha),(\rho-\gamma\alpha+\beta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta+\mu,-\alpha)\end{matrix}\right] \quad (3.4)$$

(iii) If we put $\gamma = 1$ in equation (3.3) and (3.4) then we obtain the following MSM fractional integral formula of R function

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi}\{(t-v)^{\rho}\,R_{\alpha,\beta}(u,v,x)\} = (x-v)^{\alpha+\rho+\xi-\beta-\mu-\mu'-1} \times$$

$${}_4\psi_4\left[\begin{matrix}(1,1),(\rho+\alpha-\beta,\alpha),(\rho+\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha),\\(\rho+\alpha-\beta+\eta'-\mu',\alpha)\\ ;u(x-v)^{\alpha}\\(\alpha-\beta,\alpha),(\rho+\alpha-\beta+\eta',\alpha),(\rho+\alpha+\xi-\beta-\mu-\mu',\alpha),\\(\rho+\alpha+\xi-\beta-\mu'-\eta,\alpha)\end{matrix}\right] \quad (3.5)$$

And

$$\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}\{(t-v)^{-\rho+1}\,R_{\alpha,\beta}(u,v,x)\}\right)(x) = (x-v)^{\alpha+\xi-\rho-\beta-\mu-\mu'} \times$$

$${}_4\psi_4\left[\begin{matrix}(1,1),(\rho-\alpha+\beta-\eta,-\alpha),(\rho-\alpha-\xi+\beta+\mu+\eta',-\alpha),\\(\rho-\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha)\\ ;u(x-v)^{\alpha}\\(\alpha-\beta,\alpha),(\rho-\alpha+\beta,-\alpha),(\rho-\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha),\\(\rho-\alpha+\beta-\eta+\mu,-\alpha)\end{matrix}\right] \quad (3.6)$$

(iv) If we put $v = \beta = 0$ in equation (3.5) and (3.6) then we get the following MSM fractional integral formula of F function

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi}\{t^{\rho}\,F_{\alpha}(u,x)\} = (x)^{\alpha+\rho+\xi-\mu-\mu'-1} \times$$

$${}_4\psi_4\left[\begin{matrix}(1,1),(\rho+\alpha,\alpha),(\rho+\alpha+\xi-\mu-\mu'-\eta,\alpha),(\rho+\alpha+\eta'-\mu',\alpha)\\ ;u(x)^{\alpha}\\(\alpha,\alpha),(\rho+\alpha+\eta',\alpha),(\rho+\alpha+\xi-\mu-\mu',\alpha),(\rho+\alpha+\xi-\mu'-\eta,\alpha)\end{matrix}\right] \quad (3.7)$$

And

$$\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}\{t^{-\rho+1}F_{\alpha}(u,x)\}\right)(x) = (x)^{\alpha+\xi-\rho-\mu-\mu'} \times$$

$$ {}_4\psi_4\left[\begin{matrix}(1,1),(\rho-\alpha-\eta,-\alpha),(\rho-\alpha-\xi+\mu+\eta',-\alpha),(\rho-\alpha-\eta-\xi+\mu+\mu',-\alpha)\\ ;u(x)^{\alpha}\\ (\alpha,\alpha),(\rho-\alpha,-\alpha),(\rho-\alpha-\xi+\mu+\mu'+\eta',-\alpha),(\rho-\alpha-\eta+\mu,-\alpha)\end{matrix}\right] \quad (3.8)$$

(v) If we take p=q=1 , $a_i = b_i$, $\beta = \alpha - 1, \gamma = 1, v = 0$ in equation (2.3) and (2.4), then we obtain following MSM fractional integral formula of Mittag – Leffler function $\left(I_{0+}^{\mu,\mu',\eta,\eta',\xi}\, t^{\rho}\, E_{\alpha}(ux^{\alpha})\right)(x) = x^{\rho+\xi-\mu-\mu'} \times$

$$ {}_4\psi_4\left[\begin{matrix}(1,1),(\rho+1,\alpha),(\rho+\xi-\mu-\mu'-\eta+1,\alpha),(\rho+\eta'-\mu'+1,\alpha)\\ ;u(x)^{\alpha}\\ (1,\alpha),(\rho+\eta',+1,\alpha),(\rho+\xi-\mu-\mu'+1,\alpha),(\rho+\xi-\mu'-\eta+1,\alpha)\end{matrix}\right] \quad (3.9)$$

And $\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}\, t^{-\rho+1}\, E_{\alpha}(ux^{\alpha})\right)(x) = x^{1-\rho-\mu-\mu'+\xi} \times$

$$ {}_4\psi_4\left[\begin{matrix}(1,1),(\rho-1-\eta,-\alpha),(\rho-\xi-1+\mu+\eta',-\alpha),(\rho-1-\eta-\xi+\mu+\mu',-\alpha)\\ ;u(x)^{\alpha}\\ (1,\alpha),(\rho-1,-\alpha),(\rho-\xi-1+\mu+\mu'+\eta',-\alpha),(\rho-1-\eta+\mu,-\alpha)\end{matrix}\right] \quad (3.10)$$

(vi) If we apply equation (1.8) in equation (2.3) and (2.4) , then we get following MSM fractional integral formula of Miller – Ross function

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi}\{t^{\rho}\, E_x(\beta,u)\}(x) = x^{\rho+\xi+\beta-\mu-\mu'} \times$$

$$ {}_4\psi_4\left[\begin{matrix}(1,1),(\rho+1+\beta,1),(\rho+1+\xi+\beta-\mu-\mu'-\eta,1),\\ (\rho+1+\beta+\eta'-\mu',1)\\ ;ux\\ (1+\beta,1),(\rho+1+\beta+\eta',1),(\rho+1+\xi+\beta-\mu-\mu',1),\\ (\rho+1+\xi+\beta-\mu'-\eta,1)\end{matrix}\right] \quad (3.11)$$

And $\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}\{t^{-\rho+1}\, E_x(\beta,u)\}\right)(x) = x^{1+\xi-\rho+\beta-\mu-\mu'} \times$

$$ {}_4\psi_4\left[\begin{matrix}(1,1),(\rho-1-\beta-\eta,-1),(\rho-1-\xi-\beta+\mu+\eta',-1),\\ (\rho-1-\beta-\eta-\xi+\mu+\mu',-1)\\ ;ux\\ (1+\beta,1),(\rho-1-\beta,-1),(\rho-1-\xi-\beta+\mu+\mu'+\eta',-1),\\ (\rho-1-\beta-\eta+\mu,-1)\end{matrix}\right] \quad (3.12)$$

(vii) If we take p=q=1 , $a_i = b_i$,$v = 0$ in equation (2.3) and (2.4), then we obtain following MSM fractional integral formula of Wright function

$$I_{0+}^{\mu,\mu',\eta,\eta',\xi}\left\{t^{\rho}\,{}_1\psi_1(x)\left[\begin{matrix}(\gamma,1);\\ ux^{\alpha}\\ (\gamma\alpha-\beta,\alpha);\end{matrix}\right]\right\}(x) = x^{\rho+\xi-\mu-\mu'} \times$$

$$ {}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho+\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha),\\ (\rho+\gamma\alpha-\beta+\eta'-\mu',\alpha);\\ u(x)^{\alpha}\\ (\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha-\beta+\eta',\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu',\alpha),\\ (\rho+\gamma\alpha+\xi-\beta-\mu'-\eta,\alpha);\end{matrix}\right] \quad (3.13)$$

And

$$I_{0-}^{\mu,\mu',\eta,\eta',\xi}\{t^{-\rho+1}\,{}_1\psi_1\,(x)\left[\begin{matrix}(\gamma,1);\\(\gamma\alpha-\beta,\alpha);\end{matrix}\;ux^{\alpha}\right]\}(x)=(x)^{1+\xi-\rho-\mu-\mu'}\times$$

$${}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho-\gamma\alpha+\beta-\eta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha)\\(\gamma\alpha-\beta,\alpha),(\rho-\gamma\alpha+\beta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta+\mu,-\alpha)\end{matrix}\;ux^{\alpha}\right] \tag{3.14}$$

(viii) Equation (3.13) and (3.14) convert MSM fractional integral formula of H function as

$$\left(I_{0+}^{\mu,\mu',\eta,\eta',\xi}\{t^{\rho}\,H_{1,2}^{1,1}\left[-ux^{\alpha};\begin{matrix}(1-\gamma,1);\\(0,1),(1-\gamma\alpha+\beta,\alpha);\end{matrix}\right]\}\right)(x)=x^{\rho+\xi-\mu-\mu'}\times$$

$${}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho+\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu'-\eta,\alpha),\\(\rho+\gamma\alpha-\beta+\eta'-\mu',\alpha);\\(\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha-\beta+\eta',\alpha),(\rho+\gamma\alpha+\xi-\beta-\mu-\mu',\alpha),\\(\rho+\gamma\alpha+\xi-\beta-\mu'-\eta,\alpha);\end{matrix}\;u(x)^{\alpha}\right] \tag{3.15}$$

And $\left(I_{0-}^{\mu,\mu',\eta,\eta',\xi}t^{-\rho+1}\,H_{1,2}^{1,1}\left[-ux^{\alpha};\begin{matrix}(1-\gamma,1);\\(1-\gamma\alpha+\beta,\alpha);\end{matrix}\right]\right)(x)=x^{1+\xi-\rho-\mu-\mu'}\times$

$${}_4\psi_4\left[\begin{matrix}(\gamma,1),(\rho-\gamma\alpha+\beta-\eta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta-\xi+\mu+\mu',-\alpha)\\(\gamma\alpha-\beta,\alpha),(\rho-\gamma\alpha+\beta,-\alpha),(\rho-\gamma\alpha-\xi+\beta+\mu+\mu'+\eta',-\alpha),\\(\rho-\gamma\alpha+\beta-\eta+\mu,-\alpha)\end{matrix}\;ux^{\alpha}\right] \tag{3.16}$$

4 The MSM Fractional differentiation of K_4 function

In this section, we have established the MSM fractional differential formula of K_4 function in terms of generalized Wright function ,therefore to fulfill this purpose we need following Lemma [21]-

Lemma 3 Let $\mu,\mu',\eta,\eta',\xi,\rho\in C$ and $R(\rho)>\max\{0, R(-\mu-\mu'-\eta'+\xi), R(-\mu+\eta)\}$ then

$$\left(D_{0+}^{\mu,\mu',\eta,\eta',\xi}t^{\rho-1}\right)(x)=x^{\rho+\mu+\mu'-\xi-1}\,\frac{\Gamma(\rho)\Gamma(\rho-\xi+\mu+\mu'+\eta')\Gamma(\rho+\mu-\eta)}{\Gamma(\rho-\xi+\mu+\mu')\Gamma(\rho-\xi+\mu+\eta')\Gamma(\rho-\eta)} \tag{4.1}$$

Lemma 4 Let $\mu,\mu',\eta,\eta',\xi\in C$ and $R(\rho)>\max.\{R(-\eta'), R(\mu'+\eta-\xi), R(\mu+\mu'-\xi)+[R(\xi)]+1\}$ then

$$\left(D_{0-}^{\mu,\mu',\eta,\eta',\xi}t^{-\rho}\right)(x)=x^{-\rho-\xi+\mu+\mu'}\,\frac{\Gamma(\rho+\eta')\Gamma(\rho+\xi-\mu-\mu')\Gamma(\rho+\xi-\mu'-\eta)}{\Gamma(\rho+\xi-\mu-\mu'-\eta)\,\Gamma(\rho-\mu'+\eta')\,\Gamma(\rho)} \tag{4.2}$$

Theorem 3. Let $\mu,\mu',\eta,\eta',\xi,\rho,\alpha,\beta,\gamma\in C$ be such that $R(\xi)>0$, $R(\rho+\gamma\alpha-\beta)>\max.\{0, R(\eta-\mu), R(-\mu-\mu'-\eta'+\xi)\}$ and $R(\alpha\gamma-\beta)>0$ then for $x>0$, we obtain following result –

$$\left(D_{0+}^{\mu,\mu',\eta,\eta',\xi}(t-v)^{\rho}\,K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)}\right)(x)=\frac{\prod_{j=1}^{q}\Gamma(b_j)\;(x-v)^{\gamma\alpha+\rho-\xi-\beta+\mu+\mu'-1}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)}\times$$

$${}_{p+4}\psi_{q+4}\left[\begin{matrix}(a_i,1)_{i=1}^{p},(\gamma,1),(\rho+\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha-\xi-\beta+\mu+\mu'+\eta',\alpha),\\(\rho+\gamma\alpha-\beta-\eta+\mu,\alpha)\\(b_j,1)_{j=1}^{q},(\gamma\alpha-\beta,\alpha),(\rho+\gamma\alpha-\beta-\eta,\alpha),(\rho+\gamma\alpha-\xi-\beta+\mu+\mu',\alpha),\\(\rho+\gamma\alpha-\xi-\beta+\mu+\eta')\end{matrix}\;;u(x-v)^{\alpha}\right] \tag{4.3}$$

Theorem 4 . Let $\mu, \mu', \eta, \eta', \xi, \rho, \alpha, \beta, \gamma \in C$ be such that $R(\xi) > 0$, $R(1+\gamma\alpha-\rho-\beta) < 1+\min\{R(\eta'), R(-\mu-\mu'+\xi), R(-\mu'-\eta+\xi)\}$ and $R(\alpha\gamma-\beta) > 0$ then for $x > 0$, we obtain following result –

$$\left(D_{0-}^{\mu,\mu',\eta,\eta',\xi} (t-v)^{-\rho+1} K_4^{(\alpha,\beta,\gamma),(u,v);(p,q)} \right)(x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)\ (x-v)^{\gamma\alpha-\xi-\rho-\beta+\mu+\mu'}}{\prod_{i=1}^{p} \Gamma(a_i)\ \Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta+\eta',-\alpha), (\rho-\gamma\alpha+\xi+\beta-\mu-\mu',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta+\xi-\mu',-\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta,-\alpha), (\rho-\gamma\alpha+\xi+\beta-\mu-\mu'-\eta,-\alpha), \\ (\rho-\gamma\alpha+\beta-\mu'+\eta',-\alpha) \end{matrix} \right] \quad (4.4)$$

Remark 1: If $\mu = \mu+\eta, \eta = -\xi, \xi = \mu$ *and* $\mu' = \eta' = 0$ then all the results in section 2 ,3 and 4 reduce corresponding results of K_4 function via Saigo fractional integral and differential operators . Similarly , if $\eta = 0$ then all results convert for Erde'lyi – kober fractional integral and differential operators.

5 The Caputo-type MSM Fractional differentiation of K_4 function

In this section, we have established the Caputo-type MSM fractional differential formula of K_4 function in terms of generalized Wright function so for this purpose , we need following Lemma -

Lemma 5 Let $\mu, \mu', \eta, \eta', \xi, \rho \in C$ and $m = [R(\xi)] + 1$, $R(\rho) - m > \max\{0, R(-\mu-\mu'-\eta'+\xi), R(-\mu+\eta)\}$ then

$$ {}^{c}D_{0+}^{\mu,\mu',\eta,\eta',\xi} t^{\rho-1}(x) = x^{\rho+\mu+\mu'-\xi-1} \frac{\Gamma(\rho)\Gamma(\rho-\xi+\mu+\mu'+\eta'-m)\Gamma(\rho+\mu-\eta-m)}{\Gamma(\rho-\xi+\mu+\mu')\Gamma(\rho-\xi+\mu+\eta'-m)\Gamma(\rho-\eta-m)} \quad (5.1)$$

Lemma 6 Let $\mu, \mu', \eta, \eta', \xi \in C$ and $R(\rho) + m > \max.\{R(-\eta'), R(\mu'+\eta-\xi), R(\mu+\mu'-\xi) + [R(\xi)] + 1\}$ then

$$ {}^{c}D_{0-}^{\mu,\mu',\eta,\eta',\xi} t^{-\rho}(x) = x^{-\rho-\xi+\mu+\mu'} \frac{\Gamma(\rho+\eta'+m)\Gamma(\rho+\xi-\mu-\mu')\Gamma(\rho+\xi-\mu'-\eta+m)}{\Gamma(\rho+\xi-\mu-\mu'-\eta+m)\ \Gamma(\rho-\mu'+\eta'+m)\ \Gamma(\rho)} \quad (5.2)$$

Theorem 5. Let $\mu, \mu', \eta, \eta', \xi, \rho, \alpha, \beta, \gamma \in C$ be such that $R(\xi) > 0$, $R(\rho+\gamma\alpha-\beta) - m > \max.\{0, R(\eta-\mu), R(-\mu-\mu'-\eta'+\xi)\}$ and $R(\alpha\gamma-\beta) > 0$ then for $x > 0$, we obtain following result –

$$ {}^{c}D_{0+}^{\mu,\mu',\eta,\eta',\xi} \left((t-v)^{\rho} K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)} \right)(x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)\ (x-v)^{\gamma\alpha+\rho-\xi-\beta+\mu+\mu'-1}}{\prod_{i=1}^{p} \Gamma(a_i)\ \Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho+\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\xi-m-\beta+\mu+\mu'+\eta',\alpha), \\ (\rho+\gamma\alpha-\beta-\eta+\mu-m,\alpha); \\ u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta-\eta-m,\alpha), (\rho+\gamma\alpha-\xi-\beta+\mu+\mu',\alpha), \\ (\rho+\gamma\alpha-\xi-\beta+\mu+\eta'-m,\alpha); \end{matrix} \right]$$

(5.3)

Theorem 6 . Let $\mu, \mu', \eta, \eta', \xi, \rho, \alpha, \beta, \gamma \in C$ and $m = [R(\xi)] + 1$ be such that $R(\rho) + m > \max.\{R(-\eta'), R(\mu'+\eta-\xi), R(\mu+\mu'-\xi) + m\}$ and $R(\alpha\gamma-\beta) > 0$ then for $x > 0$, we have –

$$ {}^{c}D_{0-}^{\mu,\mu',\eta,\eta',\xi} \left((t-v)^{-\rho+1} K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)} \right)(x) = \frac{\prod_{j=1}^{q} \Gamma(b_j)\ (x-v)^{\gamma\alpha-\xi-\rho-\beta+\mu+\mu'}}{\prod_{i=1}^{p} \Gamma(a_i)\ \Gamma(\gamma)} \times$$

$$ {}_{p+4}\psi_{q+4} \left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta+\eta'+m,-\alpha), (\rho-\gamma\alpha+\xi+\beta-\mu-\mu',-\alpha), \\ (\rho-\gamma\alpha+\beta-\eta+\xi-\mu'+m,-\alpha) \\ ; u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta,-\alpha), (\rho-\gamma\alpha+\xi+\beta-\mu-\mu'-\eta+m,-\alpha), \\ (\rho-\gamma\alpha+\beta-\mu'+\eta'+m,-\alpha) \end{matrix} \right] \quad (5.4)$$

If we change $\mu = \mu + \eta \,, \eta = -\xi \,, \xi = \mu \; and \; \mu' = \eta' = 0$ in theorem (5) and (6) then we get following corollary for the Saigo fractional differential operator [20] such as

Corollary 1 Let $\mu, \eta, \xi, \rho \in C$ and m= $[R(\mu)] + 1$, be such that $R(\xi) > 0, R(\rho + \gamma\alpha - \beta) - m > max.\{0, R(-\mu - \eta - \xi)\}$ and $R(\alpha\gamma - \beta) > 0$ then the generalized Caputo fractional differentiation $D_{0+}^{\mu,\eta,\xi}$ of K_4 function is given by

$$ {}^{c}D_{0+}^{\mu,\eta,\xi}\left((t-v)^{\rho}\, K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\,(x-v)^{\gamma\alpha+\rho-\beta+\eta-1}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times $$

$$ {}_{p+3}\psi_{q+3}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho+\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta+\xi+\mu+\eta-m,\alpha); \\ u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta+\xi-m,\alpha), (\rho+\gamma\alpha-\beta+\eta,\alpha); \end{matrix}\right] \quad (5.5) $$

Corollary 2 Let $\mu, \eta, \xi, \rho \in C$ and m= $[R(\mu)] + 1$, be such that $R(\rho) + m > max.\{R(-\mu-\xi), R(\eta) + m\}$ and $R(\alpha\gamma - \beta) > 0$ then the generalized Caputo fractional differentiation ${}^{c}D_{0-}^{\mu,\eta,\xi}$ of K_4 function is given by

$$ {}^{c}D_{0-}^{\mu,\eta,\xi}\left((t-v)^{-\rho+1}\, K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\,(x-v)^{\gamma\alpha-\rho-\beta+\eta}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times $$

$$ {}_{p+3}\psi_{q+3}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta-\eta,-\alpha), (\rho-\gamma\alpha+\beta+\xi+\mu+m,-\alpha); \\ u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta-\eta+\xi+m,-\alpha), (\rho-\gamma\alpha+\beta,-\alpha); \end{matrix}\right] \quad (5.6) $$

If we put $\eta = 0$ in equation (5.5) and (5.6) then we obtain corollary 3 and 4 for the Erde'lyi – kober { [7],[13] } fractional differential operator such as

Corollary 3 Let $\mu, \xi, \rho \in C$ and m= $[R(\mu)] + 1$, be such that $R(\xi) > 0, R(\rho + \gamma\alpha - \beta) - m > max.\{0, R(-\mu - \xi)\}$ and $R(\alpha\gamma - \beta) > 0$ then the Caputo-type Erde'lyi – kober fractional differentiation ${}^{c}D_{0+}^{\mu,\xi}$ of K_4 function is given by

$$ {}^{c}D_{0+}^{\mu,\xi}\left((t-v)^{\rho}\, K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\,(x-v)^{\gamma\alpha+\rho-\beta-1}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times $$

$$ {}_{p+2}\psi_{q+2}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho+\gamma\alpha-\beta+\xi+\mu-m,\alpha); \\ u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho+\gamma\alpha-\beta+\xi-m,\alpha); \end{matrix}\right] \quad (5.7) $$

Corollary 4 Let $\mu, \xi, \rho \in C$ and m= $[R(\mu)] + 1$, be such that $R(\rho) + m > max.\{R(-\mu-\xi), m\}$ and $R(\alpha\gamma - \beta) > 0$ then the Caputo-type Erde'lyi – kober fractional differentiation ${}^{c}D_{0-}^{\mu,\xi}$ of K_4 function is given by

$$ {}^{c}D_{0-}^{\mu,\xi}\left((t-v)^{-\rho+1}\, K_4^{(\alpha,\beta,\gamma),(u,v),(p,q)}\right)(x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)\,(x-v)^{\gamma\alpha-\rho-\beta}}{\prod_{i=1}^{p}\Gamma(a_i)\;\Gamma(\gamma)} \times $$

$$ {}_{p+2}\psi_{q+2}\left[\begin{matrix} (a_i,1)_{i=1}^{p}, (\gamma,1), (\rho-\gamma\alpha+\beta+\xi+\mu+m,-\alpha); \\ u(x-v)^{\alpha} \\ (b_j,1)_{j=1}^{q}, (\gamma\alpha-\beta,\alpha), (\rho-\gamma\alpha+\beta+\xi+m,-\alpha); \end{matrix}\right] \quad (5.8) $$

Remark 2 : If we take $\eta = -\mu$ in corollaries (1) and (2) respectively then these results convert Caputo- type Riemann-Liouville and Weyl fractional differential operators of K_4 functions. Similarly , if $\eta = -\mu$ then all the results derive in this paper will convert to the results for Riemann- Liouville and Weyl fractional integral and differential operators.

Conclusion : In this paper , we have established various results of K_4 function and their special cases Via MSM fractional integral and derivative operators. We have also discussed the effect of the Caputo type MSM fractional derivatives on the K_4 function. All the results established in this paper are general in nature and useful for further research.

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