A Study on the Mathematical Sequential Synthesis Process Adopted by Students in Solving Physics problems

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ABSTRACT

Mathematical manipulations in reference to our research work are the mathematical operations/procedures which stand between the problem statement and the end solution of a physics problem. They play a significant role in the dynamics of problem-solving process that requires multiple steps of simplification/rearrangement of equations towards constructing the solution. However, the application of the relevant physics learning is imperative and does not outweigh the emphasis of mathematical manipulations. Students are required to possess a number of sub-skills to be able to navigate through the numerous equations in a problem-solving process. A sub skill is that which is required and utilized in a specific phase of a systematic problem-solving process. Assessing individual's level of sub skills rather than rating overall problem-solving ability is necessary for researchers and teachers to determine effective strategies for improving problem solving. In our study, mathematical manipulations are an aspect of the dynamics of the problem-solving process. The outcome of our research study revealed lack of students' skill sets to process the mathematical sequences to arrive at the end solution.

Literature on research on problem-solving

Although mathematics is an essential component of physics, manipulation of equations, in other words 'mathematical processing' in a given problem-solving context often occurs mechanistically, either with or without success. One kind of mathematical processing would be to write equations that describe a physical situation and solve the problem [1] , the second kind, to select relevant equations correctly and manipulate a formula or combine a number of concepts to solve a problem [2].The third refers to simplification of equations to obtain the solution. Students view equations as computational tools and do not ascribe meaning to the mathematical symbols and tools as used in the physics domain. Studies have focussed essentially on how students make sense of the symbols in problem-solving [3,4,5,6], a crucial aspect of problem-solving. In addition, problem solvers should be able to use the relevant characteristics of a problem to activate and blend knowledge elements that will help them solve the problem [7,8]. Research has also demonstrated that using mathematical concepts, tools and procedures in pure mathematics contexts is different from their application in the physics domain [9-13]. Student's difficulties when using mathematical tools, such as integrals, differentials and partial derivatives, in the physics context were also identified [14-19].

In physics where formal mathematical expressions are prevalent, we want students to use math tools as experts do in tackling physics problems. There is limited investigation cited in literature explicitly on how students apply/use the required mathematical rules/tools to navigate through the numerous equations in the problem-solving process. Nevertheless, a previous study by Ibrahim et al. [9] is of relevance in the context of our study on testing students' mathematical skills in physics problem-solving. The objective of the study by Ibrahim et al. [9] was to examine students' mathematical performance on quantitative sequential and simultaneous tasks with varying mathematical complexity of synthesis problems. Sequential synthesis tasks required a chronological application of pertinent concepts, and simultaneous synthesis tasks required a concurrent application of the pertinent concepts. The authors analysed students' responses on formulation (identification of all the pertinent concepts underlying the task and generation of all the equations to mathematically express the identified concepts), combination (combination of their formulated equations associated with the pertinent concepts) and simplification (identification of variables and rearrangement of equations such that the required variable is written as a function of the others) of equations. This study indicated that mathematical complexity negatively influences the students' mathematical performance on the sequential and simultaneous synthesis problems. However, for the sequential synthesis tasks, mathematical complexity affected only the students' simplification of equation for obtaining the variable of interest. Other research studies on students' reasoning when manipulating the mathematics of physics problems have been elicited using the framework of epistemic games [20-22]. It is significant that researchers have reported on students' ability to perform mathematical-processing tasks that lead a student towards the solution of a physics problem, which are often discounted in class room teaching. Our investigation was primarily directed at testing students' math skills in solving problems (mathematics and physics) that required students to reflect on the sequential mathematical processing of equations to obtain symbolic solutions. Research study of the type mentioned is very rare in the Indian context of physics education.

Methodology

The first part of our study was to catalogue a few mathematical manipulations that are often exercised in learning physics and design an appropriate questionnaire and the second part was our investigative study of research work was directed at testing student math skills employed towards simplification of equations which is a component of the dynamics of problem-solving. Our study was primarily to test the robustness of standard mathematics skill-sets required for simplification/rearrangement of equations to arrive at the end-solution. Problem-types that were chosen for the investigation of this kind did not include the formulaic plug-and chug approach. We designed a questionnaire with problems that required manipulations of equations (at least 3 steps) to arrive at the end result in the two stated categories. We framed two problems IA and IB in mathematics and four problems in physics (II-V). Problems IV ii) and V ii) were add-on problems that required the use of solutions of IV i) and V i) respectively. The physics problems were framed in different branches of physics and in the context of a specific physics problem, students were required to use equations/concepts mainly from the same branch. From our experience over the years, one persistent weakness for solving a problem is domain vulnerability. Hence, each respondent was instructed to choose a problem of his/her choice. Fig 1 illustrates the research methodology that was developed for our investigation.

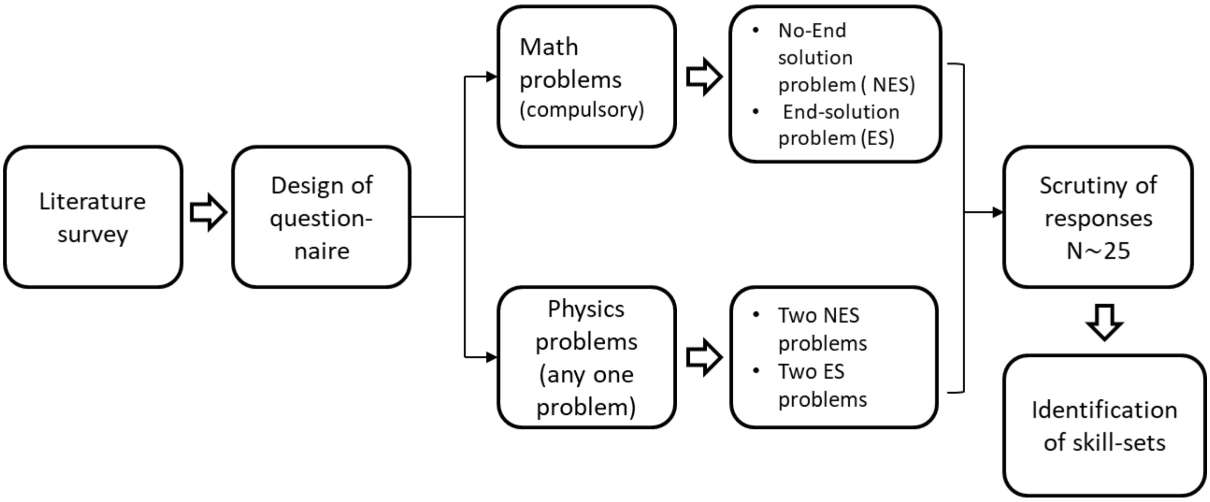


Fig 1: Schematic diagram of research methodology related to mathematical manipulations

The group of 27 students (respondents in our earlier investigations) pursuing Master's course with 'Application of theoretical concepts in physics' as the choice of an elective paper were the respondents in our study. The respondents were presented with the questionnaire at the same time and were instructed to provide answers to math problems IA and IB and solve any one physics problem of their choice among physics problems II, III, IV and V. They were allowed to solve problems at their own pace. The average time taken by a participant of the test was 50 minutes. In part A of the chapter, we present the catalogue of a few mathematical manipulations that students use in content learning and problem-solving as well, the complexity factors related to a problem, the designed questionnaire, design aspects and in part B, we present the results, representative select-responses and discussion related to two physics problems. For the current study, we explored the application of math tools by students in restructuring mathematical equations in mathematics and physics problems (not analogous) to obtain symbolic solutions.

PART A

Physicists use the 'language' of mathematics extensively. Rearrangement of equations as a sequence of mathematical statements through the use of tools is but common in all branches of physics. We sought to make a catalogue of a few mathematical manipulations that are used in physics content-learning and problem-solving. Following are a common few mathematical manipulation that are used individually or in combination.

Catalogue of a few mathematical manipulation

- Writing a secondary equation for a chosen physical quantity from a primary equation (example: writing $V = nRT/P$ from the equation of state)
- Substituting physical quantity/quantities in an equation by the required form/s (example: substituting C_P - C_V as equal to R)
- Various mathematical operations related to exponential, logarithmic, trigonometric functions
- Using mathematical tools such as integration and differentiation
- Algebraic restructuring of equations (Ex: Merging two equations or elimination of a variable from two equations)

The complexity level in the context of manipulation of equations in a Physics problem can depend on one or more of the following factors and to varying degrees.

i. Familiarity of equations related to a physics concept
ii. Adequacy of skill-sets
iii. Number of symbols used in problem-statement

iv. Number of simplification steps required

While the first two factors are relevant to the individual's cognitive domain, factors iii) and iv) are relevant to specifics in a problem. However, the factors are limited to three in a mathematics context.

PART B

In solving a physics problem, number of simplification steps is a factor closely associated with the number of substitutions of physical quantities that are required; unlike in a mathematics problem. Students may struggle to make multiple substitutions that connect different physical quantities in different equations that has a bearing on the number of symbols in the interplay of substitutions and equations. As there could be different approaches to solve a problem, specially no-end solution (NES) problems, it is hard to specify the number of simplification steps. Another aspect is that, the primary equation would be significant in solving a problem as to a certain degree, it provides the cue to initiate the mathematical processing. Here, we define a primary equation as the equation that figures in the problem statement. The two physics problem statements are (IV AND V):

IV. For an intrinsic semiconductor, the electron concentration in conduction band is $n_e = N_C\, e^{-(E_C - E_F)/kT}$ and the hole concentration is $n_p = N_v\, e^{-(E_F - E_V)/kT}$.

N_c is the effective density of states in conduction band & N_v is the effective density of states in the valence band.

i. Write the expression for Fermi energy E_F.
ii. State the condition under which the Fermi energy level is in the middle of the forbidden gap.

V. A particle is in a potential $V(x) = C_1x^2 - C_2x$, where C_1 & C_2 are constants.

i. Reduce the potential to the form: $V(x) = A(x - x_0)^2 - B$, where A, B and x_0 are to be determined in terms of C_1 & C_2.
ii. If the particle can be considered as a linear harmonic oscillator; write the energy Eigen values.

Table 1 provides an overview of the type of physics problems that we chose to formulate with reference to number of symbols, nature of simplification steps, presence of primary equation/s and end-solution in a problem statement. A tick mark in the table indicates-presence and a wrong mark indicates-absence of the aspect under the heading in a column.

Problem number	Number of symbols in problem statement	Nature of simplification steps	Primary equation in problem statement	End-solution in problem statement
IV i)	9	Algebraic, Logarithm & exponentiation	✓	×
IV ii)	nil	Deduction using result of IV i)	×	×
V i)	6	Algebraic	✓	✓

V ii)	nil	Deduction using result of V i)	×	×

Table 1: Problem number, number of symbols, nature of simplification steps, presence of primary equation/s and end-solution in a problem statement

Problem IV i)

Problem IV i) was formulated in the branch of solid-state physics. The two possible approaches to solve the physics problem are: i) mentioning the mathematical equality of electron concentration in conduction band and hole concentration in valence band in an intrinsic semiconductor and performing mathematical manipulations to write an equation for E_F. ii) simplifying the two equations discretely in the problem statement to eliminate the exponential terms and subsequently performing mathematical manipulations to write an equation for E_F. While the former approach is based on conceptual understanding in physics, the second is based on mathematical reasoning. Following are a few written solutions.

Response IV-1

IV i)

$n_e = N_c e^{-(E_c - E_F)/kT}$ $\quad n_p = N_v e^{-(E_F - E_v)/kT}$

$$E_F = \frac{n_p - n_e}{2}$$

$$= \frac{N_v e^{-(E_F - E_v)/kT} - N_c e^{-(E_c - E_F)/kT}}{2}$$

$$E_F = \frac{[N_v - N_c)\, e^{[-2E_F - E_v - E_c]/kT}}{2}$$

Response IV -2

(IV) In intrensic semiconductor both holes and electrons are equal then we can write.

$n = p$

$$(2m_e^*)^{3/2} \exp\left(\frac{E_F - E_c}{kT}\right) = (2m_h^*)^{3/2} \exp\left(\frac{E_v - E_F}{kT}\right)$$

$$\left(\frac{m_h^*}{m_e^*}\right)^{3/2} = \exp\left(\frac{E_F + E_F - (E_c + E_v)}{kT}\right)$$

$$\frac{2E_F}{kT} = \frac{E_c + E_v}{kT} + \frac{3}{2} \ln\left(\frac{m_h^*}{m_e^*}\right)$$

$$\boxed{E_F = \frac{E_c + E_v}{2} + \frac{3}{4} kT \ln\left(\frac{m_h^*}{m_e^*}\right)}$$

where m_h^* – effective mass of holes

m_e^* – effective mass of electrons.

In the Intrensic semiconductor at room temperature Fermi Level Lie b/w the conduction and valence band. ie

$$\boxed{E_F = \frac{E_c + E_v}{2}}$$

Response IV-3

IV i) Fermi energy $E_F = \frac{E_c - E_v}{2} = \frac{E_c+E_v}{2}$ $n_e = n_p$

$\Rightarrow N_c e^{-\left(\frac{E_c-E_F}{kT}\right)} = N_v e^{-(E_F-E_v)/kT}$

$N_c e^{\frac{2E_F}{kT}} = N_v e^{-\left(\frac{E_c+E_v}{kT}\right)}$

Solving for E_F we get

$\boxed{E_F = \frac{E_c+E_v}{2}}$

ii) The Condition when the Fermi energy level is in the middle of the forbidden gap when $E_c = E_v$ $\boxed{E_F = 1}$

Response IV-4

IV Given $n_e = N_c e^{-\left(\frac{E_c-E_F}{KT}\right)}$ and $n_p = N_v e^{-\frac{(E_F-E_v)}{KT}}$

where $N_C = 2\left(\frac{2\pi m_e^* kT}{h^3}\right)^{3/2}$ and $2\left(\frac{2\pi m_h^* kT}{h^3}\right)^{3/2}$.

$E_g = \frac{n_e+n_p}{2} = \frac{2kT}{n_e}$.

$\boxed{n_e = \frac{E_g}{2kT}}$ where energy gap lyes b/w middle of the valence and conduction band.

Sixty nine percent of the students who attempted to solve could nor navigate through the manipulations due to the following reasons:

- Unable to use neither the mathematical equation related to the physics concept nor the mathematical reasoning to commence solving the problem
- Mentioning equations that were needless which increased the complexity
- Mentioning the equation for Fermi energy prior to performing mathematical manipulations
- Not 'reading' the meanings of the symbols precisely

Following are two sample correct responses (I-5 & I-6). A point to note in the two responses is the use of log (logarithm to base 10) in place of ln (natural logarithm).

Response IV-5

iv) for an Intrinsic S.C

$n_e = n_p$

$\Rightarrow N_c e^{-\left(\frac{E_c-E_F}{KT}\right)} = N_v e^{-\frac{(E_f-E_v)}{KT}}$

$e^{-\frac{E_c}{KT}} e^{\frac{E_f}{KT}} e^{\frac{E_f}{KT}} \cdot e^{-\frac{E_v}{KT}} = \frac{N_v}{N_c}$

$e^{\frac{2E_f}{KT}} e^{-\left(\frac{E_c+E_v}{KT}\right)} = \frac{N_v}{N_c}$

$e^{\frac{2E_f}{KT}} = \frac{N_v}{N_c} e^{\left(\frac{E_c+E_v}{KT}\right)} \Rightarrow \frac{2E_f}{KT} = \log\left(\frac{N_v}{N_c}\right) + \left(\frac{E_c+E_v}{KT}\right)$

$\Rightarrow E_f = \frac{KT}{2}\left[\log\left(\frac{N_v}{N_c}\right) + \frac{(E_c+E_v)}{KT}\right]$

Response IV-6

iv) is for an intrinsic semiconductor

$$N_e = N_p = n_i$$

$$N_c \exp\left\{-\frac{(E_c - E_f)}{kT}\right\} = N_v \exp\left\{-\frac{(E_f - E_v)}{kT}\right\}$$

$$\frac{N_c}{N_v} \exp\left\{\frac{1}{kT}\left[-E_c + E_f + E_f - E_v\right]\right\} = 1$$

$$\exp\left\{\frac{-(E_c + E_v)}{kT} + \frac{2E_f}{kT}\right\} = \frac{N_v}{N_c}$$

taking log on b.s

$$-(E_c + E_v) + 2E_f = \log\left\{\frac{N_v}{N_c}\right\} kT$$

$$2E_f = +(E_c + E_v) + \log\left\{\frac{N_v}{N_c}\right\} kT$$

$$E_f = \frac{(E_c + E_v)}{2} + \log\left\{\frac{N_v}{N_c}\right\} \frac{kT}{2}$$

Problem IV ii) was an add-on problem that was deduction-based relevant to physics. It required students to infer the condition under which the Fermi energy level is in the middle of the forbidden gap i.e. the condition when Temperature T = 0K from the equation obtained in problem I i). Mathematically, the condition can be obtained when the term $\frac{kT}{2} ln \frac{N_V}{N_C}$ is Zero. It is expected that students deduce the condition in a physics context for profound learning by stating: at absolute zero temperature.

Problem V i)

The mathematical manipulations required in Problem V i) did not have a bearing to the physics context intrinsically. Students were required to reorganize the primary equation to obtain coefficient of x^2 as unity and manipulate the resulting equation by adding and subtracting like terms to obtain a perfect square. The problem statement comprised of the end-solution to cue students towards the required mathematical manipulations. Following are a few written responses.

Response V-1

$$V(x) = c_1 x^2 - c_2 x.$$

$$V(x) = A(x - x_0)^2 - B$$

$$= A(x^2 + x_0^2 - 2x x_0) - B$$

$$= Ax^2 + Ax_0^2 - 2x x_0 - B$$

$(x-1)^2$

$x^2 + 1 - 2x$

5 i) $V(x) = c_1 x^2 - c_2 x.$

$$= c_1 x^2 - c_2 x + \frac{c_2^2}{4} - \cancel{c_2}, -\frac{c_1^2}{4} + \cancel{c_2}$$

$$= c_1\left(x - \frac{1}{2}\right)^2 - \frac{c_1^2}{4} - c_2 x$$

It can be represented as,

$$V(x) = c_1\left(x - \frac{1}{2}\right)^2 - \frac{c_1^2}{4},$$

where, $A = c_1$

$B = \frac{c_1^2}{4} - c_2 x$

$x_0 = \frac{1}{2}$

ii) $V(x) = A(x - x_0)^2 - B.$

for Linear Harmonic oscillator,

$$E = \frac{1}{2} m v^2 + A(x - x_0)^2 - B$$

Response V-2

$$c_1 x^2 - \frac{2c_2 x\, x_0 c_1}{2x_0 c_1} \qquad A^2x^2 + B^2 + 2A$$

$$A^2 = c_1$$

$$A(x-x_0)^2 = Ax^2 + Ax_0^2 - 2Axx_0 \qquad B = \frac{c_2}{2\sqrt{c_1}}$$

$$= c_1x^2 + \quad + 2$$

$$A(x-x_0)^2 - B$$

$$A(x^2 + x_0^2 - 2xx_0) - B$$

$$c_1x^2 - c_2x$$

$$(\sqrt{c_1}x)^2 - \frac{2c_2\sqrt{c_1}x}{2\sqrt{c_1}} + \left(\frac{c_2}{2\sqrt{c_1}}\right)^2 - \left(\frac{c_2}{2\sqrt{c_1}}\right)^2$$

$$\left(c_1x - \frac{c_2}{2\sqrt{c_1}}\right)^2 - \frac{c_2}{2\sqrt{c_1}}$$

$$c_1^2\left(x - \frac{c_2}{2c_1\sqrt{c_1}}\right)^2 - \frac{c_2}{2\sqrt{c_1}}$$

Response V-3

$$V(x) = A(x-x_0)^2 - B \qquad V(x) = c_1x^2 - c_2x$$

$$V(x) = A(x^2 + x_0^2 - 2xx_0) - B \qquad A, B, c_1, c_2$$

$$V(x) = Ax^2 + Ax_0^2 - 2Axx_0 - B$$

$$V(x) = A(x-x_0)^2 - B.$$

Response V-4

(5)
$$V(x) = A(x^2 + x_0^2 - 2xx_0) - B$$
$$= Ax^2 + Ax_0^2 - A2xx_0 - B.$$
$$= .$$

Response V-5

(V)

$$V(x) = V(x_0) + \frac{dV}{dx}\Big|_{x_0}(x-x_0) + \frac{d^2V}{dx^2}\frac{(x-x_0)^2}{2!}$$

$$= c_1x_0^2 - c_2x_0 + (2c_1x_0 - c_2)(x-x_0) + c_1(x-x_0)^2$$

$$= c_1(x-x_0)^2 + (2c_1x_0 - c_2)(x-x_0) + c_1x_0^2 - c_2x_0$$

~~Comparing~~ this

Response V-6

Given $V(x) = C_1x^2 - C_2x$.

where C_1 and C_2 are constants

(i) $dV(x) = 2C_1x - C_2$.

at $x = 0$ $C_2 = 0$

$x = 1$ $C_1 = 1/2$.

$= \frac{A}{C_1x^2} - \frac{B}{C_2x}$

$V(x) = AC_2x - B(C_1x^2)$

then it will be interms of the

$dV(x) = 2C_1x - C_2$

then $V(x) = A(x - x_0)^2 - B$.

where A and B are constants and x_0 is also a constant.

Response V-7

V) ~~$V(x) = A(x-x_0)^2$~~ $V(x) = C_1x^2 - C_2x$

$V(x) = 0$ at $x = x_0$

$\Rightarrow C_1x^2 - C_2x = 0$

$x_0(C_1x_0 - C_2) = 0$

$\Rightarrow \boxed{x_0 = 0}$ & $\boxed{x_0 = C_2/C_1}$

$\left.\frac{dV}{dx}\right|_{x_0} = 0$

$\Rightarrow 2C_1x_0 - C_2 = 0$

$\Rightarrow 2C_1x_0 = C_2$

$\Rightarrow x_0 = \frac{C_2}{2C_1}$

Scrutiny of responses IV-1 ,2 ,3 & 4 reveals that students' approach to use the equation $V(x) = A(x - x_0)^2 - B$ as the primary equation was improper. The inclusion of B in the primary equation itself did not lead the students to the required stages of mathematical manipulations. Students who provided responses labelled 1 and 2 worked hard with the rearrangement and with the right intent to add and subtract like terms to obtain a perfect square. However, the mathematical manipulations were not precise enough to rewrite the potential energy function as required. Students must be able to identify the primary equation in the problem-statement and work towards the required form.

Two students evaluated the first derivative of the potential energy function with respect to the position coordinate; one equated both the potential and first derivative to zero while the other worked with values of x (x=0 and x=1) (Responses 6 & 7). They could probably have seen the relevance of the first derivative of the potential energy function with respect to x that occurs in the Taylor series expansion used in the mathematical formulation of small oscillations in classical mechanics in this problem situation. This correspondence was evident in response 5. Students failed to take the cue from the mention of energy eigen values in the problem statement of IV ii). Just as making a connection between relevant schema is of utmost importance, filtering unwanted knowledge elements in a problem situation is equally important.

Following is the only one correct response that comprised of the required mathematical manipulations.

Response V-8

i) $V(x) = C_1 x^2 - C_2 x$

$V(x) = A(x - x_0)^2 - B$

$\Rightarrow V(x) = (\sqrt{C_1}x)^2 - \frac{2C_2\sqrt{C_1}x}{2\sqrt{C_1}} + \left(\frac{C_2}{2\sqrt{C_1}}\right)^2 - \left(\frac{C_2}{2\sqrt{C_1}}\right)^2$

$\therefore V(x) = \left(\sqrt{C_1}x - \frac{C_2}{2\sqrt{C_1}}\right)^2 - \left(\frac{C_2}{2\sqrt{C_1}}\right)^2$

$= C_1\left(x - \frac{C_2}{2C_1}\right) - \left(\frac{C_2}{2\sqrt{C_1}}\right)^2$

$\Rightarrow A = C_1, \; B = \frac{C_2}{2\sqrt{C_1}}$

$x_0 = \frac{C_2}{2C_1}$ //

Problem V ii)

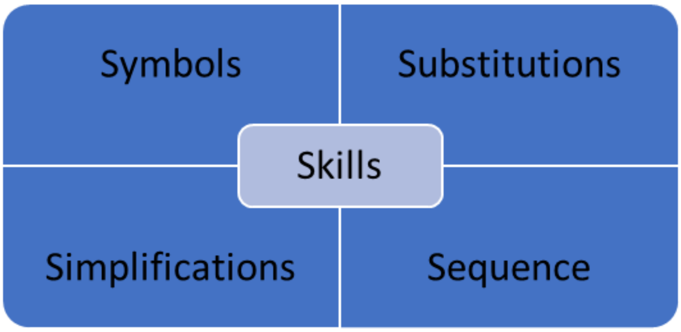
This was an add-on problem that required the result of problem Vi) to write the energy Eigen values of a linear harmonic oscillator. The respondent who provided the required form of the potential energy in problem Vi) did not infer the energy eigen values. Two incorrect equations that found a mention were: i) $E = \left(n + \frac{1}{2}\right)\hbar\omega$ and ii) $E = \frac{1}{2}mv^2 + A(x - x_0)^2 - B$

Summary and Conclusion

Mathematical manipulations in a problem should be executed with mathematical reasoning to favour students the right mathematical processing of equations in problem-solving. In our study, we found that that majority of the students were short of the skills required for mathematical processing that included mathematical rearrangement, substitution and simplification of equations in solving problems.

A significant observation in our study is that students fail to adopt an alternate strategy when one method or approach does not become definite to obtain the solution. This observation in our study is pertinent to problem IV i). Students could have used an alternative approach of purely mathematical reasoning without labelling a physics context to obtain the solution. Simplification steps often involve logarithm and exponentiation. A lack of these procedural skills seemed to impede their progress towards providing the correct solution or reaching a logical end-point. Physics problem V i) required rearrangement of the primary equation initially and algebraic manipulations later to write a perfect square term. Students failed to deduce the required equation as they incorrectly expanded the square term in the problem statement. Indeed, the solution path that involves mathematical manipulations may be structurally different and what we have attempted is to understand the underlying procedural errors.

Overall, scrutiny of the responses largely reveals that the respondents did not seek a correct rational approach of mathematical manipulations as required. Problem-solving expertise should include a well-knit fabric of skills, correct sequence, symbol-sense, appropriate substitutions and precise simplifications.



Substitutions, whenever required, should be a consequence of conceptual reasoning and simplifications should be aided with robust math skills. An important sign of physics students' proficiency in physics problem-solving is their combining the symbols and structures of mathematics with their conceptual understanding.

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