A Fixed Point Theorem To The Relation Between Normal Maps To Weakly Zamfirescu Maps In The Weak Concept

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Abstract : − Zamfirescu gave a fixed point theorem that generalizes the classical fixed point theorem by Banach , Kannan and chatterjea. In this paper we follow the idea of David Ariza-Ruiz and Antonio Jimenez-Melado to extend Zamfirescu's fixed point theorem to the class of weakly zamfirescu maps .A continuous method for this class of maps is also given the weak concept.

Keyword- **Fixed point, zamfirescu mapping ,weakly contractive mapping, continuation method.**

Introduction: -

In [9] Zamfirescu obtained a very intresting fixed point theorem on complete metric space by combining the result of Banach space [1] Kannan [7] and chatterjea [6]. We introduce the concept of weakly Zamfirescu maps and we study the existing independence between some types such as Contractions maps, Kannan maps, Chatterjea maps and its respective weak concepts. (see[1],[2],[3]and[5].) Moreover we expend Zamfirescu's fixed point theorem[9] to the class of weakly zamfirescu maps and then we prove a continuation method for this class of maps, extending various known results also we obtain for weakly zamfirescu mappings a simple expression of Cauchy modulus and modulus of convergence (see[4]).
(Banach,1922) There is $\alpha \in [0,1)$ s.t.

$d(Tx,Ty)\leq\alpha\, d(x,y)$ for all $x,y\in X$……………..(C)

(Kannan,1968) There is $K\in[0,1)$ s.t.

$d(Tx,Ty)\leq \frac{K}{2}[d(x,Tx)+d(y,Ty)$ for all $x,y\in X$………………(K)

(Chattejea,1972) There is $\xi\in[0,1)$ s.t.

$d(Tx,Ty)\leq\frac{\xi}{2}[d(x,Tx)+d(y,Ty)]$ for all $x,y\in X$………………(Ch)

The conditions (C),(K),(Ch) are independent as it will be showed in section 2 (see also[2] and [5]).

After these three results many papers have been generalizing some of the conditions (C), (K) or (Ch) obtained a fixed point theorem for the class of maps $T:X\rightarrow X$ for which there exist $\xi\in[0,1)$s.t.

$d(T(x),T(y))\leq\xi \max\{d(x,y),\frac{1}{2}[d(x,T(x))+d(y,T(y))],\frac{1}{2}[d(x,T(y))+d(y,T(x))]\}$……….(Z)

A mapping satisfying(Z) is commonly called a Zamfirescu map.

In general,Contractive maps, Kannan maps and Chatterjea maps are independence and relation between these these concept through out the examples

Example1:- ((C)∄(K),(Ch))

T:D→ R^2 ,

$T(x_1,x_2)=\alpha(-x_1,x_2)$, where $\alpha\epsilon[\frac{1}{\sqrt{3}},1)$ and

$D=\{(x_1,x_2)\in R^2:\|(x_1,x_2)\|_2\leq 1\}$.

Example2:-((K)∄(C),(Ch))

$Tx=\begin{cases}\frac{1}{3} & if 0<x\leq 1,\\ 0 & if\ x=1.\end{cases}$

Example3:- ((Ch)∄(C),(K)).

$Tx=\begin{cases}1 & if\ 0<x\leq 1,\\ \frac{2}{3} & if\ x=1.\end{cases}$

2.WEAKLY ZAMFIRESCU MAPS:-

Although Rhoades[8]showed the condition (C),(K)and (Ch).We can change the constant with a certain function which depends on x and y.In this way we obtain a generalization of the previous concepts for example-

T:X→X is a weakly contractive map if there is a function $\propto: X\times X\to[0,1]$ which is compactly less than 1,i.e.

θ(a,b):=sup$\{\alpha(x,y); a\leq d(x,y)\leq b\}<1$ $\forall\ 0<a\leq b.$

s.t

d(Tx,Ty)≤$\propto(x,y)d(x,y)$ $\forall x,y\in X$

Remark:-As in the case of a contractive map ,any weakly contractive map has at most one fixed point

We show that throughout the following examplesceletion between contractive maps and weakly contractive maps.

Example 4 :-

Consider the subset D=$[0,\frac{\pi}{2}]$ of the metric space X=R with the usual metric d(x,y)= |x-y|

and let T:D→ X be the function defined as Tx=sin(X).

then T is a weakly contractive map but not a contractive map.

Remark :- the class of weakly contractive maps is larger than the class of contractive maps.

Defination 5:- we say that a mapping T :D → X is a kannan map if there exists K∈ (0,1) such that.

d(Tx,Ty)≤ $\frac{K}{2}[d\ (x,Tx)+d(y,Ty)]$ for all x,yϵD.

Rhoades proved that the concepts of contractive map and kannan map are independent (see [ii],theorem1,(iii))

After this concept has been generalized in [2] obtaining the so called weakly kannan maps.

Defination 6**:-** we say that T:D→ X is a weakly kannan map if there exists K:D×D→ [0,1] satisfying that.

(a,b)= sup$\{k(x,y): a\leq d(x,y)\leq b\}<1$.

For every 0< $a\leq b$ s.t for all x,y∈ D

d(T(x),(y))≤ $\frac{k(x,y)}{2}$ $[d(x,T(x)+d(y,T(y))]$.

Remark :- kannan [7] showed that if a kannan map has a fixed point then it a is unique. Using the same reasoning we have that any weakly kannan map has at most one fixed point. the following example shows that the class of weakly map is larger than the class of kannan maps.

Example 7:- let D=[0,∞) be a subset of the metric space X=R with the usual metric d(x,y)=|x-y| the map T:D→X defined as.

$$Tx=\frac{1}{3}\log(1+e^x)$$

Is a weakly kannan map but not a kannan map.And after that.

Example 8**:-** ((wc)∄(wk))

Let T be the mapping of example 5 i.e, T:[0,$\frac{\pi}{2}$]→ R with Tx=sin(x)

We know that T is a weakly contractive map how ever ,T is a weakly kannan map.

Example 9**:-** ((wk)∄(wc))

Is a weakly kannan map but not a weakly contractive map.Now we show that the relation between chatterjea map.The class of weakly chatterjea maps is larger than the class of chatterjea maps.

Example10 **:-**

$$Tx=\begin{cases}\frac{2}{3}x, & if 0 \le x < 1\\ 0, & if x = 0\end{cases}$$

Is a weakly chatterja map but not a chatterjea map.

Example 11**:-** ((Wc)∄(Wch))

Fix w> 0,the map T[0,∞] →R given as Tx= $\frac{w}{w+x}^2$

Is a weakly contractive map but not a weakly chatterjea map.

3.Relation between zamfirescu maps &weakly zamfirescu maps. In 1972 zamfirescu combining condition (c),(k)and(ch)defined a new type of maps.

Suppose that there is a constant θ[0,1)s.t.

d(Tx,Ty)≤ ξmax{d(x,y),$\frac{1}{2}$[d(x,Tx)+d(y,Ty)],$\frac{1}{2}$[d(x,Tx)+d(y,Ty)]}for all x,y ϵ X

A map satisfying this condition is commonly called a zamfirescu map.in the natural way we can define the class of weakly zamfirescu map. In order to this,we change the constant ξ with a certain thus we get to the following definition.

Defination 12**:-**

Let D be a nonempty subset of a metric space (X,d) and T:D→ X a map.We say that T is a weakly Zamfirescu map if there exists a function ξ :D × D → [0 ,1] with

θ(a,b)=sup{ξ(x,y):a≤d(x,y)≤b}<1 for all 0<a≤b s.t

d(Tx,Ty)≤ ξ(x,y) max{d(x,y),$\frac{1}{2}$[d(x,Tx)+d(y,Ty)],$\frac{1}{2}$[d(x,Tx)+d(y,Ty)]}for all x,y ϵ X

We first give some theorems in which some properties of those function f satisfying (Zw) are established.

Theorem 13**:-** Let (X,d)be a metric space and D c X .If T:D→X is a weakly zamfirescu map then T has at most one fixed point in D.

Proof:- Suppose that u and v are fixed points of T with u≠v .Then $\alpha(u,v) \leq \theta(\frac{h}{2},h)<1$

Where h=d(u,v)>0 so by (Zw) have

$d(u,v)=d(T(u),T(v)) \leq \alpha(u,v)\, M_T(u,v)= \alpha(u,v)$

Recall that a self-mapping T on a metric space (X,d) is said to be asymptotically regular at $x_0 \in X$ $\lim_{n\to\infty} T^n(x_0),T^{n+1}(x_0)) = 0$.

continuity of weakly Zamfirescu map. T may be discontinuous at some points, the following result shows that the discontinuity cannot at a fixed points for T.)

Theorem 14:- Let (X, d) be a metric space and T : X→X a weakly zamfirescu map. If T has a fixed point say, then T is continuous at this points.

Proof:- let$\{x_n\}$ be a convergent sequence to u = T(u) for any n ϵ N we have

$d(T(x_n),T(u)) \leq \alpha(x_n,u)\, M_T(x_n,u)$

$\leq \max\{d(x_n,u);\frac{1}{2}[d(x_n,T(x_n))+d(u,T(u))],\frac{1}{2}[d(x_n,T(u))+d(u,T(x_n))]$

$= \max\{ d(x_n,u);\frac{1}{2}d(x_n,T(x_n))+,\frac{1}{2}[d(x_n,u)+d(T(u),T(x_n))]\}$

$\leq \max\{ d(x_n,u);\frac{1}{2}[d(x_n,u)+ d(T(u),T(x_n))],\frac{1}{2}[d(x_n,u)+d(T(u),T(x_n))]\}$

Hence for all nϵN,

$0 \leq d(T(u),T(x_n)) \leq d(x_n,u),$

So that $\{T(x_n)\}$converges to T(u),There T is continuous at u.

Now began to prove the main result of this section.

Theorem 15:-

Let (X,d) be a complete metric space and T : X → X a weakly zamfirescu map.Then T has a unique fixed point uϵx and at this point u the mapping T is continous more over for each $x_0 \epsilon X$ the sequence $\{T^n(x_n)\}$ converges to u.

Proof : Let $x_n \epsilon X$ and define $x_{n+1}=T(x_n)$ for nϵN we may assume that $d(x_0,x_1)>0$ because otherwise we have finised. We shall prove then$\{x_n\}$is a cauchy sequence and that its limit is a fixed point for T.To do it. Let us prove that

$d(x_{n+k+1},x_{n+1}) \leq \alpha(x_{n+k},x_n)\, d(x_{n+k},x_n)+2d(x_{n+1},x_n)$ for all n,kϵN.

Let n,kϵN by(Zw)

$d(x_{n+k+1},x_{n+1})=d(T(x_{n+k}),T(x_n))$

$\leq \alpha(x_{n+k},x_n) M_T(x_{n+k},x_n)$

where

$M_T(x_{n+k},x_n)=\max\{d(x_{n+k},x_n),\frac{1}{2}[d(x_{n+k},T(x_{n+k}))+d(x_n,T(x_n))],\frac{1}{2}[d(x_{n+k},T(x_n))+d(x_n,T(x_{n+k}))]\}$

$=\max\{d(x_{n+k},x_n),\frac{1}{2}[d(x_{n+k},x_{n+k+1})+d(x_n,x_{n+1})],\frac{1}{2}[d(x_{n+k},x_{n+1})+d(x_n,x_{n+k+1})]\}$

We consider the following three cases.

case1:-If $M_T(x_{n+k},x_n)=d(x_{n+k},x_n)$

case2:-If $M_T(x_{n+k},x_n)= \frac{1}{2}[d(x_{n+k},x_{n+k+1})+d(x_n,x_{n+1})]+ d(x_n,x_{n+1})]$

then

$d(x_{n+k+1},x_{n+1}) \leq \alpha\frac{(x_{n+k},x_n)}{2}[d(x_{n+k},x_{n+k+1})+d(x_n,x_{n+1})]$

Applying $d(x_n,x_{n+1}) \leq \alpha(x_{n-1},x_n)\, d(x_{n-1},x_n)$,

$d(x_{n+k},x_{n+k+1}) \leq d(x_n,x_{n+1})$

so

$d(x_{n+k+1},x_{n+1}) \leq \alpha(x_{n+k},x_n)d(x_{n+k},x_n)+2d(x_{n+1},x_n)$ for all n,kϵN,

case 3:-If $M_T(x_{n+k},x_n) = \frac{1}{2}[d(x_{n+k},x_{n+1})+d(x_n,x_{n+k+1})]$,then

$d(x_{n+k},x_{n+k+1}) \leq \alpha\frac{(x_{n+k},x_n)}{2}[d(x_{n+k},x_{n+1})+d(x_n,x_{n+k+1})]$

$\leq\alpha\frac{(x_{n+k},x_n)}{2}[d(x_{n+k},x_{n+1})+d(x_n,x_{n+1})+ d(x_{n+1},x_{n+k+1})]$

Then

$(1- \alpha\frac{(x_{n+k},x_n)}{2})\, d(x_{n+k+1},x_{n+1}) \leq \alpha\frac{(x_{n+k},x_n)}{2}[d(x_{n+k},x_{n+1})+ d(x_{n+1},x_n)]$

i.e.

$d(x_{n+k+1},x_{n+1}) \leq \frac{(x_{n+k},x_n)}{2-\alpha(xn+1,xn)}[d(x_{n+k},x_{n+1})+d(x_{n+1},x_n)]$

$\leq \alpha(x_{n+k},x_n)\, [d(x_{n+k},x_{n+1})+d(x_{n+1},x_n)]$

$\leq \alpha(x_{n+k},x_n)\, [d(x_{n+k},x_{n+1})+2d(x_n , x_{n+1})]$

$\leq \alpha(x_{n+k},x_n)\, d(x_{n+k},x_n)+2d(x_{n+1},x_n)$

To prove that { x_n } is a Cauchy sequence suppose that $\epsilon > 0$ and use proposition let (X,d) be a metric space. If T:X→X is a weakly zamfirescu map . Then T is asymptotically regular at each point in Xto obtain N ϵ N such that

$d(x_{N+1},x_N) < \frac{1}{6}(1-\theta(\frac{\epsilon}{2}, \ \epsilon)).\ \epsilon$ --(1)

We will prove inductively that $d(x_{n+1},x_n) < \varepsilon$ for all KϵN . It is obious for K = 1 and assuming

$d(x_{N+k},x_N) < \varepsilon$ let us see $d(x_{N+k+1},x_N) < \varepsilon$

Note that using

$d(x_{n+k+1},x_{n+1}) \leq \alpha(x_{n+k},x_n)d(x_{n+k},x_n)+2d(x_{n+1},x_n)$ for all n,kϵN,

we have that

$d(x_{N+K+1},x_N) \leq d(x_{N+K+1},x_{N+1})+d(x_{N+1},x_N)$

$\leq \alpha(x_{N+K},x_N)d(x_{N+K},x_N)+3d(x_{N+1},x_N)$ ---(2)

Thus If $d(x_{N+k},x_N) < \frac{\varepsilon}{2}$

It follows from (1) and (2) that

$d(x_{N+K+1},x_{N+1}) \leq d(x_{N+K},x_N)+3d(x_{N+1},x_N)$

$< \frac{\varepsilon}{2} +3.\frac{1}{6}(1-\theta\left(\frac{\varepsilon}{2}, \varepsilon\right), \varepsilon)$

$\leq \varepsilon$

And if $d(x_{N+1},x_N) \geq \frac{\varepsilon}{2}$,

Appiying the induction hypothesis, we have that $\varepsilon(x_{N+K},x_N) \leq \theta\left(\frac{\varepsilon}{2}, \varepsilon\right)$.

Then from (1) and (2) we conclude that

$d(x_{N+K+1},x_{N+1}) \leq \alpha\ (x_{N+K},x_N)d(x_{N+K},x_N)+3d(x_{N+1},x_N)$

$$< \theta\ \left(\frac{\varepsilon}{2},\varepsilon\right).\varepsilon + 3.\frac{1}{6}(1-\theta\left(\frac{\varepsilon}{2},\varepsilon\right).\varepsilon$$

$$\leq \varepsilon.$$

since (X,d) is complete then $\{x_n\}$is convergent,say to u∊X.That u is a fixed point for T follows from standard arguments which we include for sake of completeness for n∊N.We have that

$$d(u,T(u)) = \lim_{n\to\infty} d(x_{n+1,}T(u))$$

$$= \lim_{n\to\infty} d(T(x_n),T(u))$$

$$\leq \lim_{n\to\infty} sup\alpha(x_{n,}u)M_T(x_{n,}u)$$

$$\leq \lim_{n\to\infty} sup\ M_T(x_{n,}u)$$

$$= \frac{1}{2}d(u,T(u))$$

Since

$$\lim_{n\to\infty} sup\ M_T(x_{n,}u) = \lim_{n\to\infty} sup\ \max\{d(x_{n,}u),\tfrac{1}{2}[(x_{n,}(T(x_n)+d(u,T(u))],\tfrac{1}{2}[d\left(x_{n,}T(u)\right)+d(u,T(x_n))]\}$$

$$= \lim_{n\to\infty} sup\ \max\{d(x_{n,}u),\tfrac{1}{2}[d(x_{n,}x_{n+1})+d(u,T(u))],\tfrac{1}{2}[d\left(x_{n,}T(u)\right)+\ d(u,x_{n+1})]\}$$

$$= \max\{0,\tfrac{1}{2}[(0+d(u,T(u))],\tfrac{1}{2}[d\big(u,T(u)\big)+0]\}$$

By theorem 13 and 14, u is the unique fixed point of T and T is continuous at u.

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