VISCO-ELASTIC FLOW UNDER TORSIONAL OSCILLATION

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ABSTRACT

The flow of a visco-elastic fluid has been analysed induced by the torsional oscillations of a disk about an axis normal to its plane. The case of high frequency has been considered in this problem. The structure of outer boundary layer has been examined matching inner and outer solutions. Kármán-Pohlhausen method has been applied while integrating the equations. The behaviour of different flow parameters has been analysed with table.

1. INTRODUCTION:

The periodical movement of a shaft is called Torsional oscillations. It takes very important part in the studies of fluid mechanics and power transmission system. Theodore Von Kármán [1], the Hungarian-American mathematician and Aero-dynamic engineer did some pioneering works on fluid flows due to rotations of disks. This followed by numerous other studies with extension of these ideas to time dependent rotator oscillations of one or two disks including works of Benney[2], Rosenblat[3] and so on. Rao and Kasiviswanath [4] have studied the flow and heat transfer due to torsional oscillations of two disks at different speeds. Puri [5] studied the unsteady flow of an elastic-viscous fluid past an infinite plate. Das, Maji, Jana and Seth [6] studied flow induced by torsional oscillations of a disk in a rotating visco-elastic fluid. Shrivastava [7]considered the torsional oscillations of a second-order fluid when the fluid is of an infinite extent as well as the case when it is bounded by another stationary plate by expanding the velocity components and the pressure in powers of the amplitude of oscillation of the plate.

In the present paper we are going to investigate the behaviour of the flow parameters when the flow is caused by the torsional oscillation of a disk, the axis of rotation of the disk being normal to the plane of the disk and the frequency of oscillation being high. We have taken the visco-elastic fluid with short memories in our investigation. The constitutive equation of the fluid considered in this problem is

$$\sigma^{ij} = -pg_{ij} + 2\eta_0 e^{ij} - 2C\frac{\delta}{\delta t}e^{ij}$$

Where

$e_{ij} = v_{i,j} + v_{j,i}$, η_0 is limiting viscosity of small rate of shear, g_{ij} is the metric tensor with respect to fixed coordinate system x_i , C is the visco-elastic parameter, v_i is the velocity vector and $\frac{\delta}{\delta t}$ denotes the convected derivative of a tensor quantity.

2. EQUATION:

Considering the geometrical configuration of the problem we consider the cylindrical polar coordinates (r, θ, z) with axis of rotation being the initial line, the disk being on the plane $z = 0$ and the liquid occupying the space $z > 0$. The boundary conditions of the problem will be as

$$u = 0,\ v = rW\cos(\omega t)\ , w = 0 \quad at\ z = 0$$

$$u \to 0, \; v \to 0. \qquad as\; z \to \infty \;, \qquad (2.i)$$

Where u, v, w are velocity components, ω and W are the frequency and angular speed of the oscillating motion of the disk respectively.

The equations of motion in radial and transverse directions are as follows.

$$\frac{\partial^2 F}{\partial x \partial y} + \epsilon\left[\left(\frac{\partial F}{\partial x}\right)^2 - 2F\frac{\partial^2 F}{\partial x^2} - G^2\right] = \frac{1}{2}\frac{\partial^3 F}{\partial x^3} - K\left[\frac{\partial^4 F}{\partial x^3 \partial y} - \epsilon\left\{2F\frac{\partial^4 F}{\partial x^4} - 4\frac{\partial F}{\partial x}\frac{\partial^3 F}{\partial x^3} + \left(\frac{\partial^2 F}{\partial x^2}\right)^2 - G\frac{\partial^2 G}{\partial x^2} - 3\left(\frac{\partial G}{\partial x}\right)^2\right\}\right]$$
(2.ii)

$$\frac{\partial G}{\partial x} + 2\epsilon\left(\frac{\partial F}{\partial x}G - F\frac{\partial G}{\partial x}\right) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2} - K\left[\frac{\partial^3 G}{\partial x^2 \partial y} - \epsilon\left\{2F\frac{\partial^3 G}{\partial x^3} + \frac{\partial^3 F}{\partial x^3}G - 2\frac{\partial F}{\partial x}\frac{\partial^2 G}{\partial x^2} + 4\frac{\partial^2 F}{\partial x^2}\frac{\partial G}{\partial x}\right\}\right] \qquad (2.iii)$$

Where, $u = rW\frac{\partial F}{\partial x}, \quad v = rWG(x,y), \quad w = -2W\left(\frac{2\vartheta_1}{\omega}\right)^{\frac{1}{2}}F(x,y)$ (2.iv)

$$x = z\sqrt{\frac{\omega}{2\vartheta_1}} \;, \quad y = \omega t, \quad \vartheta_1 = \frac{2\eta_0}{\rho},$$

$K = \frac{CW}{\eta_0}$ and $\epsilon = \frac{W}{\omega}$

The boundary conditions now becomes

$$F = \frac{\partial F}{\partial x} = 0, \; G = \cos y \qquad \text{at} \;\; x = 0$$

$$\frac{\partial F}{\partial x}, G \;\to\; 0 \qquad \text{at} \;\; x \to \infty \qquad (2.v)$$

3. Solution:

Let us consider the high frequency case i.e. $\epsilon \ll 1$. We substitute the following series

$$F(x,y) = \sum_{i=0}^{\infty} \epsilon^i F_i(x,y) \qquad (3.i)$$

$$G(x,y) = \sum_{i=0}^{\infty} \epsilon^i G_i(x,y) \qquad (3.ii)$$

into the equations (2.ii) and (2.iii).

Equating the coefficients of like powers of ϵ , we obtain the following partial differential equations

$$F_{0_{xy}} = \frac{1}{2}F_{0_{xxx}} - KF_{0_{xxxy}} \;, \qquad (3.iii)$$

$$F_{1_{xy}} + \left(F_{0_x}\right)^2 - 2F_0 F_{0_{xx}} - {G_0}^2 = \frac{1}{2}F_{1_{xxx}} - K\left[F_{1_{xxxy}} - 2F_0F_{0_{xxxx}} - \left(F_{0_{xx}}\right)^2\right] + 4F_{0_x}F_{0_{xxx}} + G_0G_{0_{xx}} + 3\left(G_{0_x}\right)^2 , \quad (3.iv)$$

etc.

$$G_{0_y} = \frac{1}{2}G_{0_{xx}} - K\,G_{0_{xxy}} \quad (3.v)$$

$$G_{1_y} + 2\left[F_{0_x}G_0 - F_0\,G_{0_x}\right] = \frac{1}{2}G_{1_{xx}} - K\left[G_{1_{xxy}} - 2F_0\,G_{0_{xxx}} + F_0\,G_{0_{xx}} - 4F_{0_{xx}}\,G_{0_x}\right], \quad (3.vi)$$

etc.

In view of the substitutions (3.i) and (3.ii), the boundary conditions (2.v) take the suitable form as

$$F_i = F_{i_x} = 0,\ G_0 = \cos y,\ G_{i+1} = 0 \text{ at } x = 0, \qquad \text{for } i = 0,1,2,\ldots$$

$$F_{i_x} \to 0,\ G_i \to 0 \qquad \text{as } \to \infty, \qquad \text{for } i = 0,1,2,\ldots \quad (3.vii)$$

Solutions of the equations (3.iii) to (3.vi) have been obtained subject to the boundary conditions (3.vii) as

$$F_0\,(x,y) = 0 \quad (3.viii)$$

$$G_0\,(x,y) = e^{-Py}\cos(x - Qy) \quad (3.ix)$$

$$F_1(x,y) = \frac{1}{4P^2}\{1 - 2K(2P^2 + Q^2)\}\left[y + \frac{1}{2P}(e^{-2Py} - 1)\right] + J(y)e^{2ix} \quad (3.x)$$

$$G_1(x,y) = 0 \quad (3.xi)$$

where the fluctuating part $J(y)$ of F_1 is given by

$$J(y) = \left[2(m+in)\sqrt{1+16K^2}\{4K(m+in)^2 - 1\}/(q+is)\right]\left[1 - e^{\frac{-(q+is)y}{\sqrt{1+16K^2}}}\right] - [\{4K(m+in)^2 - 1\}/ 8\{i(m+in) - (1-4iK)(m+in)^3\}]\left[1 - e^{-2(m+in)y}\right]; \quad (3.xii)$$

where,

$$m^2, n^2 = \left[\sqrt{1+4K^2} \mp 2K\right]/(1+4K^2)$$

and $q^2, s^2 = 2\left[\sqrt{1+16K^2} \mp 4K\right]$

Using of these solutions in the velocity components we can conclude that as $F_0\,(x,y) = 0$, the first order solution has no component of radial or axial velocity, only the transverse velocity given by

$$v = rWexp\left(-mz\sqrt{\frac{\omega}{2\vartheta_1}}\right)\cos\left\{\omega x - nz\sqrt{\frac{\omega}{2\vartheta_1}}\right\} \quad (3.xiii)$$

It may be noted that the steady part of the radial component of velocity persists outside a shear wave or inner boundary layer of thickness of order $\left(\frac{\vartheta_1}{m^2\omega}\right)^{\frac{1}{2}}$. The axial velocity persists outside the region thereby conforming continuity. At the edge of the inner boundary layer, the steady parts of the velocity components u and w are

$$u_s = r\epsilon WD\left[1 - exp\left\{-2mz\sqrt{\frac{\omega}{2\vartheta_1}}\right\}\right]$$

$$w_s = \frac{\epsilon WD}{4m^3}\sqrt{\frac{2\vartheta_1}{\omega}}\left[1 - mz\sqrt{\frac{2\omega}{\vartheta_1}} - e^{-mz\sqrt{\frac{\omega}{2\vartheta_1}}}\right] \quad (3.xiv)$$

where $D = \frac{1}{4m^2}\{1 - 2K(2m^2 + n^2)\}$ (3.xv)

Let us study the structure of the outer boundary layer in detail. We are going to match inner and outer expansions. Since there is no oscillatory potential flow in the outer layer, the derivation of the equation is quite obvious. The terms in the inner boundary layer must match at each stage with that of the outer boundary layer. Thus in order to effect a match with inner solution of o(ϵ) , the first term of the outer solution for u_s is taken as of (ϵ). Thus for studying the flow in outer layer we write

$$F(y) = R(\zeta)\,,\, G(y) = S(\zeta)\,,\, y = \epsilon^{-1}\zeta \quad (3.xvi)$$

The transformation expresses the fact that the thickness of the outer layer is of order ϵ^{-1} times that of the inner. Clearly $S = 0$ and the equation for R will be

$$R^{///} = 2\left(R^{/^2} - 2RR^{//}\right) + K_1\left(4R^/R^{///} - 2RR^{iv} - R^{//^2}\right) \quad (3.xvii)$$

with $R^/(\infty) = 0$ as evident from the condition that outer solution matches with inner solution as $y \to \infty$.

Here $K_1 = K\epsilon^2$ and the prime denotes differentiation with respect to ζ.

With the help of (3.xiv) and (3.xvi) and using the velocity components we deduce that $R \to D\zeta$ as $\to 0$.

Thus

$$R(0) = 0,\ R^/(0) = D \quad (3.xviii)$$

Let us solve the equation (3.xvii) by Kármán-Pohlhausen method. We introduce

$$\zeta = \delta\xi,\ R = \delta S \quad (3.xix)$$

Where δ is the non-dimensional boundary layer thickness of the outer boundary layer. With trhis substitution, equation (3.xvii) takes the form

$$S^{///} = 2\left(S^{/^2} - 2SS^{//}\right) + K_1\left(4S^/S^{///} - 2SS^{iv} - S^{//^2}\right) \quad (3.xx)$$

Where differentiation is with respect to ξ .

Considering the smoothness of the solution over outer boundary layer edge, we assume the form

$$S^{/} = D(1 + b\xi)(1 - \xi)^4 \qquad (3.xxi)$$

where b is a constant to be determined. To determine b and δ we have momentum integral equation as

$$-S^{//}(0) = 6\delta^2 \int_0^1 S^{/2} d\xi + K_1 \left[6a^2(4 - b) - 7\int_0^1 S^{//2} d\xi\right] \qquad (3.xxii)$$

The condition that equation (3.xx) is satisfied as $\xi \to 0$ is found as

$$S^{///}(0) = 2\delta^2 S^{/}(o)^2 + K_1\left[4S^{/}(0)S^{///}(0) - S^{//}(0)^2\right] \qquad (3.xxiii)$$

4. Results and Discussion:

Using (3.xxi), (3.xxii) and (3.xxiii), we can determine the constants b and δ for different values of K_1. We have calculated the values of , D, δ , $R^{//}(0)$ for $K_1 = 0, .002, .005$. These values of the parameters for different values of visco-elastic parameter are given in the table-1. From the Table-1, we can conclude that absolute vale of $R^{//}(0)$ decreases with the increase in the visco-elastic parameter K_1 . The correctness of the calculations can be guessed by the fact that whereas the exact value of $R^{//}(0)$ is - 0.207 for Newtonian fluid i.e. $K_1 = 0$ the same has been calculated as – 0.205.

Let us consider again the inner solution. The term of order $o(\varepsilon)$ in the substituted series (3.i) is given as

$$F_2(x) = c_2 \ x^2 \qquad (3.xxiv)$$

Where c_2 is yet unknown. The constant was assumed to be zero by Rosenblat (4) and Srivastava (14). By matching (3.xxiv) with the appropriate term in the outer solution we get

$$c_2 = R^{//}(0) \neq 0 \ .$$

Thus the terms of even order in ε do not vanish in the inner solution.

Table - 1

K_1	0	.002	.005
b	0	-0.0003	-0.0007
D	0.249	0.2413	0.2398
δ	4.971	4.9801	4.9987
$R^{//}(0)$	-0.2052	-0.2041	-0.1998

REFERENCES

1. T. Von Kármán, Uber laminar und turbulent Reibung, ZAMM, 1 (1921) 233 – 251

2. D.J. Benny, The flow induced by a disk oscillating in its own plane, J. Fluid Mechanics, 18 (1964) 385 – 391.

3. S. Rosenblat, Torsional oscillations of a plane in a viscous fluid, J. Fluid Mech., 6 (1959) 206 – 220.

4. A. R. Rao and S. R. Kasiviswanathan, Flow and heat transfer due to torsionally oscillating Eccentric Disks Rotating at different speeds, J. Math. Phy. Sci., 25 (1991)98 – 118.

5. P. Puri, On unsteady flow of an elastic-viscous fluid past an infinite plate with variable suction, Appl. Scientific Research, 28 (1973) 111 – 133.

6. S. Das, S.L. Maji, R. N. Jana, G. S. Seth, Flow induced by torsional oscillations of a disk in a rotating Visco-elastic fluid, International J. of Comp. Application, 58 (2012) 975 – 8887.

7. A. C. Srivastava, Torsional Oscillations of an infinite plate in second order fluids, J. Fluid Mech, 17 (1963) 171 – 181.