OBSERVATIONS ON TWO SPECIAL PELL EQUATIONS

$y^2 = 30x^2 + 19$ and $y^2 = 12x^2 - 44$

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Abstract : A peer search is made on the integral solutions of positive and negative pell equations given by $y^2 = 30x^2 + 19$ and $y^2 = 12x^2 - 44$ in two sections A and B respectively. A few interesting relations among the solutions for the above two pell equations are presented. Also, employing the integer solutions of the equations under consideration, integer solutions for other choices of hyperbola and parabola are exhibited along with the corresponding pell equation. Further, a few interesting relations between special polygonal numbers generated through the solutions of the equation under consideration are presented.

IndexTerms–Positive pell equation, negative pell equation, non-homogeneous binary quadratic equation, integer solutions.

I. INTRODUCTION

Number theory is one of the most fascinating and enlivening subjects occupying a vital place in the history of mathematics. In particular, the theory of diophantine equation has occupied a significant position in the subject of number theory as it invigorates the interest towards research. The beauty of diophantine equations and system is that the number of unknowns is bigger than the number of equations. They have infinitely many real and integral solutions. One can easily understand that diophantine equations offer an unlimited field for research by reason of their variety [1,2].

It is worth mentioning here, that, the pell equation is a class of non-homogenous binary quadratic diophantine equation of the form $y^2 = Dx^2 \pm 1$, where $D > 0$ and square free integer .In particular ,the binary quadratic diophantine equation having the form $y^2 = Dx^2 + N, (N > 0)$ is referred as the positive pell equation and the equation of the form $y^2 = Dx^2 - N, (N > 0)$ is known as negative pell equation. In this context, one may refer [3-8].

This paper has two section A and B. In section A, the positive pell equation $y^2 = 30x^2 + 19$ is considered for obtaining its non-zero distinct integer solutions and section B illustrates the process of determining non-zero distinct integer solution to the negative pell equation $y^2 = 12x^2 - 44$.

A few interesting relations among the solutions for the above two pell equations are presented. Also, employing the integer solutions of the equations under consideration, integer solutions for other choices of hyperbola and parabola are exhibited along with the corresponding pell equations. Further, a few interesting relation between special polygonal numbers generated through the solutions of the equation under consideration are presented.

II. NOTATIONS

$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$ = polygonal number of rank n with sides m.

$S_n = 6n(n-1) + 1$ = star number of rank n.

III. METHOD OF ANALYSIS

SECTION A: Positive Pell Equation $y^2 = 30x^2 + 19$

The positive pell equation representing hyperbola under consideration is

$y^2 = 30x^2 + 19$ (1.1)

The smallest positive integer solutions of (1.1) are

$x_0 = 1, y_0 = 7$

The obtain the other solutions of (1.1), consider the pellian equation

$y^2 = 30x^2 + 1$ (1.2)

whose initial solution is given by

$\tilde{x}_0 = 2, \tilde{y}_0 = 11$

The general solution $(\tilde{x}_n, \tilde{y}_n)$ of (1.2) is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{30}} g_n$$

$$\tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = \left(11 + 2\sqrt{30}\right)^{n+1} + \left(11 - 2\sqrt{30}\right)^{n+1}$$
$$g_n = \left(11 + 2\sqrt{30}\right)^{n+1} - \left(11 - 2\sqrt{30}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1.1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{7}{2\sqrt{30}} g_n$$

$$y_{n+1} = \frac{7}{2} f_n + \frac{30}{2\sqrt{30}} g_n$$

The recurrence relations satisfied by the values of x and y are respectively.

$$x_{n+1} - 22x_{n+2} + x_{n+3} = 0, n = 1,2,3......$$
$$y_{n+1} - 22x_{n+1} + x_{n+3} = 0, n = 1,2,3.......$$

Some numerical examples of x_n and y_n satisfying (1.1) are given in the Table 1 below:

Table: 1 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	7
0	25	137
1	549	3007
2	12053	66017
3	264617	1449367

From the above table, we observe some interesting relations among the solutions which are presented below:

- both x_{n+1} and y_{n+1} values are odd.
- One can generate second order Ramanujan number by choosing x and y values suitably. A few illustrations are given below.

Illustration: 1

$x_2 = 549$

$= 1*549 = 3*183 = 9*61$

$\Rightarrow 275^2 - 274^2 = 93^2 - 90^2 = 35^2 - 26^2$

That is,

$275^2 - 274^2 = 93^2 - 90^2 \Rightarrow 275^2 + 90^2 = 93^2 + 274^2$

$\Rightarrow 83725$

$275^2 - 274^2 = 35^2 - 26^2 \Rightarrow 275^2 + 26^2 = 35^2 + 274^2$

$\Rightarrow 76301$

$93^2 - 90^2 = 35^2 - 26^2 \Rightarrow 93^2 + 26^2 = 35^2 + 90^2$

$\Rightarrow 9325$

Thus, 83725,76301,9325 are second order Ramanujan number.

Illustration: 2

$x_3 = 12053$

$= 1*12053 = 17*709$

$\Rightarrow 6027^2 - 6026^2 = 363^2 - 346^2$

$\Rightarrow 6027^2 + 346^2 = 363^2 + 6026^2$

$36444445 = 36444445$

Thus, 36444445 is second order Ramanujan number.

Illustration: 3

$y_3 = 66017$

$= 1*66017 = 7*9431$

$\Rightarrow 33009^2 - 33008^2 = 4719^2 - 4712^2$

$\Rightarrow 33009^2 + 4712^2 = 4719^2 + 33008^2$

1111797025 = 1111797025

Thus, 1111797025 is second order Ramanujan number

- Let $\{u_{n+1}\}$ and $\{v_{n+1}\}$ be two sequences of positive integer defined by

$u_{n+1} = \frac{x_{n+1}+1}{2}$ and $v_{n+1} = \frac{y_{n+1}-1}{2}$

Note that

- $t_{3,v_{n+1}} = t_{242,u_{n+1}} - u_{n+1} + 49$
- $t_{3,v_{n+1}} = 20S_{u_{n+1}} + 29$
- $t_{3,v_{n+1}} = t_{122,u_{n+1}} + t_{110,u_{n+1}} + 6u_{n+1}^2 - 8u_{n+1} + 49$
- $t_{3,v_{n+1}} - t_{122,u_{n+1}} - t_{110,u_{n+1}} + 8u_{n+1} - 49$ is a nasty number

1. Relations among the solutions are given below.

- $60x_{n+3} + y_{n+2} - 11y_{n+3} = 0$
- $60x_{n+2} + 11y_{n+2} - y_{n+3} = 0$
- $60x_{n+1} + 241y_{n+2} - 11y_{n+3} = 0$
- $1320x_{n+3} + y_{n+1} - 241y_{n+3} = 0$
- $1320x_{n+1} + 241y_{n+1} - y_{n+3} = 0$
- $60x_{n+3} + 11y_{n+1} - 241y_{n+2} = 0$
- $60x_{n+2} + y_{n+1} - 11y_{n+2} = 0$
- $60x_{n+1} + 11y_{n+1} - y_{n+2} = 0$
- $x_{n+2} + 2y_{n+3} - 11x_{n+3} = 0$
- $x_{n+1} + 44y_{n+3} - 241x_{n+3} = 0$
- $11x_{n+2} + 2y_{n+2} - x_{n+3} = 0$
- $241x_{n+2} + 2y_{n+1} - 11x_{n+3} = 0$
- $120x_{n+2} + y_{n+1} - y_{n+3} = 0$
- $241x_{n+1} + 44y_{n+1} - x_{n+3} = 0$
- $11x_{n+1} + 2y_{n+3} - 241x_{n+2} = 0$
- $x_{n+1} + 2y_{n+2} - 11x_{n+2} = 0$
- $11x_{n+1} + 2y_{n+1} - x_{n+2} = 0$
- $x_{n+1} + 4y_{n+2} - x_{n+3} = 0$

2. Each of the following expressions is a nasty number:

- $\frac{3}{209}[7x_{2n+4} - 3007x_{2n+2} + 836]$
- $\frac{6}{19}[7x_{2n+3} - 137x_{2n+2} + 38]$
- $\frac{12}{19}[7y_{2n+2} - 30x_{2n+2} + 19]$
- $\frac{12}{209}[7y_{2n+3} - 750x_{2n+2} + 209]$
- $\frac{12}{4579}[7y_{2n+4} - 16470x_{2n+2} + 4579]$
- $\frac{6}{19}[137x_{2n+4} - 3007x_{2n+3} + 38]$
- $\frac{12}{209}[137y_{2n+2} - 30x_{2n+3} + 209]$
- $\frac{12}{19}[137y_{2n+3} - 750x_{2n+3} + 19]$
- $\frac{12}{209}[137y_{2n+4} - 16470y_{2n+3} + 209]$
- $\frac{12}{4579}[3007y_{2n+2} - 30x_{2n+4} + 4579]$

- $\frac{12}{209}\left[3007y_{2n+3}-750x_{2n+4}+209\right]$
- $\frac{12}{19}\left[3007y_{2n+4}-16470x_{2n+4}+19\right]$
- $\frac{6}{19}\left[25y_{2n+2}-y_{2n+3}+38\right]$
- $\frac{3}{209}\left[549y_{2n+2}-y_{2n+4}+836\right]$
- $\frac{6}{19}\left[549y_{2n+3}-25y_{2n+4}+38\right]$

3. Each of the following expressions is a cubical integer:

- $\frac{1}{418}\left[7x_{3n+5}-3007x_{3n+3}+21x_{n+3}-9021x_{n+1}\right]$
- $\frac{1}{19}\left[7x_{3n+4}-137x_{3n+3}+12x_{n+2}-411x_{n+1}\right]$
- $\frac{2}{19}\left[7y_{3n+3}-30x_{3n+3}+21y_{n+1}-90x_{n+1}\right]$
- $\frac{2}{209}\left[7y_{3n+4}-750x_{3n+3}+21y_{n+2}-2250x_{n+1}\right]$
- $\frac{2}{4579}\left[7y_{3n+5}-16470x_{3n+3}+21y_{n+3}-49410x_{n+1}\right]$
- $\frac{1}{19}\left[137x_{3n+5}-3007x_{3n+4}+411x_{n+3}-9021x_{n+2}\right]$
- $\frac{2}{209}\left[137y_{3n+3}-30x_{3n+4}+411y_{n+1}-90x_{n+2}\right]$
- $\frac{2}{19}\left[137y_{3n+4}-750x_{3n+4}+411y_{n+2}-2250y_{n+2}\right]$
- $\frac{2}{209}\left[137y_{3n+5}-16470x_{3n+4}+411y_{n+3}-49410x_{n+2}\right]$
- $\frac{2}{4579}\left[3007y_{3n+3}-30x_{3n+5}+9021y_{n+1}-90x_{n+3}\right]$
- $\frac{2}{209}\left[3007y_{3n+4}-750x_{3n+5}+9021y_{n+2}-2250x_{n+3}\right]$
- $\frac{2}{19}\left[3007y_{3n+5}-16470x_{3n+5}+9021y_{n+3}-49410x_{n+3}\right]$
- $\frac{1}{19}\left[25y_{3n+3}-y_{3n+4}+75y_{n+1}-3y_{n+2}\right]$
- $\frac{1}{418}\left[549y_{3n+3}-y_{3n+5}+1647y_{n+1}-3y_{n+3}\right]$
- $\frac{1}{19}\left[549y_{3n+4}-25y_{3n+5}+1647y_{n+2}-750y_{n+3}\right]$

4. Each of the following expressions is a bi-quadratic integer:

- $\frac{1}{418}\left[7x_{4n+6}-3007x_{4n+4}+28x_{2n+4}-12028x_{2n+2}+2508\right]$
- $\frac{1}{19}\left[7x_{4n+5}-137x_{4n+4}+28x_{2n+3}-548x_{2n+2}+114\right]$
- $\frac{2}{19}\left[7y_{4n+4}-30x_{4n+4}+28y_{2n+2}-120x_{2n+2}+57\right]$
- $\frac{2}{209}\left[7y_{4n+5}-750x_{4n+4}+28y_{2n+3}-3000x_{2n+2}+627\right]$
- $\frac{2}{4579}\left[7y_{4n+6}-16470x_{4n+4}+28y_{2n+4}-65880x_{2n+2}+13737\right]$
- $\frac{1}{19}\left[137x_{4n+6}-3007x_{4n+5}+548x_{2n+4}-12028x_{2n+3}+114\right]$
- $\frac{2}{209}\left[137y_{4n+4}-30x_{4n+5}+548y_{2n+2}-120x_{2n+3}+627\right]$

- $\frac{2}{19}\left[137y_{4n+5}-750x_{4n+5}+548y_{2n+3}-3000x_{2n+3}+57\right]$
- $\frac{2}{209}\left[137y_{4n+6}-16470x_{4n+5}+548y_{2n+4}-65880x_{2n+3}+627\right]$
- $\frac{2}{4579}\left[3007y_{4n+4}-30x_{4n+6}+12028y_{2n+2}-120x_{2n+4}+1237\right]$
- $\frac{2}{209}\left[3007y_{4n+5}-750x_{4n+6}+12028y_{2n+3}-3000x_{2n+4}+627\right]$
- $\frac{2}{19}\left[3007y_{4n+6}-16470x_{4n+6}+12028y_{2n+4}-65880x_{2n+4}+57\right]$
- $\frac{1}{19}\left[25y_{4n+4}-y_{4n+5}+100y_{2n+2}-4y_{2n+3}+114\right]$
- $\frac{1}{418}\left[549y_{4n+4}-y_{4n+6}+2196y_{2n+2}-4y_{2n+4}+2508\right]$
- $\frac{1}{19}\left[549y_{4n+5}-25y_{4n+6}+2196y_{2n+3}-100y_{2n+4}+114\right]$

5. Each of the following expressions is a quintic integer:

- $\frac{1}{418}\left[7x_{5n+7}-3007x_{5n+5}+35x_{3n+5}-15035x_{3n+3}+70x_{n+3}-30070x_{n+1}\right]$
- $\frac{1}{19}\left[7x_{5n+6}-137x_{5n+5}+35x_{3n+4}-685x_{3n+3}+70x_{n+2}-1370x_{n+1}\right]$
- $\frac{2}{19}\left[7y_{5n+5}-30x_{5n+5}+35y_{3n+3}-150x_{3n+3}+70y_{n+1}-300x_{n+1}\right]$
- $\frac{2}{209}\left[7y_{5n+6}-750x_{5n+5}+35y_{3n+4}-3750x_{3n+3}+70y_{n+2}-7500x_{n+1}\right]$
- $\frac{2}{4579}\left[7x_{5n+7}-16470x_{5n+5}+35y_{3n+5}-82350x_{3n+3}+70y_{n+3}-164700x_{n+1}\right]$
- $\frac{1}{19}\left[137x_{5n+7}-3007x_{5n+6}+685x_{3n+5}-15035x_{3n+4}+1370x_{n+3}-30070x_{n+2}\right]$
- $\frac{2}{209}\left[137y_{5n+5}-30x_{5n+6}+685y_{3n+3}-150x_{3n+4}+1370y_{n+1}-300x_{n+2}\right]$
- $\frac{2}{19}\left[137y_{5n+6}-750x_{5n+6}+685y_{3n+4}-3750x_{3n+4}+1370y_{n+2}-7500x_{n+2}\right]$
- $\frac{2}{209}\left[137y_{5n+7}-16470x_{5n+6}+685y_{3n+5}-82350x_{3n+4}+1370y_{n+3}-164700x_{n+2}\right]$
- $\frac{2}{4579}\left[3007y_{5n+5}-30x_{5n+7}+15035y_{3n+3}-150x_{3n+5}+30070y_{n+1}-300x_{n+3}\right]$
- $\frac{2}{209}\left[3007y_{5n+6}-750x_{5n+7}+15035y_{3n+4}-3750x_{3n+5}+30070y_{n+2}-7500x_{n+3}\right]$
- $\frac{2}{19}\left[3007y_{5n+7}-16470x_{5n+7}+15035y_{3n+5}-82350x_{3n+5}+30070y_{n+3}-164700x_{n+3}\right]$
- $\frac{1}{19}\left[25y_{5n+5}-y_{5n+6}+125y_{3n+3}-5y_{3n+4}+7500y_{n+1}-300y_{n+2}\right]$
- $\frac{1}{418}\left[16470y_{5n+5}-y_{5n+7}+82350y_{3n+3}-5y_{3n+5}+5490y_{n+1}-10y_{n+3}\right]$
- $\frac{1}{19}\left[549y_{5n+6}-25y_{5n+7}+2745y_{3n+4}-125y_{3n+5}+5490y_{n+2}-250y_{n+3}\right]$

REMARKABLE OBSERVATIONS

I. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 2 below:

Table: 2 Hyperbolas

S.NO	Hyperbolas	(X,Y)
1	$Y^2 - 30X^2 = 698896$	$\begin{pmatrix} x_{n+3} - 549x_{n+1}, \\ 7x_{n+3} - 3007x_{n+1} \end{pmatrix}$
2	$Y^2 - 30X^2 = 1444$	$\begin{pmatrix} 14x_{n+1} - 2y_{n+1}, \\ 14y_{n+1} - 60x_{n+1} \end{pmatrix}$
3	$Y^2 - 30X^2 = 174724$	$\begin{pmatrix} 274x_{n+1} - 2y_{n+2}, \\ 14y_{n+2} - 1500x_{n+1} \end{pmatrix}$
4	$Y^2 - 30X^2 = 83868964$	$\begin{pmatrix} 6014x_{n+1} - 2y_{n+3}, \\ 14y_{n+3} - 32940x_{n+1} \end{pmatrix}$
5	$Y^2 - 30X^2 = 5776$	$\begin{pmatrix} 1098x_{n+2} - 50x_{n+3}, \\ 274x_{n+3} - 6014x_{n+2} \end{pmatrix}$
6	$Y^2 - 30X^2 = 174724$	$\begin{pmatrix} 14x_{n+2} - 50y_{n+1}, \\ 274y_{n+1} - 60x_{n+2} \end{pmatrix}$
7	$Y^2 - 30X^2 = 1444$	$\begin{pmatrix} 274x_{n+2} - 50y_{n+2}, \\ 274y_{n+2} - 1500x_{n+2} \end{pmatrix}$
8	$Y^2 - 30X^2 = 174724$	$\begin{pmatrix} 6014x_{n+2} - 50y_{n+3}, \\ 274y_{n+3} - 32940x_{n+2} \end{pmatrix}$
9	$Y^2 - 30X^2 = 83868964$	$\begin{pmatrix} 14x_{n+3} - 1098y_{n+1}, \\ 6014y_{n+1} - 60x_{n+3} \end{pmatrix}$
10	$Y^2 - 30X^2 = 174724$	$\begin{pmatrix} 274x_{n+3} - 1098y_{n+2}, \\ 6014y_{n+2} - 1500x_{n+3} \end{pmatrix}$
11	$Y^2 - 30X^2 = 1444$	$\begin{pmatrix} 6014x_{n+3} - 1098y_{n+3}, \\ 6014y_{n+3} - 32940x_{n+3} \end{pmatrix}$
12	$Y^2 - 30X^2 = 5198400$	$\begin{pmatrix} 14y_{n+2} - 274y_{n+1}, \\ 1500y_{n+1} - 60y_{n+3} \end{pmatrix}$
13	$Y^2 - 30X^2 = 2516025600$	$\begin{pmatrix} 14y_{n+3} - 6014y_{n+1}, \\ 32940y_{n+1} - 60y_{n+3} \end{pmatrix}$
14	$Y^2 - 30X^2 = 5198400$	$\begin{pmatrix} 274y_{n+3} - 6014y_{n+2}, \\ 32940y_{n+2} - 1500y_{n+3} \end{pmatrix}$
15	$Y^2 - 30X^2 = 1444$	$\begin{pmatrix} 25x_{n+1} - x_{n+2}, \\ 7x_{n+2} - 137x_{n+1} \end{pmatrix}$

II. Employing linear combinations among the solutions of (1.1), one may generate integer solutions for other choices of parabolas which are presented in Table 3 below:

Table: 3 Parabolas

S. NO	Parabolas	(X,Y)
1	$209Y - 15X^2 = 349448$	$\begin{pmatrix} x_{n+3} - 549x_{n+1}, \\ 7x_{2n+4} - 3007x_{2n+2} + 836 \end{pmatrix}$
2	$19Y - 30X^2 = 1444$	$\begin{pmatrix} 14x_{n+1} - 2y_{n+1}, \\ 14y_{2n+2} - 60x_{2n+2} + 38 \end{pmatrix}$
3	$209Y - 30X^2 = 174724$	$\begin{pmatrix} 274x_{n+1} - 2y_{n+2}, \\ 14y_{2n+3} - 1500x_{2n+2} + 418 \end{pmatrix}$
4	$4579Y - 30X^2 = 83868964$	$\begin{pmatrix} 6014x_{n+1} - 2y_{n+3}, \\ 14y_{2n+4} - 32940x_{2n+2} + 9158 \end{pmatrix}$

5	$19Y-15X^2=2888$	$\begin{pmatrix}1098x_{n+2}-50x_{n+3}, \\ 274x_{2n+4}-6014x_{2n+3}+76\end{pmatrix}$
6	$209Y-30X^2=174724$	$\begin{pmatrix}14x_{n+2}-50y_{n+1}, \\ 274y_{2n+2}-60x_{2n+3}+418\end{pmatrix}$
7	$19Y-30X^2=1444$	$\begin{pmatrix}274x_{n+2}-50y_{n+2}, \\ 274y_{2n+3}-1500x_{2n+3}+38\end{pmatrix}$
8	$209Y-30X^2=174724$	$\begin{pmatrix}6014x_{n+2}-50y_{n+3}, \\ 274y_{2n+4}-32940x_{2n+3}+418\end{pmatrix}$
9	$4579Y-30X^2=83868964$	$\begin{pmatrix}14x_{n+3}-1098y_{n+1}, \\ 6014y_{2n+2}-60x_{2n+4}+9158\end{pmatrix}$
10	$209Y-30X^2=174724$	$\begin{pmatrix}274x_{n+3}-1098y_{n+2}, \\ 6014y_{2n+3}-1500x_{2n+4}+418\end{pmatrix}$
11	$19Y-30X^2=1444$	$\begin{pmatrix}6014x_{n+3}-1098y_{n+3}, \\ 6014y_{2n+4}-32940x_{2n+4}+38\end{pmatrix}$
12	$38Y-X^2=173280$	$\begin{pmatrix}14y_{n+2}-274y_{n+1}, \\ 1500y_{2n+2}-60y_{2n+3}+2280\end{pmatrix}$
13	$836Y-X^2=83867520$	$\begin{pmatrix}14y_{n+3}-6014y_{n+1}, \\ 32940y_{2n+2}-60y_{2n+4}+50160\end{pmatrix}$
14	$38Y-X^2=173280$	$\begin{pmatrix}274y_{n+3}-6014y_{n+2}, \\ 32940y_{2n+3}-1500y_{2n+4}+2280\end{pmatrix}$
15	$19Y-30X^2=1444$	$\begin{pmatrix}25x_{n+1}-x_{n+2}, \\ 7x_{2n+3}-137x_{2n+2}+38\end{pmatrix}$

SECTION B: Negative Pell Equation $y^2=12x^2-44$

The negative pell equation representing hyperbola under consideration is

$y^2=12x^2-44$ (2.1)

The smallest positive integer solutions of (2.1) are

$x_0=2, y_0=2$

The obtain the other solutions of (2.1), consider the pellian equation

$y^2=12x^2+1$(2.2)

whose initial solution is given by

$\tilde{x}_0=2, \tilde{y}_0=7$

The general solution $(\tilde{x}_n,\tilde{y}_n)$ of (2.2) is given by

$$\tilde{x}_n=\frac{1}{2\sqrt{12}}g_n$$

$$\tilde{y}_n=\frac{1}{2}f_n$$

where

$$f_n=\left(7+2\sqrt{12}\right)^{n+1}+\left(7-2\sqrt{12}\right)^{n+1}$$

$$g_n=\left(7+2\sqrt{12}\right)^{n+1}-\left(7-2\sqrt{12}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0,y_0) and $(\tilde{x}_n,\tilde{y}_n)$, the other integer solutions of (1.1) are given by

$$x_{n+1}=f_n+\frac{1}{\sqrt{12}}g_n$$

$$y_{n+1}=f_n+\frac{12}{\sqrt{12}}g_n$$

The recurrence relations satisfied by the values of x and y are respectively.

$$x_{n+1}-14x_{n+2}+x_{n+3}=0, n=1,2,3........$$

$$y_{n+1}-14y_{n+2}+y_{n+3}=0, n=1,2,3........$$

Some numerical examples of x_n and y_n satisfying (2.1) are given in the Table 4 below:

Table: 4 Numerical Examples

n	x_{n+1}	y_{n+1}
-1	2	2
0	18	62
1	250	866
2	3482	12062
3	48498	168002

From the above table, we observe some interesting relations among the solutions which are presented below:

- both x_{n+1} and y_{n+1} values are even.
- Let $\{u_{n+1}\}$ and $\{v_{n+1}\}$ be two sequences of positive integers defined by

$u_{n+1} = \frac{x_{n+1}}{2}$ and $v_{n+1} = \frac{y_{n+1}}{2}$

Note that

- $t_{4,v_n+1} - t_{26,u_{n+1}} \equiv 0 \pmod{11}$
- $2t_{4,v_n+1} + 22$ is a nasty number
- $72t_{4,v_n+1} - 66$ is a nasty number
- $S_{v_n+1} + 6v_{n+1} + 65 = t_{146,u_n+1} + 71u_{n+1}$

1. Relations among the solutions are given below.

- $24x_{n+3} - 7y_{n+3} + y_{n+2} = 0$
- $24x_{n+2} - y_{n+3} + 7y_{n+2} = 0$
- $7y_{n+3} - 24x_{n+1} - 97y_{n+2} = 0$
- $x_{n+2} - 7x_{n+3} + 2y_{n+3} = 0$
- $x_{n+1} - 97x_{n+3} + 28y_{n+3} = 0$
- $7x_{n+2} - x_{n+3} + 2y_{n+2} = 0$
- $97y_{n+2} - 24x_{n+3} - 7y_{n+1} = 0$
- $x_{n+1} - x_{n+3} + 4y_{n+2} = 0$
- $7x_{n+1} - 97x_{n+2} + 2y_{n+3} = 0$
- $24x_{n+2} - 7y_{n+2} + 7y_{n+1} = 0$
- $x_{n+1} - 7x_{n+2} + 2y_{n+2} = 0$
- $7x_{n+3} - 97x_{n+2} - 2y_{n+1} = 0$
- $97y_{n+3} - 336x_{n+3} - y_{n+1} = 0$
- $336x_{n+2} - 7y_{n+3} + 7y_{n+1} = 0$
- $336x_{n+1} - y_{n+3} + 97y_{n+1} = 0$
- $24x_{n+1} - y_{n+2} + 7y_{n+1} = 0$
- $2y_{n+1} - x_{n+2} + 7x_{n+1} = 0$
- $x_{n+3} - 97x_{n+1} - 28y_{n+1} = 0$

2. Each of the following expressions is a nasty number:

- $\frac{3}{11}[31x_{2n+2} - x_{2n+3} + 44]$
- $\frac{6}{11}[12x_{2n+2} - y_{2n+2} + 22]$
- $\frac{3}{154}[433x_{2n+2} - x_{2n+4} + 616]$

- $\frac{6}{77}\left[108x_{2n+2} - y_{2n+3} + 154\right]$
- $\frac{6}{1067}\left[1500x_{2n+2} - y_{2n+4} + 2134\right]$
- $\frac{1}{77}\left[72x_{2n+3} - 186y_{2n+2} + 924\right]$
- $\frac{6}{1067}\left[12x_{2n+4} - 433y_{2n+2} - 2134\right]$
- $\frac{1}{44}\left[12y_{2n+3} - 108y_{2n+2} + 528\right]$
- $\frac{1}{616}\left[12y_{2n+4} - 1500y_{2n+2} + 7392\right]$
- $\frac{3}{11}\left[433x_{2n+3} - 31x_{2n+4} + 44\right]$
- $\frac{6}{11}\left[108x_{2n+3} - 31y_{2n+3} + 22\right]$
- $\frac{6}{77}\left[1500x_{2n+3} - 31y_{2n+4} + 154\right]$
- $\frac{6}{77}\left[108x_{2n+4} - 433y_{2n+3} + 154\right]$
- $\frac{6}{11}\left[1500x_{2n+4} - 433y_{2n+4} + 22\right]$
- $\frac{1}{44}\left[108y_{2n+4} - 1500y_{2n+3} + 528\right]$

3. Each of the following expressions is a cubical integer:

- $\frac{1}{22}\left[31x_{3n+3} - x_{3n+4} + 93x_{n+1} - 3x_{n+2}\right]$
- $\frac{1}{11}\left[12x_{3n+3} - y_{3n+3} + 36x_{n+1} - 3y_{n+1}\right]$
- $\frac{1}{308}\left[433x_{3n+3} - x_{3n+5} + 1299x_{n+1} - 3x_{n+3}\right]$
- $\frac{1}{77}\left[108x_{3n+3} - y_{3n+4} + 324x_{n+1} - 3y_{n+2}\right]$
- $\frac{1}{1067}\left[1500x_{3n+3} - y_{3n+5} + 4500x_{n+1} - 3y_{n+3}\right]$
- $\frac{1}{77}\left[12x_{3n+4} - 31x_{3n+3} + 36x_{n+2} - 93y_{n+1}\right]$
- $\frac{1}{1067}\left[12x_{3n+5} - 433y_{3n+3} + 36x_{n+3} - 1299y_{n+1}\right]$
- $\frac{1}{264}\left[12y_{3n+4} - 108y_{3n+3} + 36y_{n+2} - 324y_{n+1}\right]$
- $\frac{1}{3696}\left[12y_{3n+5} - 1500y_{3n+3} + 36y_{n+3} - 4500y_{n+1}\right]$
- $\frac{1}{22}\left[433x_{3n+4} - 31x_{3n+5} + 1299x_{n+2} - 93x_{n+3}\right]$
- $\frac{1}{11}\left[108x_{3n+4} - 31y_{3n+4} + 324x_{n+4} - 93y_{n+2}\right]$
- $\frac{1}{77}\left[1500x_{3n+4} - 31y_{3n+5} + 4500x_{n+2} - 93y_{n+3}\right]$
- $\frac{1}{77}\left[108x_{3n+5} - 433y_{3n+4} + 324x_{n+3} - 1299y_{n+2}\right]$
- $\frac{1}{11}\left[1500x_{3n+5} - 433y_{3n+5} + 4500x_{n+3} - 1299y_{n+3}\right]$
- $\frac{1}{264}\left[108y_{3n+5} - 1500y_{3n+4} + 324y_{n+3} - 4500y_{n+2}\right]$

4. Each of the following expressions is a bi-quadratic integer:

- $\frac{1}{22}\left[31x_{4n+4}-x_{4n+5}+124x_{2n+2}-4x_{2n+3}+132\right]$
- $\frac{1}{11}\left[12x_{4n+4}-y_{4n+4}+48x_{2n+2}-4y_{2n+2}+66\right]$
- $\frac{1}{308}\left[433x_{4n+4}-x_{4n+6}+1732x_{2n+2}-4x_{2n+4}+1848\right]$
- $\frac{1}{77}\left[108x_{4n+4}-y_{4n+5}+432x_{2n+2}-4y_{2n+3}+462\right]$
- $\frac{1}{1067}\left[1500x_{4n+4}-y_{4n+6}+6000x_{2n+2}-4y_{2n+4}+6402\right]$
- $\frac{1}{77}\left[12x_{4n+5}-31x_{4n+4}+48x_{2n+3}-124y_{2n+2}+462\right]$
- $\frac{1}{1067}\left[12x_{4n+6}-433y_{4n+4}+48x_{2n+4}-1732y_{2n+2}+6402\right]$
- $\frac{1}{264}\left[12y_{4n+5}-108y_{4n+4}+48y_{2n+3}-432y_{2n+2}+1584\right]$
- $\frac{1}{3696}\left[12y_{4n+6}-15y_{4n+4}+48y_{2n+4}-6000y_{2n+2}+22176\right]$
- $\frac{1}{22}\left[433x_{4n+5}-31x_{4n+6}+1732x_{2n+3}-124x_{2n+4}+132\right]$
- $\frac{1}{11}\left[108x_{4n+5}-31y_{4n+5}+432x_{2n+3}-124y_{2n+3}+66\right]$
- $\frac{1}{77}\left[1500x_{4n+5}-31y_{4n+6}+6000x_{2n+3}-124y_{2n+4}+462\right]$
- $\frac{1}{77}\left[108x_{4n+6}-433y_{4n+5}+432x_{2n+4}-1732y_{2n+3}+462\right]$
- $\frac{1}{11}\left[1500x_{4n+6}-433y_{4n+6}+6000x_{2n+4}-1732y_{2n+4}+66\right]$
- $\frac{1}{264}\left[108y_{4n+6}-1500y_{4n+5}+432y_{2n+4}-6000y_{2n+3}+1584\right]$

5. Each of the following expressions is a quintic integer:

- $\frac{1}{22}\left[31x_{5n+5}-x_{5n+6}+155x_{3n+3}-5x_{3n+4}+310x_{n+1}-10x_{n+2}\right]$
- $\frac{1}{11}\left[12x_{5n+5}-y_{5n+5}+60x_{3n+3}-5y_{3n+3}+120x_{n+1}-10y_{n+1}\right]$
- $\frac{1}{308}\left[433x_{5n+5}-x_{5n+7}+2165x_{3n+3}-5x_{3n+5}+4330x_{n+1}-10x_{n+3}\right]$
- $\frac{1}{77}\left[108x_{5n+5}-y_{5n+6}+540x_{3n+3}-5y_{3n+4}+1080x_{n+1}-10y_{n+2}\right]$
- $\frac{1}{1067}\left[1500x_{5n+5}-y_{5n+7}+7500x_{3n+3}-5y_{3n+5}+15000x_{n+1}-10y_{n+3}\right]$
- $\frac{1}{77}\left[12x_{5n+6}-31x_{5n+5}+60x_{3n+4}-155y_{3n+3}+120x_{n+2}-310y_{n+1}\right]$
- $\frac{1}{1067}\left[12x_{5n+7}-433y_{5n+5}+60x_{3n+5}-2165y_{3n+3}+120x_{n+3}-4330y_{n+1}\right]$
- $\frac{1}{264}\left[12y_{5n+6}-108y_{5n+5}+60x_{3n+4}-540y_{3n+3}+120y_{n+2}-1080y_{n+1}\right]$
- $\frac{1}{3696}\left[12y_{5n+7}-1500y_{5n+5}+60y_{3n+5}-7500y_{3n+3}+120y_{n+3}-15000y_{n+1}\right]$
- $\frac{1}{22}\left[433x_{5n+6}-31x_{5n+7}+2165x_{3n+4}-155x_{3n+5}+4330x_{n+2}-310x_{n+3}\right]$
- $\frac{1}{11}\left[108x_{5n+6}-31y_{5n+6}+540x_{3n+4}-155y_{3n+4}+1080x_{n+2}-310y_{n+2}\right]$
- $\frac{1}{77}\left[1500x_{5n+6}-31y_{5n+7}+7500x_{3n+4}-155y_{3n+5}+15000x_{n+2}-310y_{n+3}\right]$
- $\frac{1}{77}\left[108x_{5n+7}-433y_{5n+6}+540x_{3n+5}-2165y_{3n+4}+1080x_{n+3}-4330y_{n+2}\right]$

- $\frac{1}{11}\left[1500x_{5n+7} - 433y_{5n+7} + 7500x_{3n+5} - 2165y_{3n+5} + 15000x_{n+3} - 4330y_{n+3}\right]$
- $\frac{1}{264}\left[108y_{5n+7} - 1500y_{5n+6} + 540y_{3n+5} - 7500y_{3n+4} + 1080y_{n+3} - 15000y_{n+2}\right]$

REMARKABLE OBSERVATIONS

III. Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 5 below:

Table: 5 Hyperbolas

S.NO	Hyperbolas	(X,Y)
1	$Y^2 - 12X^2 = 1936$	$\left(x_{n+2} - 9x_{n+1}, 31x_{n+1} - x_{n+2}\right)$
2	$Y^2 - 12X^2 = 379456$	$\left(x_{n+3} - 125x_{n+1}, 433x_{n+1} - x_{n+3}\right)$
3	$Y^2 - 12X^2 = 23716$	$\left(y_{n+2} - 31x_{n+1}, 108x_{n+1} - y_{n+2}\right)$
4	$Y^2 - 12X^2 = 4553956$	$\left(y_{n+3} - 433x_{n+1}, 1500x_{n+1} - y_{n+3}\right)$
5	$Y^2 - 12X^2 = 23716$	$\left(9x_{n+1} - x_{n+2}, 12x_{n+2} - 31y_{n+1}\right)$
6	$Y^2 - 12X^2 = 4553956$	$\left(125y_{n+1} - x_{n+3}, 12x_{n+3} - 433y_{n+1}\right)$
7	$Y^2 - 12X^2 = 278784$	$\left(31y_{n+1} - y_{n+2}, 12y_{n+2} - 108y_{n+1}\right)$
8	$Y^2 - 12X^2 = 54641664$	$\left(433y_{n+1} - y_{n+3}, 12y_{n+3} - 1500y_{n+1}\right)$
9	$Y^2 - 12X^2 = 1936$	$\left(9x_{n+3} - 125x_{n+2}, 433x_{n+2} - 31x_{n+3}\right)$
10	$Y^2 - 12X^2 = 484$	$\left(9y_{n+2} - 31x_{n+2}, 108x_{n+2} - 31y_{n+2}\right)$
11	$Y^2 - 12X^2 = 23716$	$\left(9y_{n+3} - 433x_{n+3}, 1500x_{n+2} - 31y_{n+3}\right)$
12	$Y^2 - 12X^2 = 23716$	$\left(125y_{n+2} - 31x_{n+3}, 108x_{n+3} - 433y_{n+2}\right)$
13	$Y^2 - 12X^2 = 484$	$\left(125y_{n+3} - 433x_{n+3}, 1500x_{n+3} - 433y_{n+3}\right)$
14	$Y^2 - 12X^2 = 278784$	$\left(433y_{n+2} - 31y_{n+3}, 108y_{n+3} - 1500y_{n+2}\right)$
15	$Y^2 - 12X^2 = 484$	$\left(y_{n+1} - x_{n+1}, 12x_{n+1} - y_{n+1}\right)$

IV. Employing linear combinations among the solutions of (2.1), one may generate integer solutions for other choices of parabolas which are presented in Table 6 below:

Table: 6 Parabolas

S. NO	Parabolas	(X,Y)
1	$22Y-12X^2=1936$	$\left(x_{n+2}-9x_{n+1}, 31x_{2n+2}-x_{2n+3}+44\right)$
2	$308Y-12X^2=379456$	$\left(x_{n+3}-125x_{n+1}, 433x_{2n+2}-x_{2n+4}+616\right)$
3	$77Y-12X^2=23716$	$\left(y_{n+2}-31x_{n+1}, 108x_{2n+2}-y_{2n+3}+154\right)$
4	$1067Y-12X^2=4553956$	$\left(y_{n+3}-433x_{n+1}, 1500x_{2n+2}-y_{2n+4}+2134\right)$
5	$77Y-12X^2=23716$	$\left(9y_{n+1}-x_{n+2}, 12x_{2n+3}-31x_{2n+2}+154\right)$
6	$1067Y-12X^2=455396$	$\left(125y_{n+1}-x_{n+3}, 12x_{2n+4}-433y_{2n+2}+2134\right)$
7	$264Y-12X^2=278784$	$\left(31y_{n+1}-y_{n+2}, 12y_{2n+3}-108y_{2n+2}+528\right)$
8	$3696Y-12X^2=54641664$	$\left(433y_{n+1}-y_{n+3}, 12y_{2n+4}-1500y_{2n+2}+7392\right)$
9	$22Y-12X^2=1936$	$\left(9x_{n+3}-125x_{n+2}, 433x_{2n+3}-31x_{2n+4}+44\right)$
10	$11Y-12X^2=484$	$\left(9y_{n+2}-31x_{n+2}, 108x_{2n+3}-31y_{2n+3}+22\right)$
11	$77Y-12X^2=23716$	$\left(9y_{n+3}-433x_{n+2}, 1500x_{2n+3}-31y_{2n+4}+154\right)$
12	$77Y-12X^2=23716$	$\left(125y_{n+2}-31x_{n+3}, 108x_{2n+4}-433y_{2n+3}+154\right)$
13	$11Y-12X^2=484$	$\left(125y_{n+3}-433x_{n+3}, 1500x_{2n+4}-433y_{2n+4}+22\right)$
14	$264Y-12X^2=278784$	$\left(433y_{n+2}-31y_{n+3}, 108y_{2n+4}-1500y_{2n+3}+528\right)$
15	$11Y-12X^2=484$	$\left(y_{n+1}-x_{n+1}, 12x_{2n+2}-y_{2n+2}+22\right)$

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