APPLICATION OF q -SPECIAL FUNCTIONS TO PROBABILITY DISTRIBUTIONS

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Abstract:In this Paper we have obtained various form of q-analogue of distribution function. This Paper deal with various applications of special function in probability distributions .We discussed about characterization of distribution function. We find the relation between hyper geometric series and Moment generating function.

IndexTerms - q-special function, distribution function, Moment generating function.

INTRODUCTION

Some functions play important role in different area of Mathematics. There is a large theory of special functions which developed out of Mathematical physics and Statistics. Special function and q series are very active areas of research. In Mathematics, the error function is a special function of sigmoid shape that occurs in probability, statistics and partial differential equations describing diffusion.

Definition 1.1 Exponential Distribution – Exponential distribution is continuous distribution. If X is random variable with parameter $n > 0$,Then

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 \ ; otherwise \end{cases} \qquad [1]$$

Definition 1.2 Gamma Function –The Gamma function Γn is defined for n>0 as

$$\Gamma\text{n} = \int_0^{\infty} x^{n-1} e^{-x} \, dx \qquad [2]$$

Equation solve by integrating by parts, Then we have

$$\Gamma(\text{n}+1) = \text{n}\,\Gamma\text{n}$$

Definition 1.3 Gamma Distribution -A Gamma distribution is a statistical distribution. It is a continuous random variable X is said to have a gamma distribution with two parameter $\alpha > 0$ and $\lambda > 0$, Shown as $f_X(x)$ is given by

$$\text{Gamma}\ (\alpha,\lambda) = f_X(x) = \begin{cases} \dfrac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma\alpha} & ; x > 0 \\ 0 \ ; otherwise \end{cases} \qquad [3]$$

If we assume $\alpha = 1$, we obtain $$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 \ ; otherwise \end{cases} \qquad [4]$$

Then result is Gamma $(1, \lambda)$ = Exponential (λ)

Moment Generating Function of Gamma Distribution

The moment Generating Function is defined by

$$M(t) = E(e^t X) = \int_0^{\infty} e^{(t-\lambda)x} \frac{\lambda^{\alpha}}{\Gamma\alpha} x^{\alpha-1} dx \qquad [5]$$

If $\lambda = 1$,Then

$$M(t) = E(e^t X) = \int_0^{\infty} e^{(t-1)x} \frac{x^{\alpha-1}}{\Gamma\alpha} dx \qquad [6]$$

Definition 1.4 Beta Function -The beta function is also called Euler integral of first kind it is defined by

$$B\ (t,s) = \int_0^1 x^{t-1} (1-x)^{s-1} \, \text{dx} \qquad \text{for p >0 , q>0 .} \qquad [7]$$

Definition 1.5 Beta Distribution – The general formula for the probability density function of the Beta distribution is

$$\boldsymbol{f(x)} = \frac{(\boldsymbol{x-a})^{\boldsymbol{t-1}} (\boldsymbol{b-x})^{\boldsymbol{s-1}}}{\boldsymbol{B(t,s)(b-a)^{t+s-1}}} \qquad a \le x \le b \,; t,s > 0 \qquad [8]$$

Where t and s are the parameters, a and b is lower and upper bounds. in above B(t,s) is beta function define by[1.4].

If a =0 and b=1 is called standard beta distribution. Then Standard beta distribution and is also known as beta distribution of first kind is

$$f(x) = \frac{x^{t-1}(1-x)^{s-1}}{B(t,s)} \qquad 0 \le x \le 1 ; t,s > 0 \qquad [9]$$

A continuous random variable X is said to have a beta distribution of the second kind with parameter t and s. Then defined by

$$f(x) = \frac{(x)^{s-1}}{B(t,s)(1+x)^{t+s}}, \ 0 \le x < \infty, \ t,s > 0 \qquad [10]$$

And its probability distribution function is given by

$$f(x) = \int_0^\infty \frac{(x)^{s-1}}{B(t,s)(1+x)^{t+s}} dx \qquad [11]$$

Definition 1.6 q Gamma Function

The q-gamma function shown by $\Gamma_q(x)$ is an extension of the factorial function to real numbers.Jackson [1] defined q –analogue of the gamma function by

$|q| < 1 \ \ x \neq 0, -1, -2 \dots \dots$

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q;q)_\infty}{(q^x;q)_\infty} \qquad [12]$$

By [3] q-gamma function rewrite in this form

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}} \qquad [13]$$

It is well known that it satisfies

$$\Gamma_q(n) = [n-1]_q!$$

It has the following q-integral representation [3].

$$\Gamma_q(n) = \int_0^{\frac{1}{1-q}} x^{n-1} E_q^{-qx} d_q x \qquad [14]$$

Although equation (19) has been recently established by Koornwinder [4] the factor $\frac{1}{1-q}$ was traditionally omitted yielding a divergent Jackson integral.

Definition 1.7 The q-Beta function is defined

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1} d_q x = \int_0^1 x^{t-1} \frac{(xq;q)_\infty}{(xq^s;q)_\infty} d_q x \qquad [15]$$

$\Re(s) > 0$, $\Re(t) > 0$.

And we have the relation of beta and gamma function is

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} \qquad [16]$$

Definition 1.8

Diaz et al. in[2] defined the gamma q distribution and beta q distribution

The gamma q-distribution in $[0, \frac{1}{1-q}]$ is

$$\gamma_{q,a}(x) = \frac{x^{a-1} E_q^{-qx}}{\Gamma_q(a)} 1_{[0,\frac{1}{1-q}]}(x) \qquad [17]$$

The beta q-distribution in [0,1] by

$$\beta_q \quad (t,s)(x) = \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)} 1_{[0,1]}(x) \qquad [18]$$

Main Result

q analogy of Gamma and Beta Distribution

Definition-1.9 If X be a continuous random variable , it is said to have a q-Gamma distribution with parameters $\lambda > 0$ then it is probability density function is defined by

$$f_{X,q}(x) = \begin{cases} \frac{(\lambda)^{\alpha} x^{\alpha-1} e_q^{-\lambda q x}}{\Gamma_q \alpha} & ; x > 0 \\ 0 \ ; otherwise \end{cases} \qquad [19]$$

and its distribution function is defined by

$$F_{X,q}(x) = \int_0^{\frac{1}{1-q}} \frac{(\lambda)^{\alpha} x^{\alpha-1} e_q^{-\lambda q x}}{\Gamma_q \alpha} d_q x \qquad [20]$$

Theorem 1.The q gamma distribution is the probability distribution that is area under curve is unity.

Proof-By the definition of q gamma function and q analogy of gamma distribution

$$\int_0^\infty f_{X,q}(x)\, d_q x = \int_0^\infty \frac{(\lambda)^\alpha x^{\alpha-1} e_q^{-\lambda q x}}{\Gamma_q \alpha} d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^\infty x^{\alpha-1} e_q^{-\lambda q x}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^{\frac{1}{1-q}} x^{\alpha-1} e_q^{-\lambda q x}\, d_q x$$

Where $\lim_{q\to 1} \frac{1}{1-q} = \infty$,when parameter $\lambda = 1$,Then

$$\int_0^\infty f_{X,q}(x)\, d_q x = \frac{1}{\Gamma_q \alpha} \int_0^{\frac{1}{1-q}} x^{\alpha-1} e_q^{-qx}\, d_q x = \frac{\Gamma_q \alpha}{\Gamma_q \alpha} = 1 \qquad [21]$$

Theorem 2. The Mean and Var of q-gamma distribution is equal to parameter $[\alpha]_q$.

Proof –Mean of the q-gamma distribution is

$$\overline{X} = E_q(X) = \int_0^\infty x\, f_{X,q}(x)\, d_q x$$

$$= \int_0^\infty x\, \frac{(\lambda)^\alpha x^{\alpha-1} e_q^{-\lambda q x}}{\Gamma_q \alpha}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^\infty x \,.\, x^{\alpha-1} e_q^{-\lambda q x}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^{\frac{1}{1-q}} x^{\alpha} e_q^{-\lambda q x}\, d_q x$$

Where $\lim_{q\to 1} \frac{1}{1-q} = \infty$,when parameter $\lambda = 1$,Then

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \cdot \frac{\Gamma_q \alpha+1}{(\lambda)^{\alpha+1}} = \frac{[\alpha]_q\, \Gamma_q \alpha}{\lambda\, \Gamma_q \alpha}$$

$$= \frac{[\alpha]_q}{\lambda} = [\alpha]_q \qquad [22]$$

Now we find $E_q(X^2)$

$$E_q(X^2) = \int_0^\infty x^2\, f_{X,q}(x)\, d_q x$$

$$= \int_0^{\frac{1}{1-q}} x^2\, \frac{(\lambda)^\alpha x^{\alpha-1} e_q^{-\lambda q x}}{\Gamma_q \alpha}\, d_q x$$

$$= \int_0^{\frac{1}{1-q}} \frac{(\lambda)^\alpha x^{\alpha+1} e_q^{-\lambda q x}}{\Gamma_q \alpha}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^{\frac{1}{1-q}} x^{\alpha+1} e_q^{-\lambda q x}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \int_0^{\frac{1}{1-q}} x^{\alpha+2-1} e_q^{-\lambda q x}\, d_q x$$

$$= \frac{(\lambda)^\alpha}{\Gamma_q \alpha} \cdot \frac{\Gamma_q \alpha+2}{(\lambda)^{\alpha+2}}$$

$$= \frac{(\alpha+1)\Gamma_q \alpha+1}{(\lambda)^2 \Gamma_q \alpha}$$

$$= \frac{[\alpha+1]_q [\alpha]_q \Gamma_q \alpha}{(\lambda)^2 \Gamma_q \alpha}$$

$$= \frac{[\alpha+1]_q [\alpha]_q}{(\lambda)^2}$$

So, we conclude variance of the q-gamma distribution is

$$\text{Var (X,q)} = E_q(X^2) - \{E_q(X)\}^2$$

$$= \frac{[\alpha+1]_q [\alpha]_q}{(\lambda)^2} - \left\{\frac{[\alpha]_q}{\lambda}\right\}^2$$

$$= \frac{[\alpha]_q}{\lambda^2}$$

When Parameter $\lambda = 1$, Then

$$\textbf{Var (X,q)} = [\alpha]_q \quad \textbf{[23]}$$

Definition-1.10 Let X be a continuous random variable it is said to have a q-beta distribution of the first kind with two parameters t and s, If its probability density function defined by

$$f(x,q) = \beta_q(t,s)(x) = \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}$$

And its distribution function is defined by

$$F\,(x,q) = \int_0^1 \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}\, d_q x \quad [24]$$

Theorem 3 The q beta distribution also satisfies the basic properties -

q- beta distribution is the probability distribution that is the area of $\beta_q(t,s)$ is unity.

Proof: By the definition of q-beta distribution, we have

$$\int_0^1 F\,(x,q)\, d_q x = \int_0^1 \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \int_0^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x = \frac{\beta_q(t,s)}{\beta_q(t,s)} = 1 \quad [25]$$

Theorem 4 The Mean and Variance of q beta distribution is $\frac{[t]_q}{[t+s]_q}$ and $\frac{[s]_q[t]_q}{(t+s)_q^2[t+s+1]_q}$.

Proof: We know that mean of a continuous random variable can be obtained by

$$\overline{X} = E_q(X) = \int_0^1 x\,.\,f(x,q)\, d_q x$$

$$= \int_0^1 x\,.\frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \int_0^1 x^{(t+1)-1}(1-qx)_q^{s-1}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)}\, \beta_q(t+1,s)$$

$$= \frac{\Gamma_q(t+1)\,\Gamma_q s}{\Gamma_q(t+s+1)} \cdot \frac{\Gamma_q(t+s)}{\Gamma_q t\, \Gamma_q s}$$

$$= \frac{[t]_q\, \Gamma_q(t)\, \Gamma_q s}{[t+s]_q \Gamma_q(t+s)} \cdot \frac{\Gamma_q(t+s)}{\Gamma_q t\, \Gamma_q s}$$

$$= \frac{[t]_q}{[t+s]_q}$$

Now we find $E_q(X^2)$,

$$E_q(X^2) = \int_0^1 x^2\, f_{X,q}(x)\, d_q x$$

$$= \int_0^1 x^2\, \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \int_0^1 x^{t+2-1}(1-qx)_q^{s-1}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)}\, B_q(t+2,s)$$

$$= \frac{1}{\beta_q(t,s)}\, B_q(t+2,s)$$

$$= \frac{\Gamma_q(t+2)\,\Gamma_q s}{\Gamma_q(t+s+2)} \cdot \frac{\Gamma_q(t+s)}{\Gamma_q t\, \Gamma_q s}$$

$$= \frac{[t+1]_q[t]_q}{[t+s]_q[t+s+1]_q}$$

So, we conclude variance of the q-beta distribution is

$$\text{Var [X ,q]} = E_q(X^2) - \{E_q(X)\}^2$$

$$= \frac{[t+1]_q[t]_q}{[t+s]_q[t+s+1]_q} - \{\frac{[t]_q}{[t+s]_q}\}^2$$

$$= \frac{[s]_q[t]_q}{(t+s)_q^2[t+s+1]_q} \quad [26]$$

The q –k moment of a Beta random variable X is

$$\mu_{X,q}(k) = E_q\{X^k\}$$

$$= \frac{B_{q\,(t+k\,,s)}}{B_{q\,(t\,,s)}}$$

$$= \prod_{n=0}^{k-1} \frac{[t+n]_q}{[t+s+n]_q} \qquad [27]$$

Relation between Moment generating function and hyper geometric series

The Moment generating function of a q-beta random variable X is defined for any z and it is

$$M_{X,q}(Z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{B_q(t+n,s)}{B_q(t,s)} = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!_q} \prod_{k=0}^{n-1} \frac{[t+n]_q}{[t+s+n]_q}$$

Proof-By using the definition of Moment generating function, we obtain

$$M_{X,q}(Z) = E[\exp(zX)]$$

$$= \int_{-\infty}^{\infty} exp\,(zX) f_{X,q}(x)\, d_q x$$

$$= \int_0^1 exp\,(zX) \frac{x^{t-1}(1-qx)_q^{s-1}}{\beta_q(t,s)}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \int_0^1 exp\,(zX)\; x^{t-1}(1-qx)_q^{s-1}\, d_q x$$

By Taylor series expansion of exp(zX)

$$= \frac{1}{\beta_q(t,s)} \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(zx)^n}{n_q!}\right) x^{t-1}(1-qx)_q^{s-1}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \sum_{n=0}^{\infty} \frac{(z)^n}{n_q!} \int_0^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \sum_{n=0}^{\infty} \frac{(z)^n}{n_q!} \int_0^1 x^{(t+n)-1}(1-qx)_q^{s-1}\, d_q x$$

$$= \frac{1}{\beta_q(t,s)} \sum_{n=0}^{\infty} \frac{(z)^n}{n_q!}\; B_q(t+n\,,s)$$

$$= \sum_{n=0}^{\infty} \frac{(z)^n}{n_q!} \frac{B_q(t+n\,,s)}{\beta_q(t,s)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(z)^n}{n_q!} \frac{B_q(t+n\,,s)}{\beta_q(t,s)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(z)^n}{n_q!}\; \mu_{X,q}(n)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(z)^n}{n_q!} \prod_{n=0}^{k-1} \frac{[t+n]_q}{[t+s+n]_q}$$

The above formula for the moment generating function might seem impractical to compute , because it involves an infinite sum as well as product whose number of terms increase indefinitely. However the function

$$1F_1(t,t+s,z) = 1 + \sum_{n=1}^{\infty} \frac{(z)^n}{n_q!} \prod_{n=0}^{k-1} \frac{[t+n]_q}{[t+s+n]_q} \qquad [28]$$

is function is called confluent hypergeometric function of first kind ,that has been extensively studied in many branches of Mathematics .its Properties are well known and efficient algorithm for its computation are available in most software packages for scientific computation. If q tends to 1,then q-gamma and q-beta distribution tern into classical gamma and beta distribution.

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