CONVOLUTION PROPERTIES OF UNIVALENT FUNCTIONS WITH MISSING EVEN COEFFICIENTS EXTENDED BY SALAGEAN OPERATOR

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ABSTRACT

In this paper we have studied univalent functions with missing even coefficients by using Salagean differential operator. Here we have obtained some properties of $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $a_{2k+1} \geq 0$ like convolution results, coefficient bounds and Hadmard product.

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1. INTRODUCTION

Let M denote the class of normalised univalent functions of the form $f(z) = z + \sum_{k=1}^{\infty} a_k z^k$ is analytic in the unit disk, E={$z/|z| < 1$}

Let R = $\{w/w \ is \ analytic \ in \ E, w(0) = 0, |w(z)| < 1 \ in \ E\}$

Let $G(A, B)$ denote subclass of analytic functions in E, which are of the form $\frac{1+Aw(z)}{1+Bw(z)}$, -1≤A<B<1 where $w(z) \epsilon R$.

In this paper we study another subclass S^* of M which consist of functions of the form, $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $a_{2k+1} \geq 0$

Let Salagean introduced differential operator,

D^n : M → M, n∈ N by $D^0\big(f(z)\big) = f(z)$, $D^1\big(f(z)\big)$ =z $f'(z)$……..

…. $D^n\big(f(z)\big) = D(D^{n-1}f(z))$

Consider the following definitions,

$M^*(A,B) = \left\{ f/f \in M and \ \frac{D^{n+1} f(z)}{D^n f(z)} \epsilon \ G(A,B) \right\}$

$H(A,B) = \left\{ f/f \in M and \ \left(\frac{D^{n+1} f(z)}{D^n f(z)}\right)' \epsilon \ G(A,B) \right\}$

$S^*(A,B) = \left\{ f/f \in S and \ \left(\frac{D^{n+1} f(z)}{D^n f(z)}\right) \epsilon \ G(A,B) \right\}$ $C(A,B) = \left\{ f/f \in H and \ \frac{(D^{n+1} f(z))'}{D^n f(z)} \epsilon \ G(A,B) \right\}$

If $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $g(z) = z - \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$ then their convolution is defined by,

$$h(z) = f(z) * g(z)$$

$$h(z) = z - \sum_{k=1}^{\infty} a_{2k+1} b_{2k+1} z^{2k+1}$$

2. PRELIMINARY AND MAIN RESULTS

We start with following lemma which will be required for further investigation.

Lemma 1. A function, $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $a_{2k+1} \geq 0$ is in $S^*(A,B)$ iff $\sum_{k=1}^{\infty} \left(\frac{(2k+1)^n (2k(B+1)+(B-A))}{(B-A)}\right) \leq 1.$

Proof : Let $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$.

$D^n f(z) = z - \sum_{k=1}^{\infty} (2k+1)^n a_{2k+1} z^{2k+1}$ and

$$D^{n+1} f(z) = z - \sum_{k=1}^{\infty} (2k+1)^{n+1} a_{2k+1} z^{2k+1}$$

Therefore, $\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{z - \sum_{k=1}^{\infty} (2k+1)^{n+1} a_{2k+1} z^{2k+1}}{z - \sum_{k=1}^{\infty} (2k+1)^n a_{2k+1} z^{2k+1}}$

Now, $\frac{D^{n+1} f(z)}{D^n f(z)} \epsilon \ G(A,B)$ if and only if

$$\frac{z - \sum_{k=1}^{\infty} (2k+1)^{n+1} a_{2k+1} z^{2k+1}}{z - \sum_{k=1}^{\infty} (2k+1)^n a_{2k+1} z^{2k+1}} = \frac{1+AW(z)}{1+BW(z)}.$$

$$\sum_{k=1}^{\infty} -2k(2k+1)^n a_{2k+1} z^{2k+1} + w(z)[(B-A)z + \sum_{k=1}^{\infty} (2k+1)^n (A - (2k+1)B) a_{2k+1} z^{2k+1}].$$

Note that, $|w(z)| \leq 1$. Thus we get,

$$\left| \frac{\sum_{k=1}^{\infty} 2k(2k+1)^n a_{2k+1} z^{2k+1}}{(B-A)z + \sum_{k=1}^{\infty} (2k+1)^n (A-(2k+1)B) a_{2k+1} z^{2k+1}} \right| \leq 1$$

Allowing $|z| = r \to 1$ we get,

$$\frac{\sum_{k=1}^{\infty} 2k(2k+1)^n a_{2k+1}}{(B-A) + \sum_{k=1}^{\infty} (2k+1)^n (A-(2k+1)B) a_{2k+1}} \leq 1$$

$$\sum_{k=1}^{\infty} (2k+1)^n \big((B-A) + 2k(B+1)\big) a_{2k+1} \leq B - A$$

$$\sum_{k=1}^{\infty} \left(\frac{(2k+1)^n \big(2k(B+1)+(B-A)\big)}{(B-A)}\right) a_{2k+1} \leq 1 .$$

Lemma: 2

A function $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $a_{2k+1} \geq 0$ is in $C(A, B)$ if and only if $\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right) a_{2k+1} \leq 1.$

Proof: We have $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1} \in C(A, B)$

Then $\left(\frac{D^{n+1} f(z)}{D^n f(z)}\right)' \in G(A, B)$

If and only if $\frac{1-\sum_{k=1}^{\infty}(2k+1)^{n+2} a_{2k+1} z^{2k}}{1-\sum_{k=1}^{\infty}(2k+1)^{n+1} a_{2k+1} z^{2k}} = \frac{1+Aw(z)}{1+Bw(z)}$

$$\sum_{k=1}^{\infty} -2k(2k+1)^{n+1} a_{2k+1} z^{2k} + w(z)[(B-A) + \sum_{k=1}^{\infty}(2k+1)^{n+1}(A-(2k+1)B) a_{2k+1} z^{2k}] = 0$$

$$\sum_{k=1}^{\infty} 2k(2k+1)^{n+1} a_{2k+1} z^{2k} = w(z)[(B-A) + \sum_{k=1}^{\infty}(2k+1)^{n+1}(A-(2k+1)B) a_{2k+1} z^{2k}]$$

Since $|w(z)| \leq 1$ we get ,

$$\therefore \left|\frac{\sum_{k=1}^{\infty} 2k(2k+1)^{n+1} a_{2k+1} z^{2k}}{(B-A)+\sum_{k=1}^{\infty} 2k(2k+1)^{n+1}(A-(2k+1)B) a_{2k+1} z^{2k}}\right| \leq 1$$

Allowing $|z| = r \to 1$ we get ,

$$\frac{\sum_{k=1}^{\infty}(2k+1)^{n+1} 2k\, a_{2k+1}}{(B-A)+\sum_{k=1}^{\infty}(2k+1)^{n+1}(A-(2k+1)B)\, a_{2k+1}} \leq 1$$

$$\sum_{k=1}^{\infty}(2k+1)^{n+1}\left(2k(B+1)+(B-A)\right) a_{2k+1} \leq B-A$$

$$\therefore \sum_{k=1}^{\infty}\left(\frac{(2k+1)^{n+1}\left(2k(B+1)+(B-A)\right)}{B-A}\right) a_{2k+1} \leq 1$$

We define $h(z) = f(z) * g(z) = z - \sum_{k=1}^{\infty} a_{2k+1} b_{2k+1} z^{2k+1}$, $a_{2k+1}, b_{2k+1} \geq 0$

For $f(z)$ & $g(z)$ members of $s^*(A, B)$ and C(A,B).

- **Theorem 1**

If $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $g(z) = z - \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$, $a_{2k+1}, b_{2k+1} \geq 0$ are elements of classes $s^*(A, B)$ and C(A,B) then $h(z) = z - \sum_{k=1}^{\infty} a_{2k+1} b_{2k+1} z^{2k+1}$ is an element of $s^*(A_1, B_1)$ with $-1 \leq A_1 \leq B_1 \leq 1$, where $A_1 \geq -1$, $B_1 \leq \frac{A_1-2R}{1+2R}$ and these bounds for A_1& B_1 are sharp.

Proof:-

By lemma 1. We have,

$$\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right) a_{2k+1} \leq 1 \qquad (1.1)$$

Since $f \in s^*(A, B)$

$$\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right) b_{2k+1} \leq 1 \qquad (1.2)$$

We want to find A_1, B_1 , Such that

$-1 \leq A_1 < B_1 \leq 1$ for $h\square(z) = f(z) * g(z) \in s^*(A_1, B_1)$

Now , $h(z) \in s^*(A_1, B_1)$ if ,

$$\sum_{k=1}^{\infty} \left(\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{(B_1-A_1)}\right) a_{2k+1}b_{2k+1} \leq 1 \qquad (1.3)$$

$\sum_{k=1}^{\infty} u_1\, a_{2k+1}\, b_{2k+1} \leq 1$ where, $u_1 = \left(\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{(B_1-A_1)}\right)$

By Cauchy Schwarz inequality,

$$\sum_{k=1}^{\infty} \sqrt{ua_{2k+1}\, ub_{2k+1}} \leq \left(\sum_{k=1}^{\infty} ua_{2k+1}\right)^{1/2} \left(\sum_{k=1}^{\infty} ub_{2k+1}\right)^{1/2} \leq 1 \qquad (1.4)$$

Where $u = \left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right)$ (1.5)

(1.3) is true if,

$$u_1 a_{2k+1}\, b_{2k+1} \leq u\sqrt{a_{2k+1}\, b_{2k+1}} \qquad (1.6)$$

Where $u_1 = \left(\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{(B_1-A_1)}\right)$

$$u_1\sqrt{a_{2k+1}\, b_{2k+1}} \leq u$$

Also from (1.4),

$$u_1\sqrt{a_{2k+1}\, b_{2k+1}} \leq 1 \text{ for k=1,2,3……} \qquad (1.7)$$

Therefore it is enough to find u_1 such that ,

$$\frac{1}{u} \leq \frac{u}{u_1} \text{ ie } u_1 \leq u^2 \qquad (1.8)$$

Therefore

$$\left(\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{(B_1-A_1)}\right) \leq \left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right)^2 = u^2$$

$$(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big) \leq (B_1-A_1)u^2$$

$$A_1 \geq \frac{2k(2k+1)^n(B_1+1)+\big((2k+1)^n-u^2\big)B_1}{((2k+1)^n-u^2)} \qquad (1.9)$$

Take $B_1 = 1$ and $k = 1$ in (1.9) & (1.5) above.

Therefore

$$A_1 \geq \frac{2(3^n)(2)+\left(3^n-u^2\right)}{(3^n-u^2)}$$

$$A_1 \geq \frac{5\,3^n-u^2}{3^n-u^2}$$

Where , $u^2 = \left(\frac{3^n(2(B+1)+B-A)}{(B-A)}\right)^2$

$$u^2 = \left(\frac{3^n(3B-A+2)}{B-A}\right)^2$$

Therefore

$$A_1 \geq \frac{5.3^n\left(\frac{3^n(3B-A+2)}{B-A}\right)^2}{3^n\left(\frac{3^n(3B-A+2)}{(B-A)}\right)^2} \geq 1+4\left[\frac{3^n(B-A)^2}{3^n(B-A)^2-(3^n(3B-A+2))^2}\right]$$

Let us verify for a particular f and $g \in s^*(A,B)$

Consider

$$f(z) = g(z) = z - \left(\frac{3^n(3B-A+2)}{(B-A)}\right)z^3 \in s^*(A,B)$$

Their convolution is

$$h(z) = z - \left(\frac{3^n(3B-A+2)}{(B-A)}\right)^2 z^3$$

Thus $h(z) \in s^*(A_1, B_1)$ if

$$\frac{3^n(2(B_1+1)+(B_1-A_1))}{B_1-A_1} = \left(\frac{3^n(3B-A+2)}{B-A}\right)^2$$

$$\frac{3^n(3B_1-A_1+2)}{B_1-A_1} = \left(\frac{3^n(3B-A+2)}{B-A}\right)^2$$

Substitute $B_1 = 1$ to get ,

$$\frac{3^n(5-A_1)}{1-A_1} = \left(\frac{3^n(3B-A+2)}{B-A}\right)^2$$

$$A_1 - 1 = \frac{4.3^n(B-A)^2}{\left[3^n(B-A)^2-\left(3^n(3B-A+2)\right)^2\right]}$$

$$A_1 = 1 + 4R \text{ Where } R = \frac{3^n(B-A)^2}{\left[3^n(B-A)^2-\left(3^n(3B-A+2)\right)^2\right]}$$

This asserts that h(z) $\in s^*$(1+4k,1) with

$$R = \frac{3^n(B-A)^2}{\left[3^n(B-A)^2-\left(3^n(3B-A+2)\right)^2\right]}$$

- **Theorem 2**

If $f(z) \in s^*$(A,B) and g(z) $\in s^*(A', B')$. Then h(z) $\in s^*(A_1, B_1)$

Where $A_1 \geq$ -1 , $B_1 \leq \frac{A_1+2R}{1-2R}$ with

$$R = \frac{(B-A)(B'-A')}{(B-A)(B'-A')-3^n(3B-A+2)(3B'-A'+2)}$$

Proof :-

Proceeding with the argument developed in Theorem 1, we require

$$\frac{(2k+1)^n(2k(B_1+1)+(B_1-A_1))}{(B_1-A_1)} \leq \left\{\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right\}\left\{\frac{(2k+1)^n(2k(B'+1)+(B'-A'))}{(B^1-A^1)}\right\} = \alpha \qquad (1.10)$$

$$\frac{2k(B_1+1)(B_1-A_1)}{(B_1-A_1)} \leq \frac{\alpha}{(2k+1)^n}$$

$$\frac{2k(B_1+1)}{(B_1-A_1)} \leq \frac{\alpha}{(2k+1)^n} - 1$$

$$\frac{2k(B_1+1)}{(B_1-A_1)} \leq \frac{\alpha-(2k+1)^n}{(2k+1)^n} \qquad (1.11)$$

$$\frac{B_1-A_1}{2k(B_1+1)} \geq \frac{(2k+1)^n}{\alpha-(2k+1)^n}$$

$$\frac{B_1-A_1}{(B_1+1)} \geq \frac{2k(2k+1)^n}{\alpha-(2k+1)^n}$$

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2k(2k+1)^n}{\left(\left(\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right)\left(\frac{(2k+1)^n(2k(B'+1)+(B'-A'))}{(B'-A')}\right)\right)-(2k+1)^n}$$

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2k(2k+1)^n(B-A)(B'-A')}{(2k+1)^n(2k(B+1)+(B-A))\left((2k+1)^n\left(2k(B'+1)+(B'-A')\right)\right)-(2k+1)^n(B-A)(B'-A')}$$

Taking k = 1

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(3)^n(B-A)(B'-A')}{3^n(3B-A+2)3^n(3B'-A'+2)-3^n(B-A)(B'-A')}$$

$$\frac{B_1-A_1}{B_1+1} \geq \frac{2(B-A)(B'-A')}{3^n(3B-A+2)(3B'-A'+2)-(B-A)(B'-A')}$$

$$\frac{B_1-A_1}{B_1+1} \geq 2R \qquad (1.12)$$

where $R = \frac{(B-A)(B'-A')}{3^n(3B-A+2)(3B'-A'+2)-(B-A)(B'-A')}$

$$B_1 \leq \frac{A_1+2R}{1-2R} \qquad (1.13)$$

But $B_1 \geq -1$

$\therefore -1 \leq \frac{A_1+2R}{1-2R}$

$\therefore A_1 \geq -1$

Now consider,

$f(z) = z - \frac{(B-A)}{(3B-A+2)3^n} z^3 \in s^*(A,B)$

$g(z) = z - \frac{(B'-A')}{3^n(3B'-A'+2)} z^3 \in s^*(A',B')$

Then , $h(z) = f(z)g(z)$

$h(z) = z - \frac{(B-A)(B'-A')}{(3^n)^2(3B-A+2)(3B'-A'+2)} z^3$

where A_1 & B_1 can be found by the following rel^n ,

$\frac{3^n(3B_1-A_1+2)}{B_1-A_1} = \frac{(3^n(3B-A+2))(3^n(3B'-A'+2))}{(B-A)(B'-A')}$

On simplification and substituting $B_1 = 1$, We get

$\frac{(5-A_1)}{1-A_1} = \frac{(3B_1-A_1+2)3^n(3B'-A'+2)}{(B-A)(B'-A')} = \alpha$

$\frac{5-A_1}{1-A_1} =$, where $\alpha = \frac{3^n(3B_1-A_1+2)(3B'-A'+2)}{(B-A)(B'-A')}$

$A_1 = 1 + 4R$

Where $R = \frac{1}{1-\alpha}$

$R = \frac{1}{1-\frac{3^n(3B-A+2)(3B'-A'+2)}{(B-A)(B'-A')}}$

$R = \frac{(B-A)(B'-A')}{(B-A)(B'-A')- 3^n(3B-A+2)(3B'-A'+2)}$

- **Theorem 3**

If $f(z) \in C(A,B)$ & $g(z) \in C(A',B')$ then $f(z) * g(z) \in C(A_1, B_1)$

Where $A_1 \geq -1$, $B_1 \leq \frac{A_1+2R}{1-2R}$ with

$$R = \frac{(B-A)(B'-A')}{(B-A)-9(3^n(3B-A+2)(3B'-A'+2))} \quad (1.14)$$

Proof :-Let us verify with following examples,

$f(z) = z - \frac{(B-A)}{3.3^n(3B-A+2)} z^3 \in C(A,B)$

$g(z) = z - \frac{(B'-A')}{3.3^n(3B'-A'+2)} z^3 \in C(A',B')$

Then ,

$h(z) = f(z) * g(z)$

$h(z) = z - \frac{(B-A)(B'-A')}{9(3^n)^2(3B-A+2)(3B'-A'+2)} z^3 \in C(A_1, B_1)$

If ,

$\frac{3^n(2(B_1+1)+(B_1-A_1))}{(B_1-A_1)} = \frac{9(3^n)^2(3B-A+2)(3B'-A'+2)}{(B-A)(\,B'-A')}$

Take $B_1 = 1$

$\frac{(5-A_1)}{1-A_1} = \frac{9(3^n)(3B-A+2)(3B'-A'+2)}{(B-A)(\,B'-A')} = \alpha$

$\frac{(5-A_1)}{1-A_1} =$, Where $\alpha = \frac{9(3^n)(3B-A2)(3B'-A'+2)}{(B-A)(\,B'-A')}$

$A_1 = 1 + \frac{4}{1-\alpha}$

$A_1 = 1 + \frac{4}{1-\left(\frac{9(3^n)(3B-A2)(3B'-A'+2)}{(B-A)(\,B'-A')}\right)}$

$$A_1 = 1 + \frac{4(B-A)(B'-A')}{(B-A)(B'-A') - 9(3^n(3B-A+2)(3B'-A'+2))}$$

$A_1 = 1 + 4R$ Where ,

$$R = \frac{(B-A)(B'-A')}{(B-A)(B'-A') - 9(3^n(3B-A+2)(3B'-A'+2))}$$

- **Theorem 4**

If $f(z) \in s^*(A,B)$, g(z) $\in$ $s^*(A',B')$ then $f(z) * g(z) \in C\,(A_1, B_1)$ where $A_1 \geq$ -1 and $B_1 \leq \frac{A_1+2R}{1-2R}$ with

$R = \frac{3(B-A)(B'-A')}{3^n(3B-A+2)(3B'-A'+2)-3(B-A)(B'-A')}$ (1.15)

Proof:-

$$f(z) * g(z) \in C\,(A_1, B_1)\ if\ ,$$

$\frac{(2k+1)^{n+1}(2k(B_1+1)+(B_1-A_1))}{B_1-A_1} \leq \left\{\frac{(2k+1)^n(2k(B+1)+(B-A))}{(B-A)}\right\}\left\{\frac{(2k+1)^n(2k(B'+1)+(B'-A'))}{(B'-A')}\right\} = \alpha$

Proceeding as in Theorem 2 with k=1 we get,

$\frac{3^{n+1}(2(B_1+1)+(B_1-A_1))}{(B_1-A_1)} \leq \alpha$

$\frac{(3B_1-A_1+2)}{B_1-A_1} \leq \frac{\alpha}{3^{n+1}}$

$\therefore \frac{(B_1-A_1)}{2(B_1+1)} \geq \frac{3^{n+1}}{\alpha-3^{n+1}} = R$

$\frac{B_1-A_1}{2(B_1+1)} \geq R$

$\therefore B_1 \leq \frac{A_1+2R}{1-2R}$

But , $B_1 \geq -1$

$\frac{A_1+2R}{1-2R} \geq -1$

$\therefore\ A_1 \geq -1$ for ,

$R = \frac{3(B-A)(B'-A')}{3^n(3B-A+2)(3B'-A'+2)-3(B-A)(B'-A')}$

- **Remark** : - Consider the functions ,

$f(z) = z - \frac{(B-A)}{3^n(3B-A+2)} z^3 \in s^*(A,B)$

$g(z) = z - \frac{(B'-A')}{3^n(3B'-A'+2)} z^3 \in s^*(A',B')$

Then $f(z) * g(z) \in C(A_1, B_1)$ with $B_1 \leq \frac{A_1+2R}{1-2R}$, $A_1 \geq -1$ with R as in (1.15). This clearly indicates that our bounds are best possible.

- **Theorem 5**

If $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$, $a_{2k+1} \geq 0$ belongs to $s^*(A,B)$ and $g(z) = z - \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$ with $|b_{2k+1}| \leq 1$ for $k \geq 1$ then $f(z) * g(z) \in M^*(A,B)$

Proof : - Since , $f(z) \in s^*(A,B)$ we have ,

$\sum_{k=1}^{\infty}\left\{(2k+1)^n\left(\frac{2k(B+1)+(B-A)}{B-A}\right)\right\} a_{2k+1} b_{2k+1} \leq \sum_{k=1}^{\infty}\left\{(2k+1)^n\left(\frac{2k(B+1)+(B-A)}{(B-A)}\right)\right\} a_{2k+1} |b_{2k+1}| \leq 1$

(1.16)

From (1.16) and the fact that , $|b_{2k+1}| \leq 1$.

This shows that ,

$f(z) * g(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$

- **Theorem 6**

If $f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1} \in C(A,B)$, $a_{2k+1} \geq 0$ and $g(z) = z - \sum_{k=1}^{\infty} b_{2k+1} z^{2k+1}$; $|b_{2k+1}| \leq 1$ for $k \geq 1$ then $f(z) * g(z) \in C(A,B)$

Same as above theorem.

- **Theorem 7**

If $f(z), g(z) \in s^*(A,B)$ then $h(z) = z - \sum_{k=1}^{\infty}\left(a_{2k+1}^2 + b_{2k+1}^2\right) z^{2k+1} \in s(A_1, B_1)$ where $A_1 \geq -1$, $B_1 \leq \frac{A_1+4R}{1-4R}$ with $R = \frac{k(2k+1)^n(B-A)^2}{(2k+1)^{2n}\left(2k(B+1)+(B-A)\right)^2-(B-A)^2}$

Proof : Since , $f(z)$, g(z) $\in s^*(A,B)$ we have ,

$\sum_{k=1}^{\infty}(2k+1)^n\left(\frac{2k(B+1)+(B-A)}{B-A}\right)a_{2k+1}\leq 1$ and

$\sum_{k=1}^{\infty}(2k+1)^n\left(\frac{2k(B+1)+(B-A)}{B-A}\right)b_{2k+1}\leq 1$

Note that,

$$\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n\big(2k(B+1)+(B-A)\big)}{B-A}\right)^2 a_{2k+1}^2\leq\left(\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n\big(2k(B+1)+(B-A)\big)}{(B-A)}\right)a_{2k+1}\right)^2\leq 1$$

(1.17)

Similarly $\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n\big(2k(B+1)+(B-A)\big)}{B-A}\right)^2 b_{2k+1}^2\leq 1$ (1.18)

Adding (1.17) & (1.18) we get,

$$\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n\big(2k(B+1)+(B-A)\big)}{B-A}\right)^2\left(a_{2k+1}^2+b_{2k+1}^2\right)\leq 2 \qquad (1.19)$$

Now $f(z)*g(z)\in s(A_1,B_1)$ if ,

$$\sum_{k=1}^{\infty}\left(\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{B_1-A_1}\right)\left(a_{2k+1}^2+b_{2k+1}^2\right)\leq 1$$

(1.19) implies that it is enough to show that ,

$$\frac{(2k+1)^n\big(2k(B_1+1)+(B_1-A_1)\big)}{(B_1-A_1)}\leq\frac{1}{2}\left(\frac{(2k+1)^n\big(2k(B+1)+(B-A)\big)}{(B-A)}\right)^2=\frac{u^2}{2}$$

$$\frac{2k(B_1+1)}{B_1-A_1}\leq\frac{u^2}{2(2k+1)^n}-1$$

$$\frac{B_1+1}{B_1-A_1}\leq\frac{u^2-2(2k+1)^n}{4k(2k+1)^n}$$

$$\frac{B_1-A_1}{B_1+1}\geq\frac{4k(2k+1)^n}{u^2-2(2k+1)^n}=\beta(k)$$

Note that , $\beta(k)$ decreases as k increases on substituting $k=1$ and simplifying we get ,

$\frac{B_1-A_1}{B_1+1}\geq 4R$, where $R=\frac{k(2k+1)^n}{u^2-2(2k+1)^n}$

$\therefore\ B_1\leq\frac{A_1+4R}{1-4R}$

But $B_1\geq -1$

$\therefore\ -1\leq\frac{A_1+4R}{1-4R}$

$\therefore\ A_1\geq -1$

With ,

$$R=\frac{k(2k+1)^n}{\left((2k+1)^n\left(\frac{2k(B+1)+(B-A)}{(B-A)}\right)\right)^2-2(2k+1)^n}$$

$$R=\frac{k(B-A)^2}{(2k+1)^{2n}\big(2k(B+1)+(B-A)\big)^2-2(B-A)^2}$$

References

[1] S. M. Khairnar, and Meena More, Convolution Properties of Univalent Functions with Missing Coefficient of Alternating Type, Int. Journal of Math. Analysis, Vol. 2,2008, no. 12, 569 – 580.

[2] P. L. Duren, Univalent Functions, Grundlehren Math. Wiss., Vol. 259, Springer, New York 1983.

[3] K. S. Padmanabhan and M. S. Ganeshan, Convolution of certain class of univalent functions with negative coefficients, Indian J. Pure Appl. Math. Sept.19 (9) (1988), 880 – 889.

[4] Yi Ling Cang and Jin Lin Liu, On certain univalent functions with missing coefficients, Advances in difference equations 2013, 2013:89, doi:10.1186/1687 – 1847 – 2013 – 89.

[5] L. V. Ahlfors, Complex Analysis, McGraw Hill BookCo.,Inc; New York 1983.

[6] K. K. Dixit and Y. K. Mishra, Generalization of certain classes of univalent functions with negative and missing coefficients, Acta Ciencia Indica. Mathematics 01/1998;24(1)

[7] S.M. Khairnar and S.R. Kulkarni, On Hadmard product of univalent functions with missing second coefficients, Acta Ciencia Indica. Mathematics 01/2002;28(4).