A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE MISSING EVEN COEFFICIENTS DEFINED BY SALAGEAN DERIVATIVE

AMRUTA PATIL[1], S. M. KHAIRNAR[2] AND B.R. AHIRRAO[3]
[1] Department of Mathematics, AISSMS,
Institute of Information Technology,
Shivajinagar, Pune - 411001,, India
[2] Department of Engineering Sciences,
D. Y. Patil School of Engineering, Lohegaon-411047, Pune, India
[3] Department of Mathematics,
Z. B. Patil College, Dhule - 424002, India.

Abstract
In this article we have define a subclass $s(\gamma,\alpha,\mu,\lambda)$ of univalent functions with negative odd coefficients defined by Salagean derivative operator in the unit disk $U=\{z\in C:|z|<1\}$. We have obtained different properties like coefficient inequality distribution theorem, radii of starlikeness, convexity, close to convexity and close to starlikeness and hadamard product for class $s(\gamma,\alpha,\mu,\lambda)$.

Keywords : *Univalent function, Salagean derivative, Distribution theorem, Closure theorem.*

1. Introduction

A denote the class of function of the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k (1)$$

which are analytic and univalent in open unit disk $U=\{z\in C:|z|<1\}$.
$f(z)$ is given by (1) and $g(z)$ is in class A defined by

$$g(z)=z+\sum_{k=2}^{\infty} b_k z^k, (2)$$

then Hadamard product of f and g is

$$(f*g)(z)=z+\sum_{k=2}^{\infty} (a_k b_k) z^k, ‘\ z\in U. (3)$$

Let S denote the subclass of A consisting of a function of the form,

$$f(z)=z-\sum_{k=1}^{\infty} a_{2k+1} z^{2k+1},\ a_{2k+1}\geq 0. (4)$$

aim to study the subclass $S(\gamma,\alpha,\mu,\lambda)$ consisting of function $f\in S$ and satisfying,

$$\left|\frac{\gamma\left((D(z))'-\frac{D(z)}{z}\right)}{\alpha(D(z))'+(1-\gamma)\frac{D(z)}{z}}\right|<\mu,\quad z\in U (5)$$

for $0\leq\gamma<1, 0\leq\alpha<1, 0<\mu<1$ and $D(z)$ is Salagean derivative defined by

$$D^0 f(z)=f(z),\quad F^1 f(z)=zf^1(z)\cdots$$

$$\cdots D(z)=D(D^{\lambda-1}f(z))=z-\sum_{k=1}^{\infty}(2k+1)^{2k+1}z^{2k+1}. (6)$$

2. Coefficient Inequality

In the following theorem we obtain a necessary and sufficient condition for function to be class $S(v,\alpha,\mu,\lambda)$.

Theorem 2.1 : Let function f defined by (4). Then $f\in S(\gamma,\alpha,\lambda)$ if and only if

$$\sum_{k=1}^{\infty}[2k\gamma+\mu(\alpha(2k+1)+(1-\gamma))](2k+1)^{2k+1}\le\mu(\alpha+(1-\gamma)),$$

where $0<\mu<1, 0\le\gamma<1, 0\le\alpha<1$ and $\gamma>-1$. The result is sharp for the function

$$f(z)=z-\frac{\mu(\alpha+1-\gamma)}{[2k\gamma+\mu((2k+1)\alpha+(1-\gamma))](2k+1)^{\lambda}}z^{2k+1},\ \ k\ge1.$$

Proof : Suppose that the inequality holds true and $|z|=1$. Then we obtain

$$\left|\gamma(D(z))'-\frac{D(z)}{z}\right|-\mu\left|\alpha(D(z))'+(1-\gamma)\frac{D(z)}{z}\right|$$

$$=\left|\sum_{k=1}^{\infty}\gamma(-2k)(2k+1)^{2k+1}z^{2k}\right|$$

$$\le\sum_{k=1}^{\infty}(2k+1)^{\lambda}[2k\gamma+\mu((2k+1)\alpha+1-\gamma)]a_{2k+1}-\mu(\alpha+(1-\gamma))\le0.$$

Hence by maximum modulus principle $f\in S(\gamma,\alpha,\mu,\lambda)$.

assume that $f\in S(\gamma,\alpha,\mu,\lambda)$ so that

$$\left|\frac{\gamma(D())'-\frac{D(z)}{z}}{\alpha(D(z))'+(1-\gamma)\frac{D(z)}{z}}\right|<\mu,\ \ z\in U.$$

Hence

$$\left|\gamma(D(z))'-\frac{D(z)}{z}\right|<\mu\left|\alpha(D(z))'+(1-\gamma)\frac{D(z)}{z}\right|.$$

Thus,

$$\sum_{k=1}^{\infty}[2k\gamma+\mu(\alpha(2k+1)+(1-\gamma))](2k+1)^{2k+1}\le\mu(\alpha+(1-\gamma)).$$

Therefore,

$$a_{2k+1}\le\frac{\mu(\alpha+(1-\gamma))}{[2k\gamma+\mu(\alpha(2k+1)+(1-\gamma))](2k+1)^{\lambda}}.$$

Corollary 2.1 : Let the function $f\in S(\gamma,\alpha,\mu,\lambda)$ then

$$a_{2k+1}\le\frac{\mu(\alpha+(1-\gamma))}{[2(\gamma+\mu(\alpha(2k+1)+(1-\gamma)))](2k+1)^{\lambda}}.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $S(\gamma,\alpha,\mu,\lambda)$.

Theorem 3.1 : Let function $f\in S(\gamma,\alpha,\mu,\lambda)$ then,

$$|z|-\frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}}\le|f(z)|\le|z|+\frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}}|z|^{3}.(7)$$

The result is sharp to attained

$$f(z)=z-\frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}}z^{3}.$$

Proof : Let

$$f(z) = z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1}$$

$$|f(z)| = \left| z - \sum_{k=1}^{\infty} a_{2k+1} z^{2k+1} \right|$$

$$\leq |z| + \sum_{k=1}^{\infty} a_{2k+1} |z^{2k+1}|$$

$$|f(z)| \leq |z| + |z|^3 \sum_{k=1}^{\infty} a_{2k+1}.$$

By Theorem 2.1, we get

$$\sum_{k=1}^{\infty} a_{2k+1} \leq \frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}}. \quad (8)$$

Thus

$$|f(z)| \leq |z| + \frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}} |z|^3.$$

Also

$$|f(z)| \geq |z| - \frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}} |z|^3.$$

Therefore,

$$|z| - \frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}} |z|^3 \leq |f(z)| \leq |z| + \frac{\mu(\alpha+(1-\gamma))}{(2\gamma+\mu(3\alpha+1-\gamma))3^{\lambda}}.$$

Theorem 3.2 : Let $f \in S(\gamma,\alpha,\mu,\lambda)$ then,

$$1 - \frac{\mu(\alpha+(1-\gamma))}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}} |z|^2 \leq |f'(z)| \leq 1 + \frac{\mu(\alpha+(1-\gamma))}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}} |z|^2$$

with equality for,

$$f(z) = z - \frac{\mu(\alpha+(1-\gamma))}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}} z^3.$$

Proof : Note that

$$3^{\lambda}[2\gamma+\mu(3\alpha+1-\gamma)] \sum_{k=1}^{\infty} (2k+1) a_{2k+1}$$

$$\leq \sum_{k=1}^{\infty} (2k+1)[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{2k+1}$$

$$\leq \mu(\alpha+1-\gamma).$$

$$\therefore \quad \sum_{k=1}^{\infty} (2k+1) a_{2k+1} \leq \frac{\mu(\alpha+1-\gamma)}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}}. \quad (9)$$

Theorem 2.1,

$$|f'(z)| = \left| 1 - \sum_{k=1}^{\infty} (2k+1) a_{2k+1} z^{2k} \right|$$

$$\leq \quad 1+|z|^2 \sum_{k=1}^{\infty}(2k+1)a_{2k+1}.$$

From (9)

$$|f'(z)|\leq 1+\frac{\mu(\alpha+1-\gamma)}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}}|z|^2.(10)$$

Similarly,

$$|f'(z)|=\left|1-\sum_{k=1}^{\infty}(2k+1)a_{2k+1}z^{2k}\right|$$

$$|f'(z)|\geq 1-\frac{\mu(\alpha+1-\gamma)}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}}\ |z|^2.(11)$$

By combining (10) and (11) we get,

$$1-\frac{\mu(\alpha+1-\gamma)}{[2\gamma+\mu(3\alpha+1-\gamma)]3^{\lambda}}|z|^2\leq|f'(z)|\leq 1+\frac{\mu(\alpha+1-\gamma)}{[2\alpha+\mu(3\alpha+1-\gamma)]}|z|^2.$$

4. Radii of Starlikeness, Convexity, Close to Convexity and Close to Starlikeness

Theorem 4.1 : Let $f\in S(\gamma,\alpha,\mu,\lambda)$ then f is starlike in $|z|<R_1$ of order $\delta, 0\leq\delta<1$ where,

$$R_1=\inf_k\left\{\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+\mu((2k+1)\alpha+1-\gamma)]}{(2k+1-\delta)\mu(\alpha+1-\gamma)}\right\}^{1/2k},\quad k\geq 1.(12)$$

Proof : Let f is starlike of order $=del, 0\leq\delta<1$ if $Re\left\{\frac{zf'(z)}{z}\right\}>\delta.$

is enough to show that,

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{1-\sum_{k=1}^{\infty}(2k+1)a_{2k+1}z^{2k}}{1-\sum_{k=1}^{\infty}a_{2k+1}z^{2k}}-1\right|$$

$$= \left|\frac{-\sum_{k=1}^{\infty}2ka_{2k+1}z^{2k}}{1-\sum_{k=1}^{\infty}a_{2k+1}z^{2k}}\right|$$

$$\leq \frac{\sum_{k=1}^{\infty}2ka_{2k+1}|z|^{2k}}{1-\sum_{k=1}^{\infty}a_{2k+1}|z|^{2k}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\delta.$$

That is

$$\frac{\sum_{k=1}^{\infty}(2k)a_{2k+1}|z|^{2k}}{1-\sum_{k=1}^{\infty}a_{2k+1}|z|^{2k}}\leq 1-\delta$$

$$\sum_{k=1}^{\infty}(2k+1-\delta)a_{2k+1}|z|^{2k}\leq 1-\delta$$

$$\therefore \sum_{k=1}^{\infty}\frac{(2k+1-\delta)}{(1-\delta)}a_{2k+1}|z|^{2k}\leq 1. (13)$$

By Theorem 2.1, equation (13) becomes true if,

$$\frac{(2k+1-\delta)}{(1-\delta)}|z|^{2k}\leq \frac{(2k+1)^{\lambda}[2k\gamma+\mu((2k+1)\alpha+1-\gamma)]}{\mu(\alpha+1-\gamma)}$$

or if

$$|z|\leq \left[\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+\mu((2k+1)\alpha+1-\gamma)]}{(2k+1-\delta)\mu(\alpha+(1-\gamma))}\right]^{\frac{1}{2k}},\ k\geq 1. (14)$$

Theorem 4.2 : Let $f\in S(\gamma,\alpha,\mu,\lambda)$. Then f is convex in $|z|<R_2$ of order $\delta, 0\leq\delta<1$ where

$$R_2=\inf_k\left\{\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+\mu((2k+1)\alpha+1-\gamma)]}{(2k+1)(2k+1-\delta)\mu(\alpha+1-\gamma)}\right\},\ k\geq 1. (15)$$

Proof : Let f is convex in $|z|<R_2$ of order $\delta, 0\leq\delta<1$ if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta.$$

Thus it is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{k=1}^{\infty}2k(2k+1)a_{2k+1}z^{2k}}{1-\sum_{k=1}^{\infty}(2k+1)a_{2k+1}z^{2k}}\right|$$

$$\leq \frac{\sum_{k=1}^{\infty}2k(2k+1)a_{2k+1}|z|^{2k}}{1-\sum_{k=1}^{\infty}(2k+1)a_{2k+1}|z|^{2k}}.$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta \text{ if } \sum_{k=1}^{\infty}\frac{(2k+1)(2k+1-\delta)a_{2k+1}|z|^{2k}}{1-\delta}\leq 1. (16)$$

Hence by Theorem 2.1, (16) will be true if

$$\frac{(2k+1)(2k+1-\delta)|z|^{2k}}{1-\delta}\leq \frac{[2k\gamma+\mu((2k+1)\alpha+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)}, k\geq 1$$

or if,

$$|z|^{2k}\leq\left[\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+((2k+1)\alpha+1-\gamma)]}{(2k+1)(2k+1-\delta)\mu(\alpha+1-\gamma)}\right],\ k\geq 1. (17)$$

Theorem 4.3 : Let $f\in S(\gamma,\alpha,\mu,\lambda)$. Then f is close to convex in $|z|<R_3$ of order $\delta, 0\leq\delta<1$, where,

$$R_3=\inf_k\left\{\frac{(1-\delta)(2k+1)^{\lambda+1}[2k\gamma+\mu(\alpha(2k+1)+1-\alpha)]}{\mu(\alpha+1-\gamma)}\right\}^{1/2k},\quad k\geq 1.(18)$$

Proof : Let f is close to convex in $|z|<R_3$ of order $\delta, 0\leq\delta<1$ if $Re\{f'(z)\}>\delta$.

is enough to show that,

$$\begin{aligned}|f'(z)-1| &= \left|-\sum_{k=1}^{\infty}(2k+1)a_{2k+1}z^{2k}\right|\\ &\leq \sum_{k=1}^{\infty}(2k+1)a_{2k+1}|z|^{2k}.\end{aligned}$$

Thus

$$|f'(z)-1|\leq 1-\delta$$

if

$$\sum_{k=1}^{\infty}\frac{(2k+1)a_{2k+1}|z|^{2k}}{1-\delta}\leq 1.(19)$$

Hence by Theorem 2.1, (19) will be true if

$$\frac{(2k+1)|z|^{2k}}{1-\delta}\leq\frac{(2k+1)^{\lambda}[2k\gamma+\mu(\alpha(2k+1)+1-\alpha)]}{\mu(\alpha+1-\gamma)}$$

or if

$$|z|\leq\left[\frac{(1-\delta)(2k+1)^{\lambda+1}[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)]}{\mu(\alpha+1-\gamma)}\right]^{1/2k}, k\geq 1.(20)$$

Theorem 4.4 : Let $f\in S(\gamma,\alpha,\mu,\lambda)$. Then f is close to starlike in $|z|<R_4$ of order $\delta, 0\leq\delta<1$ where

$$R_4=\inf_k\left\{\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)]}{\mu(\alpha+1-\gamma)}\right\}^{1/2k},\quad k\geq 1.(21)$$

Proof : Let $f\in S(\gamma,\alpha,\mu,\lambda)$ is close to starlike in $|z|<R_4$ of order $\delta, 0\leq\delta<1$, if

$$Re\left\{\frac{f(z)}{z}\right\}>\delta.$$

It is enough to show that,

$$\left|\frac{f(z)}{z}-1\right|=\sum_{k=1}^{\infty}a_{2k+1}z^{2k}|.$$

$$\left|\frac{f(z)}{z}-1\right|\leq\sum_{k=1}^{\infty}a_{2k+1}|z|^{2k}.$$

Thus

$$\left|\frac{f(z)}{z}-1\right|\leq 1-\delta \text{ if } \sum_{k=1}^{\infty}a_{2k+1}|z|^{2k}\leq 1-\delta.(22)$$

Hence by Theorem 2.1, (22) will be true if

$$\frac{|z|^{2k}}{1-\delta}\leq\frac{[2k\gamma+\mu(\alpha(2k+1)+1-\alpha)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)}$$

or if

$$|z|\leq\left[\frac{(1-\delta)(2k+1)^{\lambda}[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)]}{\mu(\alpha+1-\gamma)}\right]^{1/2k},\quad k\geq 1.(23)$$

5. Closure Theorem

Theorem : Let $f_i \epsilon \ S(\gamma, \alpha, \mu, \delta), i = 1,2, \dots, s$. Then

$$g(z)=\sum_{i=1}^{s} c_i f_i(z) \in S(\gamma,\alpha,\mu,\lambda).$$

For $f_i(z)=z-\sum_{k=1}^{\infty} a_{k,i} z^{2k+1}$ where $\sum_{i=1}^{s} c_i=1$.

Proof :

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_i f_i(z) \\ &= z-\sum_{k=1}^{\infty}\sum_{i=1}^{s} c_i a_{k,i} z^{2k+1} \\ &= z-\sum_{k=1}^{\infty} e_k z^{2k+1} \end{aligned}$$

where

$$e_k=\sum_{i=1}^{s} c_i a_{k,i}.$$

Thus $g(z)\in S(\gamma,\alpha,\mu,\lambda)$ if

$$\sum_{k=1}^{\infty} \frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)} e_k \le 1$$

that is if

$$\sum_{k=1}^{\infty}\sum_{i=1}^{s} \frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha_1-\gamma)} c_i a_{k,i}$$

$$= \sum_{i=1}^{s} c_i \sum_{k=1}^{\infty} \frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)} a_{k,i} \le \sum_{i=1}^{s} c_i=1.$$

Theorem 5.2 : Let $f, g \in S(\gamma,\alpha,\mu,\lambda)$. Then

$$h(z)=Z-\sum_{k=1}^{\infty} (a_{2k+1}^2+b_{2k+1}^2) Z^{2k+1}$$

belongs to $S(\gamma,\alpha,\ell,\lambda)$ where

$$\lambda \ge \frac{4k\gamma\mu^2(\alpha+1-\gamma)}{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)]^2(2k+1)^{\lambda}-2\mu^2(\alpha+1-\gamma)(\alpha(2k+1)+1-\gamma)}.$$

Proof : Let $f, g \in S(\gamma,\alpha,\mu,\lambda)$, so by Theorem 2.1

$$\sum_{k=1}^{\infty} \left\{ \frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)} a_{2k+1} \right\}^2 \le 1$$

and

$$\sum_{k=1}^{\infty} \left\{ \frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)} b_{2k+1} \right\}^2 \le 1.$$

From above equations we get,

$$\sum_{k=1}^{\infty}\left\{\frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)}\right\}^{2}(a_{2k+1}^{2}+b_{2k+1}^{2})\leq 2.$$

$$\sum_{k=1}^{\infty}\frac{1}{2}\left\{\frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\mu(\alpha+1-\gamma)}\right\}^{2}(a_{2k+1}^{2}+b_{2k+1}^{2})\leq 1.(24)$$

But $h(z)\in S(\gamma,\alpha,\ell,\lambda)$ if and only if,

$$\sum_{k=1}^{\infty}\frac{[2k\gamma+\ell(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\ell(\alpha+1-\gamma)}(a_{2k+1}^{2}+b_{2k+1}^{2})\leq 1(25)$$

where $0<\ell<1$, however (24) implies (25) of

$$\frac{[2k\gamma+\ell(\alpha(2k+1)+1-\gamma)](2k+1)^{\lambda}}{\ell(\alpha+1-\gamma)}\leq\frac{1}{2}\left[\frac{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)](2k+1)}{\mu(\alpha+1-\gamma)}\right]^{2}$$

we get

$$\ell\geq\frac{4k\gamma\mu^{2}(\alpha+1-\gamma)}{[2k\gamma+\mu(\alpha(2k+1)+1-\gamma)]^{2}(2k+1)^{\lambda}-2\mu^{2}(\alpha+1-\gamma)(\alpha(2k+1)+1-\gamma)}.$$

References

[1] Padmanabhan K. S. and Ganeshan M. S., Convolution of certain class of univalent functions with negative coefficients, Indian J. Pure and Appl. Math., 19(9) (Sept. 1998), 880-889.

[2] Andrews G. E., Askey R.and Roy R., Special Functions, Cambridge University Press, Cambridge, U.K.(1999).

[3] Darus M., Some subclasses of analytic functions, J. Math. Comp. Sci. (Math.Ser.), 16(8) (2003), 121-126.

[4] Atshan W. G. and Kulkarni S. R., On a new class of analytic functions with negative coefficients, Analele Universităţii Oradea, Fasc. Mathematica, Tom, XVI (2009), 43-51.

[5] Atshan W. G. and Ruaa Muslim Abd, New class of univalent functions with negative coefficients defied by Ruscheweyh derivative, Gen. Math. Notes, 18(2) (2013), 77-91.