Half Logistic - Rayleigh Distribution: Properties

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Abstract: In this paper, we introduce a new two parameter Half Logistic Rayleigh Distribution (HL-RD). Which propound a more adjustable mode for modelling simulated data. The recommended distribution reveals increasing, decreasing and bath-shaped probability Density, Distribution and Hazard Rate Functions. Some distributional properties of new model are investigated which include the Density Function, Distribution Function (DF), Quantile Function(QF), Moments, Moment Generating Function (MGF), Cumulative Generating Function (CGF) . The parameters involved in the model are estimated using Maximum Likelihood Estimation (MLE) method.

Keywords: Half Logistic Rayleigh Distribution, Moments, MGF, CGF, QF, MLE.

1. Introduction:

The commonly used Lifetime distributions (Weibul, Logistic, Lomax, etc.,) have a restricted range of performance. Such type of distributions cannot give a better fit to model for all practical situations. Recently, several authors have developed a number of new families of statistical models by applying different techniques. Various techniques have been introduced in the literature to derive new flexible models as discussed by Lai (2013). Al-Awadhi and Ghitany (2001) introduced the discrete Poisson–Lomax distribution by using the Lomax distribution as a mixing distribution for the Poisson parameter. Beta-Pareto has been presented by Akinsete, Famoye and Lee (2008). Uniform Exponential Distribution (UED) and Exponential Pareto distribution (EPD) were introduced by Abed Al-Kadim and Abdalhussain Boshi(2013). Sharma and Shanker (2013) used a mixture of exponential (q) and gamma (2;q) to create a two-parameter Lindley distribution.

In this paper, we introduce new linear compound distribution as Half Logistic Rayleigh Distribution (HLRD) and discuss some of its properties, such as Distribution, Hazard Function, Quantile and Random Generation Moments, and its Moment Generating Function (MGF), Cumulative Generating Function (CGF). We estimate parameters by Maximum Likelihood method and also define Asymptotic Confidence bounds for Half Logistic Rayleigh Distribution. Finally we use Simulation study about the parameters estimation.

2. The Probability density and Distribution functions of the HLRD

2.1 HLRD Specifications

In this section, we define new Scale (θ) and Shape (λ)two parameter distribution called Half Logistic Rayleigh Distribution with parameters θ and λ. The Probability Density Function (PDF), Cumulative Distribution Function (CDF), Survival Function (SF) and Hazard Function (HF) of the new model HLRD are respectively defined as follows**:**

A random variable X ~ HLRD(θ,λ) has Probability density function and is in the form

$$f(x;\theta,\lambda) = \frac{4\lambda(x-\theta)e^{-\lambda(x-\theta)^2}}{\left[1+e^{-\lambda(x-\theta)^2}\right]^2}; x>\theta, \lambda>0 \qquad ...(1)$$

Where $x > \theta,\ \lambda > 0$

A random variable X ~HLRD (θ,λ) correspond Cumulative Distribution Function is in the form

$$F(x;\theta,\lambda)=\frac{1-e^{-\lambda(x-\theta)^2}}{1+e^{-\lambda(x-\theta)^2}};x>\theta,\lambda>0 \qquad ...(2)$$

Where x > θ, λ > 0

2.2Limits of the HLRD function

The limit of the Probability Density Function is given by

$$x\xrightarrow{\lim}0\, f(x;\theta,\lambda)=\frac{4\lambda(-\theta)e^{-\lambda(-\theta)^2}}{\left[1+e^{-\lambda\theta^2}\right]^2} \qquad ...(3)$$

$$x\xrightarrow{\lim}\infty f(x;\theta,\lambda)=\frac{-4\lambda\theta e^{-\lambda\theta^2}}{\left[1+e^{-\lambda\theta^2}\right]^2}=0 \qquad ...(4)$$

3. Survival and Hazard Functions

3.1Survival Function

If X ~ HLRD (θ,λ), then Survival Function of HLRD is given by

$$S(x,\theta,\lambda)=\frac{2e^{-\lambda(x-\theta)^2}}{1+e^{-\lambda(x-\theta)^2}};x>\theta,\lambda>0 \qquad ...(5)$$

3.2Hazard Function

If X ~ HLRD (θ,λ), then Hazard Function of HLRD is given by

$$h(x;\theta,\lambda)=\frac{2\lambda(x-\theta)}{\left[1+e^{-\lambda(x-\theta)^2}\right]};x>\theta,\lambda>0 \qquad ...(6)$$

Here (θ,λ) are Location and Scale parameters.

4. Quantile Function and Random Generation

4.1Quantile Function

If x ~ HLRD (θ,λ), then Quantile Function is obtained as follows

$$Q(x;\ \theta,\lambda)=\theta+\frac{1}{\sqrt{\lambda}}\sqrt{\log p}$$

$$\theta+\sqrt{\frac{\log p}{\lambda}} \qquad ...(7)$$

4.2Random Generation

Substitute F(x) = u in (2) , we obtain

$$x=\theta+\frac{1}{\sqrt{\lambda}}[\log Y]^{1/2} \qquad ...(8)$$

Equation (8) can be used to simulate HLRvariable.

5 Statistical Properties of HLRD

5.1. Moments

The following theorem gives moments of HLRD

Theorem:

The rth moment about the origin of X ~ HLRD (θ,λ) is given by

$$\mu_r^1 = E(x^r) = 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{r}{k}\binom{-2}{j}\frac{\theta^{r-k}}{\left(\sqrt{\lambda}\right)^k}\left[\frac{\Gamma\left(\frac{k+2}{2},0\right)}{(-j-1)^{k/2}(j+1)}-\frac{\Gamma\left(\frac{k+2}{2},(-j-1)\lambda\theta^2\right)}{(-j-1)^{k/2}(j+1)}\right]$$

Proof: We have

$$\mu_r^1 = E(x^r)$$

$$= \int_0^{\infty} x^r.\frac{4\lambda(x\text{-}\theta)\mathrm{e}^{-\lambda(x\text{-}\theta)^2}}{\left[1+\mathrm{e}^{-\lambda(x\text{-}\theta)^2}\right]^2}\mathrm{d}x \qquad \ldots(9)$$

$$= 2\int_{1+e^{-\lambda\theta^2}}^{1} t^{-2}\left[\log(t-1)\right]^{k/2}dt \qquad \ldots(10)$$

$$= 2\int_{\lambda\theta^2}^{0} z^{k/2}\frac{e^z}{\left(1+e^z\right)^2}dz \qquad \ldots(11)$$

$$= 2\sum_{k=0}^{\infty}\binom{r}{k}\frac{\theta^{r-k}}{\left(\sqrt{\lambda}\right)^k}\int_{\lambda\theta^2}^{0} z^{k/2}\frac{e^z}{(1+e^z)^2}dz \qquad \ldots(12)$$

$$\left(1+e^z\right)^{-2} = \sum_{j=0}^{\infty}\binom{-2}{j}\left(e^z\right)^j \qquad \ldots(13)$$

$$= 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{r}{k}\binom{-2}{j}\frac{\theta^{r-k}}{\left(\sqrt{\lambda}\right)^k}\int_{\lambda\theta^2}^{0} z^{k/2}e^{z(1+j)}dz \qquad \ldots.(14)$$

Consider

$$= \int_{\lambda\theta^2}^{0} z^{k/2}e^{z(1+j)}dz \qquad \ldots(15)$$

$$= \left[\frac{\Gamma\left(\frac{k+2}{2},0\right)}{(-j-1)^{k/2}(j+1)}-\frac{\Gamma\left(\frac{k+2}{2},(-j-1)\lambda\theta^2\right)}{(-j-1)^{k/2}(j+1)}\right] \qquad \ldots(16)$$

Substitute (16) in(14)

$$\mu_r^1 = 2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\binom{r}{k}\binom{-2}{j}\frac{\theta^{r-k}}{\left(\sqrt{\lambda}\right)^k}\left[\frac{\Gamma\left(\frac{k+2}{2},0\right)}{(-j-1)^{k/2}(j+1)}-\frac{\Gamma\left(\frac{k+2}{2},(-j-1)\lambda\theta^2\right)}{(-j-1)^{k/2}(j+1)}\right]$$

5.2. Moment Generating Function

Theorem2:- prove that

$$M_x(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t\theta)^j}{t^i\left(\sqrt{\lambda}\right)^i i!\,j!}\left[\left[\frac{\Gamma\left(\frac{i}{2}+1\right),0}{(-j-1)^{k/2}(j+1)}\right]-\left[\frac{\Gamma\left(\frac{i}{2}+1\right),(-j-1)\lambda\theta^2}{(-j-1)^{k/2}(j+1)}\right]\right]$$

If X ~ HLRD (θ,λ), Moment Generating Function is given by

$$M_x(t)=E(e^{tx})=\int_{x=0}^{\infty}e^{tx}.f(x)dx \quad \ldots(17)$$

$$=\int_0^{\infty}e^{tx}.\frac{4\lambda(x-\theta).e^{-\lambda(x-\theta)^2}}{\left[1+e^{-\lambda(x-\theta)^2}\right]^2}dx \quad \ldots(18)$$

$$=\int_0^{\infty}\frac{e^{tx}.e^{-\lambda(x-\theta)^2}}{\left[1+e^{-\lambda(x-\theta)^2}\right]^2}dx \quad \ldots(19)$$

$$=\int_{1+e^{-\lambda\theta^2}}^{1}e^{tx}\left(\frac{-dt}{t^2}\right) \quad \ldots.(20)$$

$$=-\int_{1+e^{-\lambda\theta^2}}^{1}t^{-2}.e^{tx\left[\theta+\frac{1}{\sqrt{\lambda}}\sqrt{\log(-t)}\right]}dt \quad \ldots(21)$$

$$=\int_{1}^{1+e^{-\lambda\theta^2}}t^{-2}.e^{t\left[\theta+\frac{1}{\sqrt{\lambda}}\sqrt{\log(-t)}\right]}dt \quad \ldots(22)$$

$$e^{t\left[\frac{1}{\sqrt{\lambda}}\sqrt{\log(-t)}\right]}=\sum_{i=0}^{\infty}\frac{t^i\left(\frac{1}{\sqrt{\lambda}}\sqrt{\log(-t)}\right)^i}{i!} \quad \ldots(23)$$

$$e^{t\theta}=\sum_{j=0}^{\infty}\frac{(t\theta)^j}{j!} \quad \ldots(24)$$

Substitute (23) and(24) in(22)

$$=\int_{1}^{1+e^{-\lambda\theta^2}}t^{-2}\sum_{i=0}^{\infty}\frac{t^i\left(\frac{1}{\sqrt{\lambda}}\sqrt{\log(-t)}\right)^i}{i!}\sum_{j=0}^{\infty}\frac{(t\theta)^j}{j!}dt \quad \ldots(25)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^i\left(\frac{1}{\sqrt{\lambda}}\right)^i}{i!}.\frac{(t\theta)^j}{j!}\int_{\lambda\theta^2}^{0}t^{-2}\left(\sqrt{\log(-t)}\right)^i dt \quad \ldots(26)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t\theta)^{j}}{t^{i}\left(\sqrt{\lambda}\right)^{i}i!\,j!}\left[\left[\frac{\Gamma\left(\frac{i}{2}+1\right),0}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]-\left[\frac{\Gamma\left(\frac{i}{2}+1\right),\left(-j-1\right)\lambda\theta^{2}}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]\right] \quad ....(27)$$

$$M_{x}(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t\theta)^{j}}{t^{i}\left(\sqrt{\lambda}\right)^{i}i!\,j!}\left[\left[\frac{\Gamma\left(\frac{i}{2}+1\right),0}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]-\left[\frac{\Gamma\left(\frac{i}{2}+1\right),\left(-j-1\right)\lambda\theta^{2}}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]\right]$$

5.3. Cumulative Generating Function

If X ~ HLRD (θ,λ) Cumulative Generating Function is given by

$K_{x}(t)=\log M_{x}(t)$

$$=\log\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(t\theta)^{j}}{t^{i}\left(\sqrt{\lambda}\right)^{i}i!\,j!}\left[\left[\frac{\Gamma\left(\frac{i}{2}+1\right),0}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]-\left[\frac{\Gamma\left(\frac{i}{2}+1\right),\left(-j-1\right)\lambda\theta^{2}}{\left(-j-1\right)^{k/2}\left(j+1\right)}\right]\right]\right) \quad ...(28)$$

6 Estimation of parameters of HLRD

6.1. Maximum Likelihood method of estimation

Let x_1, x_2,..., x_n be a random sample of size n from HLRD (θ,λ) , then the likelihood function L of this sample is defined as

$$\ln=\ln\left(\prod_{i=1}^{n}\left(x;\theta,\lambda\right)\right)$$

$$=\ln\prod_{i=1}^{n}\left[\frac{4\lambda(x-\theta)e^{-\lambda(x-\theta)^{2}}}{\left[1+e^{-\lambda(x-\theta)^{2}}\right]^{2}}\right] \quad ...(29)$$

$$\ln=n\ln 4\lambda+n\sum_{i=1}^{n}\ln\left(x-\theta\right)-\lambda\sum_{i=1}^{n}\left(x_{i}-\theta\right)^{2}+4\lambda\sum_{i=1}^{n}\ln\left(x_{i}-\theta\right)^{2} \quad ...(30)$$

Calculating the 1st and 2nd order partial derivative of (31) with respective to (θ,λ) and then 1st order partial derivatives equating to zero we get the following equations

$$\frac{\partial\ln}{\partial\theta}=4\lambda\sum\left(x_{i}-\theta\right)-\sum\frac{1}{\left(x_{i}-\theta\right)}\left[8\lambda+n\right] \quad ...(31)$$

$$\frac{\partial\ln}{\partial\lambda}=\frac{n}{4\lambda}+8\sum_{i=1}^{n}\ln\left(x_{i}-\theta\right)^{2}-\sum_{i=1}^{n}\left(x_{i}-\theta\right)^{2} \quad ...(32)$$

$$\frac{\partial\ln}{\partial\lambda}=0$$

$$\frac{n}{4\lambda}=\sum_{i=1}^{n}\left(x_{i}-\theta\right)^{2}-4\sum_{i=1}^{n}\ln\left(x_{i}-\theta\right)^{2}$$

$$\hat{\lambda} = \frac{n}{4\left[\sum_{i=1}^{n}(x_i-\theta)^2 - 2\sum_{i=1}^{n}\ln(x_i-\theta)^2\right]} \quad \ldots (33)$$

2nd order partial derivative is given by

$$\frac{\partial \ln}{\partial\theta\partial\lambda} = 4\sum(x_i-\theta) - \sum\frac{8}{(x_i-\theta)}$$

$$= \frac{4\left[\sum(x_i-\theta)\right]^2 - 8}{\sum(x_i-\theta)} \quad \ldots (34)$$

6.2. Asymptotic Confidence bounds:

Here we derive the asymptotic confidence bounds for unknown parameters θ, λ *when* $\theta < x$,

$\lambda > 0$ The simplest large sample approach is to assume that the MLEs (θ,λ)are approximately normal with mean (θ, λ) and covariance matrix I_0^{-1}, where I_0^{-1} is the inverse of the observed information matrix which defined as follows

$$I_0^{-1} = \begin{bmatrix} \frac{d^2\ln}{d\theta} & \frac{d^2\ln}{d\theta d\lambda} \\ \frac{d^2\ln}{d\theta d\lambda} & \frac{d^2\ln}{d\lambda} \end{bmatrix}$$

$$\frac{\partial^2 \ln}{\partial\theta^2} = -4\lambda\sum(x_i-\theta) + \sum_{i=1}^{n}(x_i-\theta)^{-2}\left[8\lambda+n\right] \quad \ldots (35)$$

$$\frac{\partial^2 \ln}{\partial\lambda^2} = \frac{-n}{4\lambda^2} \quad \ldots (36)$$

Conclusions:

In this paper, we introduced two parameter Half Logistic Rayleigh Distribution (HLRD). Some characteristics of new distribution, such as ordinary moments and generating functions are obtained.

References:

1. Abdul-Moniem I. B., “Recurrence relations for moments of lower generalized order statistics from exponentiated Lomax distribution and its characterization,” “International Journal of Mathematical Archive”, vol. 3, pp. 2144–2150, 2012. View at Google Scholar
2. Abed Al-Kadim and Abdalhussain Boshi, ”Exponential Pareto Distribution”, ”Mathematical Theory and Modeling”, vol.3,pp.135-146(2013).
3. Abd Elfattah AM, Hassan AS, Ziedan DM. Efficiency of maximum likelihood estimators under different censored Samplin schemes for Rayleigh distribution. Interstat. 2006;1-16.
4. Abd Elfattah AM, Hassan AS, Ziedan DM. Efficiency of Bayes estimator for Rayleigh distribution. Interstat. 2006;1-15.
5. Ahmed ZA, Nofal ZM, Abd El Hadi N. Exponentiated transmuted generalized Raleigh distribution: A new four parameter Rayleigh distribution. Pakistan Journal of Statistics and Operation Research. 2015;11(1):115-134.
6. Cordeiro G. M., M. Alizadeh, and P. . Marinho, “The type I half-logistic family of distributions,” Journal of Statistical Computation and Simulation, vol. 86, no. 4, pp. 707–728, 2016.
7. Dey S, Das MK. A note on prediction interval for a Rayleigh distribution: Bayesian approach. American Journal of Mathematical & Management Science. 2007;1&2:43-28. Ul Haq et al.; JAMCS, 29(1): 1-19, 2018; Article no.JAMCS.44266 19
8. Dey S. Comparison of Bayes estimators of the parameter and reliability function for Rayleigh distribution under different loss function. Malaysian Journal of Mathematical Science. 2009;3:217- 261.
9. Haq MA. Transmuted exponentiated inverse Rayleigh distribution. Journal of Statistics Applications & Probability. 2016;5(2):337-343.
10. Hendi ML, Abu SE, Alraddadi AA. A Bayesian analysis of record statistics from Rayleigh model, International Mathematical Forum. 2007;2(13):619-63
11. Howlader HA, Hossian A. On Bayesian estimation and prediction from Rayleigh based on type II censored data. Communication in Statistics; Theory & Methods. 1995;24:2249-2259.
12. Kareema Abid Al Kadhim and Ameer Dhea Il-deen Al Musawy(2017), ”Uniform-Pareto distribution”, “Journal of University of Babylon, Pure and Applied Sciences”, vol. 25, pp.1663-1673.
13. Lai C. D., “Constructions and applications of lifetime distributions,” Applied Stochastic Models in Business and Industry, vol.29, no. 2, pp. 127–140, 2013.
14. Lee E. T. and J. W. Wang, Statistical Methods for Survival Data Analysis, John Wiley and Sons, New York, NY, USA, 3rd edition, 2003.
15. Lalitha S, Mishra A. Modified maximum likelihood estimation for Rayleigh distribution. Communication in Statistics, Theory & Methods. 1996;25(2):389-401
16. Merovci F. Transmuted Rayleigh distribution. Austrian Journal of Statistics. 2016;42(1):21-31.
17. Merovci F. Transmuted generalized Rayleigh distribution. Journal of Statistics Applications & Probability. 2014;3(1):9-20.
18. Masood Anwar and Amna Bibi(2018), “The Half-Logistic Generalised Weibull Distribution” , “Journal of Probability and Statistics”, vol.2018,pp.1-12.
19. Masood Anwar and Jawaria Zahoor(2018),The “Half-Logistic Lomax Distribution for Life time Modeling” , “Journal of Probability and Statistics”, vol.2018, pp.1-12
20. Rényi A. On measures of entropy and information. In: Proceedings of the 4th Fourth Berkeley Symposium on Mathematical Statistics & Probability. 1961;547-561. University of California Press, Berkeley.
21. SHARMA, M., SHANKER, R., (2013). A two-parameter Lindley distribution for modeling waiting and survival times data, Applied Mathematics, 4, 363-368.