# Existence of Solution for the First Order Functional Differential Equation in Banach Algebra with Maxima

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***Abstract:*** In this paper we proves the existence for the first order functional differential equations in Banach algebra with maxima. A step by step procedure (algorithm) for the solution is developed and it is shown that certain sequence of successive approximations converges, monotonically to the solution of the related perturbed differential equations under some suitable mixed hybrid conditions.

***Index Terms:*** Hybrid differential equation, Initial value problem(IVP) Approximation theorem, functional differential equations with maxima, existence theorem.

## 1. Introduction

The importance of the functional differential equations with maxima lies in the real world problems of automatic regulation of the technical systems and such differential equations is a special class of functional differential equations n which the present state of unknown function related to the systems based upon the maximum value of the earlier state in some earlier interval of time. See Mangomedov[11, 12] and the references therein. Again, Myshkis [15] signalized the need to study the differential equations with maxima and since then several classes of ordinary and partial differential equations with maxima have been discussed in the literature for different qualitative aspects of the solutions. A handful particulars on the topic appears in the monograph of Bainov and Hristova [1] and the research papers by Otrocol and Rus[11], Dhage and Otrocol [10] and the references therein. Also hybrid differential equations are build up in Dhage[2] to cover different dynamic systems of the real world problem.

In this research work we blend these two ideas together and studied the hybrid differential equations with maxima for existence and numerical aspects of the solutions. The originality of our paper lies in the fact that our problem as well as our solution is new to the literature in the theory of functional differential equations with maxima.

## 2. Statement of the Problem

Given a closed and bounded interval $I=[l_0,l_0+a]$ of the real line R for some $l_0,a\in R$ with $l_0\geq 0, a>0$ ,consider the initial value problem (IVP) of first order functional differential equation (in short FDE) with maxim viz.,

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t),X(t))}\right]=g\left[t,x(t),X(t)\right],\qquad t\in I, \qquad \text{……………………..(2.1)}$$

$$x(t_0)=\eta_0\in\Re_+,$$

where $f:I\times\Re\times\Re\to\Re\setminus\{0\}\, and\ g:I\times\Re\times\Re\to\Re$ are continuous functions and

$$X(t)=\max_{t_0\leq\zeta\leq t}x(\zeta)\ \ for\ some\ t\in I.$$

By the solution of the FDE(2.1) we mean a function $x\in C(I,\Re)$ that satisfies

(i) $t\mapsto\dfrac{x(t)}{f(t,x(t),X(t))}$ is differentiable function, and

(ii) $x$ satisfies the equation in (2.1) on I,

where $C(I,\Re)$ is the space of continuous real valued functions defined on I. In the following section we give some basic definitions and results which will be used in the subsequent parts of the paper.

## 3. Auxiliary results

In this paper, unless and until mentioned, it follows that, let E denote a partially ordered real normed linear space with an order relation $\leq$ and the norm $\|\cdot\|$ in which the usual addition and the scalar multiplication by positive real numbers are preserved by $\leq$. Some details of a partially ordered normed linear space appear in Heikkila and Lakshmikantham [12] and the references therein. Two elements a and b in E are said to be comparable if either the relation $a\leq b\ or\ b\leq a$ hold.

A non-empty subset *C* of E is called a chain or totally ordered if all the elements of *C*are comparable. It is known that set E is regular if a non-decreasing sequence $\{a_n\}$ in set E is such that $a_n\to a^*\ as\ n\to\infty$, then $a_n\leq a^*\,(resp.\ a_n\geq a^*)\ \ for\ all\ n\in N$ .The conditions guaranteeing the regularity of set E may be found in Heikkila and Lakshmikantham [12] and the references therein.

We require the following definitions in the sequel.

**Definition 3.1.** A mapping $\Im:E\to E$ is called isotone or nondecreasing if it preserves the order relation $\leq$,that is , if $x\leq y\Rightarrow\Im x\leq\Im y\ \ for\ all\ x,y\in E$ .

**Definition 3.2.** A $\Im: E \to E$ is called partially continuous at a point $a \in E$ if for $\in > 0$ there exists a $\delta > 0$ Such that $\|\Im x - \Im y\| < \in$ whenever x is comparable to a and $\|x - a\| < \delta$. $\Im$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that $\Im$ is partially continuous on E, then it is continuous on every chain C contained in E.

**Definition 3.3.** A non-empty subset Ś of the partially ordered Banach space(POBS) E is called partially bounded if every chain C in Ś is bounded. An operator $\Im: E \to E$ is called partially bounded if every chain C in T(E) is bounded. $\Im$ is called uniformly partially bounded if all chains C in $\Im(E)$ are bounded by the unique Constant. $\Im$ is called bounded if T(E) is a bounded subset of E.

**Definition 3.4.** A non-empty subset Ś of the POBS E is called partially compact if every chain C in Ś is compact. An operator $\Im: E \to E$ is called partially compact if every chain C in $\Im(E)$ is relatively compact subset of E. $\Im$ is called uniformly partially compact if $\Im(E)$ is a uniformly partially bounded and partially compact on E. $\Im$ is called partially totally bounded if for any bounded subset Ś of E , $\Im$ ( Ś) is a relatively compact subset of E. If $\Im$ is partially continuous and partially totally bounded, then it is called partially continuous on E.

**Remark :** Suppose that $\Im$ is a nondecreasing operator on E into itself. Then $\Im$ is a partially bounded or partially compact if $\Im(C)$ is a bounded or relatively compact subset of E for each chain C in E.

**Definition3.5.** The order relation $\leq$ and the metric d on a non-empty set E are said to be comparable if $\{y_n\}$ is a monotonic sequence , that is, monotonic decreasing or monotonic increasing sequence in E and if a subsequence $\{y_{n_k}\}$ *of* $\{y_n\}$ converges to y* implies that the original sequence $\{y_n\}$ converges to y*.Similarly, given a partially ordered normed linear space $(E, \leq, \|\cdot\|)$, the order relation $\leq$ and the norm $\|\cdot\|$ are said to be comparable if $\leq$ metric d defined through the norm and the $\|\cdot\|$ are comparable.

**Definition 3.6.** An upper semi –continuous and nondecreasing function $\varphi: \Re_+ \to \Re_+$ is called a ᴅ-function provided $\varphi(0) = 0$. Let $(E, \leq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\Im: E \to E$ is called partially nonlinear $\mathscr{D}$ -Lipschitz if there exists a $\mathscr{D}$ -function $\varphi: \Re_+ \to \Re_+$ such that

$$\|\Im x - \Im y\| \leq \varphi(\|x - y\|) \qquad (3.1)$$

for all comparable elements $x, y \in E$. If $\varphi(r) = kr, k > 0$, then $\Im$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \leq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E : x \geq \theta, \textit{where } \theta \textit{ is the zero element of } E\}$$

and

$$\mathscr{K} = \{E^+ \subset E : uv \in E^+ \textit{ for all } u, v \in E^+\}. \qquad (3.2)$$

The elements of the set $\kappa$ are called the positive vectors in E. The following lemma follows immediately from the definition of the set $\kappa$ which is often times used in the hybrid fixed point theory of Banach algebra and applications to nonlinear differential and integral equations.

**Lemma 3.1 [3]** If $u_1, u_2, v_1, v_2 \in \kappa$ *are such that* $u_1 < v_1$ *and* $u_2 < v_2$, *then* $u_1 u_2 \leq v_1 v_2$.

**Definition 3.7.** An operator $\Im: E \to E$ is said to be positive if the range $R(\Im)$ *of* $\Im$ *is such that* $R(\Im) \subseteq \mathscr{K}$

**Theorem 3.1.[Dhage[3]]** Let $(E, \leq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that every compact chain of E is Compatible Banach Space . Let $\mathscr{A}, \mathscr{B}: E \to \mathscr{K}$ *and* $\mathscr{C}: E \to E$ be three nondecreasing operators such that

(a) $\mathscr{A}$ and $\mathscr{C}$ are partially bounded and partially nonlinear $\mathscr{D}$ -Lipschitz with $\mathscr{D}$ -functions $\psi_{\mathscr{A}}$ and $\psi_{\mathscr{C}}$ respectively.

(b) $\mathscr{B}$ is partially continuous and uniformly partially compact,

(c) $0 < M\psi_{\mathscr{A}}(r) + M\psi_{\mathscr{C}}(r) < r, r > 0$, where $M = Sup\{\|\mathscr{B}(C)\| : C \textit{ is a chain in } E\}$, and

(d) there exists an element $\alpha_0 \in X$ such that $\alpha_0 \leq \mathscr{A}\alpha_0 \mathscr{B}\alpha_0 + \mathscr{C}\alpha_0$ *or* $\alpha_0 \geq \mathscr{A}\alpha_0 \mathscr{B}\alpha_0 + \mathscr{C}\alpha_0$. then the operator equation

$$\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x \qquad (3.3)$$

has a solution x* and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathscr{A}x_n \mathscr{B}x_n + \mathscr{C}x_n, n = 0, 1, 2 \ldots$; converges monotonically to x*.

**Remark 3.2** The condition that every compact chain of E is Banach holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\leq$ and the norm $\|\cdot\|$ in it.

**Remark 3.3** We remark that hypothesis (a) of Theorem 3.1 implies that the operators $\mathscr{A}$ and $\mathscr{C}$ are partially continuous and consequently all the operators $\mathscr{A}$ , $\mathscr{B}$ and $\mathscr{C}$ in the theorem are partially continuous on E.

The regularity of E in the above Theorem 3.1 may be replaced with a stronger continuity condition of the operators $\mathscr{A}$ , $\mathscr{B}$ and $\mathscr{C}$ on E which is a result proved in Dhage[3,4]

## 4. Main results

The FDE (2.1) is considered in the function space $C(I, \Re)$ of continuous real-valued functions defined on J. We define a norm

$\|\cdot\|$ and the order relation $\leq$ in $C(I,\Re)$ by

$$\|x\| = \sup_{t\in I}|x(t)| \tag{4.1}$$

and

$$x \leq y \Leftrightarrow x(t) \leq y(t) \tag{4.2}$$

for all $t \in I$ respectively. Clearly, $C(I,\Re)$ is a Banach algebra with respect to above supremum norm and is also partially ordered with respect to the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(I,\Re)$ has some nice properties with respect to the above order relation in it. The following lemma follows by an application of Arzela –Ascoli theorem.

**Lemma 4.1** Let $\left(C(I,\Re),\leq,\|\cdot\|\right)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (4.1) and (4.2) respectively. Then every partially compact subset S of $C(I,\Re)$ is Banach.

**Proof.** The proof of the lemma is given in Dhage and Dhage[6,7,8,9] and so we omit the details of it. We use the following definition for proving main result

**Definition 4.1** A function $u \in C(I,\Re)$ is said to be a lower solution of the FDE (2.1) if the function $t \mapsto \dfrac{x}{f(t,u(t),U(t))}$ is differentiable and satisfies

$$\left.\begin{array}{c}\dfrac{d}{dt}\left[\dfrac{u(t)}{f(t,u(t),U(t)}\right] \leq g(t,u(t),U(t)),\\ u(t_0)\leq \eta_0\end{array}\right\}$$

for all $t \in I, where\, U(t) = \max_{t_0\leq\xi\leq t} u(\xi)\ for\ t \in I$. Similarly , a function $v \in C(I,\Re)$ is said to be an upper solution of the FDE (2.1) if it satisfies the above property and inequalities with reverse sign.

We consider following set of assumptions in what follows:

($A_0$) The map $x \mapsto \dfrac{x}{f(t,x,x)}$ is injection for each $t \in I$.

($A_1$) $f$ defines a function $f: I\times\Re\times\Re \to \Re_+$.

($A_2$) There exists a constant $M_f > 0$ *such that* $0 < f(t,x,y) \leq M_f$ *for all* $t \in I$ *and* $x, y \in \Re$.

($A_3$) There exists a $\mathscr{D}$ -function $\varphi$ such that

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \varphi(Max\{x_1 - y_1, x_2 - y_2\}),$$

for all $t \in I$ *and* $x_1,x_2,y_1,y_2 \in \Re, x_1 \geq y_1$ *and* $x_2 \geq y_2$.

($B_1$) g defines a function $g: I\times\Re\times\Re \to \Re_+$.

($B_2$) There exists a $M_g > 0$ *such that* $g(t,x,y) \leq M_g$ *for all* $t \in I$ *and* $x, y \in \Re$.

($B_3$) $g(t,x,y)$ is nondecreasing in $x$ *and* $y$ for all $t \in I$.

($C_1$) The FDE (2.1) has a lower solution $u \in C(I,\Re)$.

**Remark 4.1.** Note that the hypothesis ($A_0$) holds in particular if the function $x \mapsto \dfrac{x}{f(t,x,x)}$ is increasing for each $t \in I$.

**Lemma 4.2.** Suppose that hypothesis ($A_0$) holds. Then a function $x \in C(I,R)$ is a solution of the FDE (2.1) if and only if it is a solution of the nonlinear functional integral equation (in short FIE) ,

$$x(t) = f(t,x(t),X(t))\left(ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x(s),X(s))ds\right) \tag{4.3}$$

for all $t \in I$, *where* $c = \dfrac{x_0 e^{\lambda t_0}}{f(t_0,\eta_0,\eta_0)}$.

**Theorem 4.1.** Assume that hypothesis ($A_0$)-($A_3$),($B_1$)-($B_3$) and ($C_1$) hold. If

$$\left(\frac{\eta_0}{f(t_0,\eta_0,\eta_0)} + M_g a\right)\varphi(r) + \omega(r) < r, r > 0. \tag{4.4}$$

Then the FDE (2.1) has a solution $x^*$ defined on $I$ and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_{n+1} = \left[f(t,x_n(t),X_n(t))\right]\left(ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x_n(s),X_n(s))ds\right), \tag{4.5}$$

where $x_1 = u$, converges monotonically to $x^*$.

**Proof.** Set $E = C(I, \mathfrak{R})$ then by Lemma 4.1 , every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define three operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ on $E$ by

$$\mathscr{A}x(t) = f(t, x(t), X(t)), t \in I, \quad (4.6)$$

$$\mathscr{B}x(t) = ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)), t \in I, \quad (4.7)$$

and

$$\mathscr{C}x(t) = f(t, x(t), X(t)), t \in I. \quad (4.8)$$

We know from the continuity of the integral, that $\mathscr{A}$ and $\mathscr{B}$ define the map $\mathscr{A}, \mathscr{B} : E \to \mathscr{K}$ .

Now by Lemma 4.2, the FDE(2.1) is equivalent to the operator equation

$$\mathscr{A}x(t)\mathscr{B}x(t) + \mathscr{C}x(t) = x(t), t \in I. \quad (4.9)$$

We shall show that the operators $\mathscr{A}$ , $\mathscr{B}$ and $\mathscr{C}$ satisfy all the conditions of theorem 3.1. This is achieved in the set of resultant steps.

**Step I:** $\mathscr{A}$ and $\mathscr{B}$ are nondecreasing on E.

Let $x, y \in E$ be such that $x \geq y$. then $x(t) \geq y(t)$ *for all* $t \in I$. Since $y$ is continuous on $[a,t]$, there exists a $\xi^* \in [a,t]$ *such that* $y(\xi^*) = \max_{a \leq \xi \leq t} y(\xi)$. By definition of $\leq$, one has $x(\xi^*) \geq y(\xi^*)$.

consequently, we obtain

$$X(t) = \max_{a \leq \xi \leq t} x(\xi) \geq x(\xi^*) \geq y(\xi^*) = \max_{a \leq \xi \leq t} y(\xi) = Y(t)$$

for each $t \in I$ . Then by hypothesis ($A_3$), we obtain

$$\mathscr{A}x(t) = f(t, x(t), X(t)) \geq f(t, x(t)Y(t)) = \mathscr{A}y(t),$$

for each $t \in I$ . This shows that $\mathscr{A}$ is nondecreasing operator on E into E. Similarly using hypothesis ($B_3$), it is shown that the operator $\mathscr{B}$ is also nondecreasing on E into itself. Thus $\mathscr{A}$ and $\mathscr{B}$ are nondecreasing positive operators on E into itself.

**Step II:** $\mathscr{A}$ is partially bounded and partially $\mathscr{D}$ -Lipschitz on E.

Let $x \in E$ be arbitrary. Then by ($A_2$),

$$|\mathscr{A}x(t)| \leq |f(t, x(t), X(t)| \leq M_f,$$

for all $t \in I$. Taking supremum over t, we get $\|\mathscr{A}x\| \leq M_f$ and so , $\mathscr{A}$ is bounded. This further implies that $\mathscr{A}$ is partially bounded on E.

Now , let $x, y \in E$ be such that $x \geq y$. Then, we have

$$|x(t) - y(t)| \leq |X(t) - Y(t)|$$

and that

$$\begin{aligned} |X(t) - Y(t)| &= X(t) - Y(t) \\ &= \max_{t_0 \leq \xi \leq t} x(\xi) - \max_{t_0 \leq \xi \leq t} y(\xi) \\ &\leq \max_{t_0 \leq \xi \leq t} [x(\xi) - y(\xi)] \\ &= \max_{t_0 \leq \xi \leq t} |x(\xi) - y(\xi)| \\ &\leq \|x - y\| \end{aligned}$$

for each $t \in I$. Thus we obtain from this

$$\begin{aligned}|Ax(t)-By(t)| &= |f(t,x(t),X(t))-f(t,y(t),Y(t))| \\ &\le \varphi\left(\max_{t_0\le\xi\le t}\{|x(t)-y(t)|,|X(t)-Y(t)|\}\right) \\ &\le \varphi(\|x-y\|),\end{aligned}$$

for all $t \in I$ *with* $x \ge y$. Hence , $\mathscr{A}$ is a partially nonlinear $\mathscr{D}$ -Lipschitz on E with a $\mathscr{D}$ -function $\varphi$ and which further implies that $\mathscr{A}$ is a partially continuous operator on E. Similarly , it can be shown that $\mathscr{C}$ is a partially nonlinear $\mathscr{D}$ -Lipschitz on E with a $\mathscr{D}$ -function $\omega$ and which is again a partially continuous operator on E.

**Step III:** $\mathscr{B}$ is partially continuous on E.

Let $\{x_n\}_{n\in N}$ be a sequence in a chain C of E such that $x_n \to x$ *for all* $n \in N$. Then by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\mathscr{B}x_n(t) &= \lim_{n\to\infty} ce^{-\lambda t} + \lim_{n\to\infty}\int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x_n(s),X_n(s))ds \\ &= ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)}\left[\lim_{n\to\infty} g(s,x_n(s),X_n(s))\right]ds \\ &= ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)}\left[\lim_{n\to\infty} g(s,x(s),X(s))\right]ds \\ &= \mathscr{B}x(t),\end{aligned}$$

for all $t \in I$. This shows that $\mathscr{B}x_n$ converges to $\mathscr{B}x$ point wise on I.

Now we will prove that $\{\mathscr{B}x_n\}_{n\in N}$ is an equicontinuous sequence of functions in E.

Let $t_1, t_2 \in E$ *with* $t_1 < t_2$. Then

$$\begin{aligned}|\mathscr{B}x_n(t_2)-\mathscr{B}x_n(t_1)| \le &\left|ce^{-\lambda t_1}-ce^{-\lambda t_2}\right| \\ &+\left|\int_{t_0}^{t_1} e^{-\lambda(t_1-s)} g(s,x_n(s),X_n(s))ds - \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s,x_n(s),X_n(s))ds\right| \\ &+\left|\int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s,x_n(s),X_n(s))ds - \int_{t_0}^{t_2} e^{-\lambda(t_2-s)} g(s,x_n(s),X_n(s))ds\right| \\ \le &\left|ce^{-\lambda t_1}-ce^{-\lambda t_2}\right| + \left|\int_{t_0}^{t_1}\left|e^{-\lambda(t_1-s)}-e^{-\lambda(t_2-s)}\right|\left|g(s,x_n(s),X_n(s))\right|ds\right| \\ &+\left|\int_{t_2}^{t_1}\left|g(s,x_n(s),X_n(s))\right|ds\right| \\ \le &\left|ce^{-\lambda t_1}-ce^{-\lambda t_2}\right| + M_g\int_{t_0}^{t_0+a}\left|e^{-\lambda(t_1-s)}-e^{-\lambda(t_2-s)}\right|ds + M_g|t_1-t_2| \\ &\to 0 \text{ as } (t_2-t_1)\to 0\end{aligned}$$

uniformly for all $n \in N$ . This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and hence this proves that $\mathscr{B}$ is partially continuous on E.

**Step IV:** $\mathscr{B}$ is uniformly partially compact operator on E.

Let C be an arbitrary chain in E. We have to show that $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element . Then there is an element $x \in C$ such that $y = \mathscr{B}x$ .Now , by hypothesis ($B_2$),

$$\begin{aligned}
|y(t)| &\leq \left| ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) ds \right| \\
&\leq \left| ce^{-\lambda t} \right| + \left| \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s)) ds \right| \\
&\leq \left| \frac{\eta_0}{f(t_0, \eta_0, \eta_0)} \right| + \int_{t_0}^{t_0+a} |g(s, x(s), X(s))| ds \\
&\leq \left| \frac{\eta_0}{f(t_0, \eta_0, \eta_0)} \right| + M_g a = M.
\end{aligned}$$

for all $t \in I$. Taking supremum over t, we obtain $\|y\| = \|\mathscr{B}x\| \leq M \ \ for\ all\ y \in \mathscr{B}(C).$

Hence, $\mathscr{B}(C)$ is uniformly bounded subset of E. Moreover, $\|\mathscr{B}(C)\| \leq M$ for all chains C in E. Hence ,

$\mathscr{B}$ is uniformly partially bounded operator on E.

Next, we will show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in I \ with\ t_1 < t_2$. Then, for any $y \in \mathscr{B}(C)$ one has

$$\begin{aligned}
|y(t_2) - y(t_1)| &= |\mathscr{B}x(t_2) - \mathscr{B}x(t_1)| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} g(s, x(s), X(s)) ds - \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds - \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x(s), X(s)) ds \right| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + \left| \int_{t_0}^{t_1} \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| |g(s, x(s), X(s)) ds| \right| \\
&\quad + \left| \int_{t_2}^{t_1} |g(s, x(s), X(s)) ds| \right| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + M_g \int_{t_0}^{t_0+a} \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| ds + M_g |t_1 - t_2|
\end{aligned}$$

$$\Rightarrow |y(t_2) - y(t_1)| \to 0 \ as\ t_2 - t_1 \to 0$$

uniformly for all $y \in \mathscr{B}(C)$.Hence $\mathscr{B}(C)$ is an equicontinuous subset of E. Now , $\mathscr{B}(C)$ is uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathscr{B}$ is a uniformly partially compact operator on E into itself.

**Step V:** u satisfies the operator inequality $u \leq \mathscr{A}u\mathscr{B}u + \mathscr{C}u.$

By hypothesis ($C_1$), the FDE (2.1) has a lower solution u defined on I. Thus we have

$$\left.\begin{aligned}
\frac{d}{dt}\left[ \frac{u(t)}{f(t, u(t), U(t))} \right] &\leq g(t, u(t), U(t)), \\
u(t_0) &\leq \eta_0,
\end{aligned}\right\} \tag{4.10}$$

for all $t \in I$. Multiplying the above inequality by the integrating factor $e^{\lambda t}$, we obtain

$$\left( e^{\lambda t} \frac{u(t)}{f(t,u(t),U(t))} \right)' \leq e^{\lambda t} g(t,u(t),U(t)), \tag{4.11}$$

for all $t \in I$. A direct integration of (4.11) from $t_0$ and $t$ yields

$$u(t) \leq f(t,u(t),U(t)) + \left( ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,u(s),U(s))ds \right), \tag{4.12}$$

for all $t \in I$. From definition of the operator $\mathscr{A}$ and $\mathscr{B}$ it follows that $u(t) \leq \mathscr{A}u(t)\mathscr{B}u(t) + \mathscr{C}u(t)$, for all $t \in I$. Hence $u \leq \mathscr{A}u\mathscr{B}u + \mathscr{C}u$.

**Step VI:** $\mathscr{D}$ -function $\varphi$ and $\omega$ satisfy the growth condition

$$0 < M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{A}}(r) < r, r > 0.$$

Finally , the $\mathscr{D}$ -function $\varphi$ of the operator $\mathscr{A}$ satisfies the inequality given in hypothesis (d) of Theorem (3.1) . Now from the estimate given in step IV, it follows that

$$M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{A}}(r) \leq \left( \frac{\eta_0}{f(t_0,\eta_0,\eta_0)} + M_g a \right) + \varphi(r) + \omega(r) < r$$

for all $r > 0$.

Thus $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem (3.1) and we apply it to conclude that the operator equation $\mathscr{A}x\mathscr{B}x + \mathscr{C}x = x$ has a solution. Consequently the integral equation and the FDE(2.1) has a positive solution $x^*$ defined on $I$. Furthermore, the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (4.5) converges monotonically $x^*$. This completes the proof.

**Remark 4.2.** The conclusion of Theorem 4.1 also remains true if we replace the hypothesis with the following :

($C_2$) The FDE(2.1) has an upper solution $v \in C(I, \mathfrak{R})$.

The proof of this can be carried out using similar procedure as mentioned above with appropriate modifications.

**Remark 4.3.** We note that if the FDE(2.1) has a lower u as well as upper solution v such that $u \leq v$, then under the given conditions of Theorem (4.1) it has corresponding solutions $x_*$ and $x^*$ and these solution satisfy $x_* \leq x^*$. Hence they are the minimal and maximal solutions of the FDE (2.1) in the vector segment $[u,v]$ of the Banach space $E = C(I, \mathfrak{R})$, where the vector segment $[u,v]$ is a set in $C(I, \mathfrak{R})$ defined by

$$[u,v] = \{x \in C(I,\mathfrak{R}) \mid u \leq x \leq y\}.$$

This is because the order relation $\leq$ defined by (4.2) is equivalent to the order relation defined by the order cone $\mathscr{K} = \{x \in C(I,\mathfrak{R}) \mid x \geq \theta\}$ which is a closed set in $C(I, \mathfrak{R})$.

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