Origin and Applications of Complex Numbers

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Abstract

This is the compilation of historical information from various sources, about the number $i = \sqrt{-1}$. The information has been put together for students of Complex Analysis who are curious about the origin and applications of the subject. Complex numbers were first conceived and defined by the Italian mathematician Gerolamo Cardano, who called them "fictitious", during his attempts to find solutions to cubic equations. This ultimately led to the fundamental theorem of algebra, which shows that with complex numbers, a solution exists to every polynomial equation of degree one or higher. Complex numbers thus form an algebraically closed field, where any polynomial equation has a root.

KEYWORDS: Complex, Polynomial, Descartes method, reducible, irreducible, de Moivre's theorem, etc.

Origin of Comlex Numbers.

1. Al-Khawarizmi $(780-850)$ in his Algebra has solution to quadratic equations of various types. Solutions agree with is learned today at school, restricted to positive solutions [9] Proofs are geometric based. Sources seem to be greek and hindu mathematics. According to G.J. Toomer, quoted by Van der Waerden,

Under the caliph al-Ma'mun (reigned $(813-833)$ al-Khwarizmi became a member of the "House of Wisdom" (Dar al-Hikma), a kind of academy of scientists set up at Baghdad, probably by Caliph Harun al-Rashid, but owing its preeminence to the interest of al-Ma'mun, a great patron of learning and scientific investigation. It was for al-Ma'mun that Al-Khwarizmi composed his astronomical treatise, and his Algebra also is dedicated to that ruler

2. The methods of algebra known to the arabs were introduced in Italy by the Latin transla- tion of the algebra of al-Khwarizmi by Gerard of Cremona $(1114-1187)$ and by the work of Leonardo da Pisa (Fibonacci)$(1170-1250)$.About 1225, when Frederick II held court in Sicily, Leonardo da Pisa was presented to the emperor. A local mathematician posed several problems, all of which were solved by Leonardo. One of the problems was the solution of the equation

$$x^3 + 2x^2 + 10x = 20$$

3. The general cubic equation

$$x^3 + ax^2 + bx + c = 0$$

can be reduced to the simpler form

$$x^3 + px + q = 0$$

through the change of variable $x' = x + \frac{1}{3}a$. This change of variable appears for the first time in two anonymous florentine manuscripts near the end of the 14th century.

If only positive coefficients and positive values of x are admitted, there are three cases, all collectively known as *depressed cubic*:

(a) $x^3 + px = q$

(b) $x^3 = px + q$

(c) $x^3 + q = px$

4. The first to solve equation (1) (and maybe (2) and (3)) was Scipione del Ferro, professor of

U. of Bologna until (1526), when he died. In his deathbed, del Ferro confided the formula to his pupil Antonio Maria Fiore. Fiore challenged Tartaglia to a mathematical contest. The night before the contest, Tartaglia rediscovered the formula and won the contest. Tartaglia in turn told the formula (but not the proof) to Gerolamo Cardano, who signed an oath to secrecy. From knowledge of the formula, Cardano was able to reconstruct the proof. Later, Cardano learned that del Ferro had the formula and verified this by interviewing relatives who gave him access to del Ferro's papers. Cardano then proceeded to publish the formula for all three cases in his Ars Magna (1545). It is noteworthy that Cardano mentioned del Ferro as first author, and Tartaglia as obtaining the formula later in independent manner.

5. A difficulty in case (2) that was not present in the solution to (1) is the possibility of having the square root of a negative number appear in the numerical expression given by the formula. Here is the derivation: Substitute $x = u + v$ into $x^3 = px + q$ to obtain

$$x^3 - px = u^3 + v^3 + 3uv(u+v) - p(u+v) = q$$

Set $3uv = p$ above to obtain $u^3 + v^3 = q$ and also $u^3v^3 = (p/3)^3$. That is, the sum and the product of two cubes is known. This is used to form a quadratic equation which is readily

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$$x = u + v = \sqrt[3]{\frac{1}{2}q + w} + \sqrt[3]{\frac{1}{2}q - w}$$

Where $w = \sqrt{(\frac{1}{2}q)^2 - (\frac{1}{3}p)^3}$

The so-called *casus irreducibilis* is when the expression under the radical symbol in w is negative. Cardano avoids discussing this case in Ars Magna. Perhaps, in his mind, avoiding it was justified by the (incorrect) correspondence between the *casus irreducibilis* and the lack of a real, positive solution for the cubic.

6. According to [9], "Cardano was the first to introduce complex number $a + \sqrt{-b}$ into algebra, but had misgivings about it." In Chapter 37of Ars Magna the following problem is posed: "To divide 10 in two parts, the product of which is 40". It is clear that this case is impossible. Nevertheless, we shall work thus: We divide 10 into two equal parts, making each 5 These we square, $making$ 25. $subtract$ 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book leaving a remainder of -15, the squar root of which added to or subtracted from 5 gives parts the product of which is 40. These will be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ Putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ making $25 - (-15)$ which is +15. Hence this product is 40.

7. Rafael Bombelli authored l'Algebra (1572, and 1579), a set of three books. Bombelli introduces a notation for $\sqrt{-1}$ and calls it "*piú di meno*".

The discussion of cubics in l'Algebra follows Cardano, but now the casus irreducibilis is fully discussed. Bombelli considered the equation

$$x^3 = 15x + 4$$

for which the Cardan formula gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli observes that the cubic has $x = 4$ as a solution, and then proceeds to explain the expression given by the Cardan formula as another expression for $x = 4$ as follows.He Sets

$$\sqrt[3]{2 + \sqrt{-121}} = a + ib$$

from which he deduces

$$\sqrt[3]{2 - \sqrt{-121}} = a - ib$$

and obtains, after algebraic manipulations, a = 2 and b = 1. Thus

$$x = a + bi + a - bi = 2a = 4$$

After doing this, Bombelli commented: " At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof."

8. Ren'e Descartes (1596 − 1650) was a philosopher whose work, La G'eom'etrie, includes his application of algebra to geometry from which we now have Cartesian geometry. Descartes was pressed by his friends to publish his ideas, and he wrote a treatise on science under the title "Discource de la method pour bein conduire sa raison et chercher la v'erit'e dans les sciences". Three appendices to this work were La Dioptrique, Les M'et'eores, and La G'eom'etrie.The treatise was published at Leiden in 1637. Descartes associated imaginary numbers with geometric impossibility.This can be seen from the geometric construction he used to solve the equation $z^2 = az - b^2$,with a and b^2 both positive. According to [1], Descartes coined the term imaginary:

> "For any equation one can imagine as many roots [as its degree would suggest], but in many cases no quantity exists which corresponds to what one imagines."

9. John Wallis (1616 − 1703) notes in his *Algebra* that negative numbers, so long viewed with suspicion by mathematicians, had a perfectly good physical explanation, based on a line with a zero mark, and positive numbers being numbers at a distance from the zero point to the right, where negative numbers are a distance to the left of zero. Also, he made some progress at giving a geometric interpretation to $\sqrt{-1}$.

10. Abraham de Moivre (1667-1754) left France to seek religious refuge in London at eighteen years of age. There he befriended Newton. In 1698 he mentions that Newton knew, as early as 1676 of an equivalent expression to what is today known as de Moivre's theorem:

$$(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$

where n is an integer. Apparently Newton used this formula to compute the cubic roots that appear in Cardan formulas, in the irreducible case. de Moivre knew and used the formula that bears his name, as it is clear from his writings - although he did not write it out explicitly.

11. L. Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ [3] and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. Euler used the formula $x + iy = r(\cos\theta + i \sin\theta)$, and visualized the roots of $z^n = 1$ as vertices of a regular polygon. He defined the complex exponential, and proved the identity $e^{i\theta}$ =

Application of imaginary numbers:

For most human tasks, real numbers (or even rational numbers) offer an adequate description of data. Fractions such as 2/3 and 1/8 are meaningless to a person counting stones, but essential to a person comparing the sizes of different collections of stones. Negative numbers such as -3 and -5 are meaningless when measuring the mass of an object, but essential when keeping track of monetary debits and credits. Similarly, imaginary numbers have essential concrete applications in a variety of sciences and related areas such as signal processing, control theory, electromagnetism, quantum mechanics, cartography, vibration analysis, and many others.

APPLICATION OF COMPLEX NO IN ENGINEERING:

Control Theory

In control theory, systems are often transformed from the time domain to the frequency domain using the Laplace transform. The system's poles and zeros are then analyzed in the complex plane. The root locus, Nyquist plot, and Nichols plot techniques all make use of the complex plane.

In the root locus method, it is especially important whether the poles and zeros are in the left or right half planes, i.e. have real part greater than or less than zero. If a system has poles that are

- in the right half plane, it will be unstable,
- all in the left half plane, it will be stable,
- on the imaginary axis, it will have marginal stability.

If a system has zeros in the right half plane, it is anon minimum phase system.

Signal analysis

Complex numbers are used in signal analysis and other fields for a convenient description for periodically varying signals. For given real functions representing actual physical quantities, often in terms of sines and cosines, corresponding complex functions are considered of which the real parts are the original quantities. For a sine wave of a given frequency, the absolute value |z| of the corresponding z is the amplitude and the argument arg (z) the phase.

If Fourier analysis is employed to write a given real-valued signal as a sum of periodic functions, these periodic functions are often written as complex valued functions of the form

ω f (t) = z

where ω represents the angular frequency and the complex number z encodes the phase and amplitude as explained above.

Improper integrals

In applied fields, complex numbers are often used to compute certain real-valued improper integrals, by means of complex-valued functions. Several methods exist to do this; see methods of contour integration.

Residue theorem

The residue theorem in complex analysis is a powerful tool to evaluate path integrals of meromorphic functions over closed curves and can often be used to compute real integrals as well. It generalizes the Cauchy and Cauchy's integral formula.

The statement is as follows. Suppose U is a simply connected open subset of the complex plane C, a1,…, an are finitely many points of U and f is a function which is defined and holomorphic on U\{a1,…,an}. If γ is a rectifiable curve in which doesn't meet any of the points ak and whose start point equals its endpoint, then

Here, Res(f,ak) denotes the residue off at ak, and n(γ,ak) is the winding number of the curve γ about the point ak. This winding number is an integer which intuitively measures how often the curve γ winds around the point ak; it is positive if γ moves in a counter clockwise ("mathematically positive") manner around ak and 0 if γ doesn't move around ak at all.

In order to evaluate real integrals, the residue theorem is used in the following manner: the integrand is extended to the complex plane and its residues are computed (which is usually easy), and a part of the real axis is extended to a closed curve by attaching a half-circle in the upper or lower half-plane. The integral over this curve can then be computed using the residue theorem. Often, the half-circle part of the integral will tend towards zero if it is large enough, leaving only the real-axis part of the integral, the one we were originally interested

Application of complex number in Computer Science. Quantum mechanics

The complex number field is relevant in the mathematical formulation of quantum mechanics, where complex Hilbert spaces provide the context for one such formulation that is convenient and perhaps most standard. The original foundation formulas of quantum mechanics – the Schrödinger equation and Heisenberg's matrix mechanics – make use of complex numbers.

The quantum theory provides a quantitative explanation for two types of phenomena that classical mechanics and classical electrodynamics cannot account for:

- Some observable physical quantities, such as the total energy of a black body, take on discrete rather than continuous values. This phenomenon is called quantization, and the smallest possible intervals between the discrete values are called quanta (singular: quantum, from the Latin word for "quantity", hence the name "quantum mechanics.") The size of the quanta typically varies from system to system.

  Under certain experimental conditions, microscopic objects like atoms or electrons exhibit wave-like behavior, such as interference. Under other conditions, the same species of objects exhibit particle-like behavior ("particle" meaning an object that can be localized to a particular region of space), such as scattering. This phenomenon is known

1. Arithmetic and logic in computer system

Arithmetic and Logic in Computer Systems provides a useful guide to a fundamental subject of computer science and engineering. Algorithms for performing operations like addition, subtraction, multiplication, and division in digital computer systems are presented, with the goal of explaining the concepts behind the algorithms, rather than addressing any direct applications. Alternative methods are examined, and explanations are supplied of the fundamental materials and reasoning behind theories and examples.

2.Recticing Software engineering in 21st century

This technological manual explores how software engineering principles can be used in tandem with software development tools to produce economical and reliable software that is faster and more accurate. Tools and techniques provided include the Unified Process for GIS application development, service-based approaches to business and information technology alignment, and an integrated model of application and software security. Current methods and future possibilities for software design are covered.

In Electrical Engineering:

The voltage produced by a battery is characterized by one real number (called potential), such as +12 volts or -12 volts. But the "AC" voltage in a home requires two parameters. One is a potential, such as 120 volts, and the other is an angle (called phase). The voltage is said to have two dimensions. A 2-dimensional quantity can be represented mathematically as either a vector or as a complex number (known in the engineering context as phasor). In the vector representation, the rectangular coordinates are typically referred to simply as X and Y. But in the complex number representation, the same components are referred to as real and imaginary. When the complex number is purely imaginary, such as a real part of 0 and an imaginary part of 120, it means the voltage has a potential of 120 volts and a phase of 90°, which is physically very real.

Application in electronics engineering

Information that expresses a single dimension, such as linear distance, is called a scalar quantity in mathematics. Scalar numbers are the kind of numbers students use most often. In relation to science, the voltage produced by a battery, the resistance of a piece of wire (ohms), and current through a wire (amps) are scalar quantities.

When electrical engineers analyzed alternating current circuits, they found that quantities of voltage, current and resistance (called impedance in AC) were not the familiar one-dimensional scalar quantities that are used when measuring DC circuits. These quantities which now alternate in direction and amplitude possess other dimensions (frequency and phase shift) that must be taken into account.

In order to analyze AC circuits, it became necessary to represent multi-dimensional quantities. In order to accomplish this task, scalar numbers were abandoned andcomplex numberswere used to express the two dimensions of frequency and phase shift at one time.

In mathematics, i is used to represent imaginary numbers. In the study of electricity and electronics, j is used to represent imaginary numbers so that there is no confusion with i, which in electronics represents current. It is also customary for scientists to write the complex number in the form a+jb.

In electrical engineering, the Fourier transform is used to analyze varying voltages and currents. The treatment of resistors, capacitors, and inductors can then be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance. (Electrical engineers and some physicists use the letter j for the imaginary unit since i is typically reserved for varying currents and may come into conflict with i.) This approach is called phasor calculus. This use is also extended into digital signal processing and digital image processing, which utilize digital versions of Fourier analysis (and wavelet analysis) to transmit, compress, restore, and otherwise process digital audio signals, still images, andvideosignals.

Introduce the formula E = I â€¢ Z where E is voltage, I is current, and Z is impedance.

Complex numbers are used a great deal in electronics. The main reason for this is they make the whole topic of analyzing and understanding alternating signals much easier. This seems odd at first, as the concept of using a mix of real and 'imaginary' numbers to explain things in the real world seem crazy!. To help you get a clear picture of how they're used and what they mean we can look at a mechanical example…

We can now reverse the above argument when considering a.c. (sine wave) oscillations in electronic circuits. Here we can regard the oscillating voltages and currents as 'side views' of something which is actually 'rotating' at a steady rate. We can only see the 'real' part of this, of course, so we have to 'imagine' the changes in the other direction. This leads us to the idea that what the oscillation voltage or current that we see is just the 'real' portion' of a 'complex' quantity that also has an 'imaginary' part. At any instant what we see is determined by aphase anglewhich varies smoothly with time.

We can now consider oscillating currents and voltages as being complex values that have a real part we can measure and an imaginary part which we can't. At first it seems pointless to create something we can't see or measure, but it turns out to be useful in a number of ways.

It helps us understand the behaviour of circuits which contain reactance (produced by capacitors or inductors) when we apply a.c. signals.

1.It gives us a new way to think about oscillations. This is useful when we want to apply concepts like the conservation of energy to understanding the behaviour of systems which range from simple a mechanical pendulums to a quartz-crystal oscillator.

Applications in Fluid Dynamics
Influid dynamics, complex functions are used to describe potential flow in two dimensions. Fractals.

Certain fractals are plotted in the complex plane, e.g. the Mandelbrot set

Fluid Dynamics and its sub disciplines aerodynamics, hydrodynamics, and hydraulics have a wide range of applications. For example, they are used in calculating forces and moments onaircraft, the mass flow of petroleum through pipelines, and prediction of weather patterns.

The concept of a fluid is surprisingly general. For example, some of the basic mathematical concepts in traffic engineering are derived from considering traffic as a continuous fluids.

Relativity

Inspecialandgeneral relativity, some formulas for the metric onspacetimebecome simpler if one takes the time variable to be imaginary. (This is no longer standard in classical relativity, but isused in an essential wayinquantum field theory.) Complex numbers are essential tospinors, which are a generalization of thetensorsused in relativity.

Applied mathematics

In differential equations, it is common to first find all complex roots r of the characteristic equation of a linear differential equation and then attempt to solve the system in terms of base functions of the form

f(t) = ert.

In Electromagnetism: Instead of taking electrical and magnetic part as a two different real numbers, we can represent it as in one complex number

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