A Study on R-Near Ring

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Abstract : In this paper we introduce the concept of $R$-near ring. We define that a near ring $N$ is a $R$-near ring if for every $a \in N$, there exists $x \in N$ such that $xax = xa$. The properties of $R$-near ring are discussed using the concept of zero divisors, ideal and subcommutativity. It is proved that every ideal and every left $N$-subgroup of any $R$-near ring without zero divisors is an $R$-near ring. It is also proved that any $R$-near ring is IFP near ring and subcommutative if it is Boolean; zero-symmetric if it is commutative also. A commutative zero symmetric Boolean near ring is always a $R$-near ring.

Keywords - $N$-subgroup, IFP, subcommutative, nil near ring, distributive, Boolean.

1.Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring.

Throughout this paper $N$ stands for a right near ring $(N,+)$, with at least two elements and '0' denotes the identity element of the group $(N,+)$ and we write $xy$ for $x, y$ for any two elements $x, y$ of $N$. Obviously $0n = 0$ for all $n \in N$. If, in addition, $n0 = 0$ for all $n \in N$ then we say that $N$ is zero symmetric. An element $a$ is said to be nilpotent if $a^k = 0$ for some positive integer $k$.

2.Preliminaries

Notation 2.1 [1]

$E$ denotes the set of all idempotents of $N$ ($a \in E$ iff $a^2 = a$)

Definition 2.2 [2]

If all nonzero elements of $N$ are left (right) cancelable, we say that $N$ fulfills the left (right) cancellation law.

Definition 2.3 [5]

An element $0 \neq x \in N$ is called a right zero divisor if there exists $0 \neq a \in N$ such that $ax = 0$.

Definition 2.4 [5]

An element $0 \neq x \in N$ is called a left zero divisor if there exists $0 \neq a \in N$ such that $xa = 0$.

Definition 2.5 [5]

A zero divisor is an element that is either a left (or) right zero divisor.

Definition 2.6 [2]

If $(N,\cdot)$ is commutative we call $N$ itself a commutative near ring.
Definition 2.7 [2]

A near ring $N$ is **Boolean** if and only if for all $x \in N$: $x^2 = x$.

Definition 2.8

An additive group $A$ of $N$ is called a **left $N$-subgroup** if $NA \subseteq A$ where $NA = \{ra | a \in A, r \in N\}$.

Definition 2.9 [2]

Let $N$ be a near ring and $P$ a $N$-group. A normal subgroup $I$ of $(N, +)$ is called an **ideal** of $N$ if (i) $IN \subseteq I$ (ii) For all $n, n_1 \in N$ and for all $i \in I$, $n(n_1 + i) - nn_1 \in I$.

Definition 2.10 [2]

$N$ is called a **nil near ring** if every element of $N$ is nilpotent.

Definition 2.11 [4]

$N$ is said to be **subcommutative** if $Na = aN$ for all $a \in N$.

Definition 2.12 [1]

$N$ is called a **$P_k$ near ring** ($P_k'$ near ring) if there exists a positive integer $k$ such that $x^kN = xNx (N x^k = xNx)$ for all $x \in N$.

Definition 2.13 [2]

$N$ is said to fulfill the **Insertion of Factors Property (IFP)** provided that for all $a, b, n$ in $N$, $ab = 0 \Rightarrow anb = 0$.

Definition 2.14 [2]

$N_d = \{n \in N | n(a + a') = na + na' \text{ for all } a, a' \in N\}$ - the set of all distributive elements of $N$. $N$ is called **distributive** if $N = N_d$.

Theorem 2.15 [2]

Let $I$ be an ideal of $N$. $N$ is nil if and only if $I$ and $N/I$ are nil.

Theorem 2.16 [2]

Let $I$ be an ideal of $N$. Then $N$ is zero symmetric if and only if $I$ and $N/I$ are zero symmetric.

3.R-near ring

Definition 3.1

$N$ is called a **$R$-near ring** if for every $a \in N$ there exists $x \in N$ such that $xax = xa$.

Theorem 3.2

Let $N$ be an $R$-near ring. (i) If $ax = 0$ then $xa = 0$. (ii) If $ax \in E$ then $xa \in E$. (iii) If the left cancellation law is valid in $N$, then $xa \in E$ implies $ax \in E$, for all $a \in N$ and for some $x \in N$.

**Proof:**

Let $a \in N$. Since $N$ is an $R$-near ring, there exists $x \in N$ such that $xax = xa \rightarrow (1)$. (i) If $ax = 0$ then from (1) we get, $xa = x0 = 0$. Thus $xa = 0$. (ii) If $ax \in E$ then $(ax)^2 = ax \rightarrow (2)$. Now $(xa)^2 = (xa)(xa) = x(ax)x = x(ax)x = xax = xa$. That is $(xa)^2 = xa$ and hence $xa \in E$. (iii) If $xa \in E$, then $(xa)^2 = xa \rightarrow (3)$. Now $x(ax)^2 = (xa)(xax) = (xa)(xa) = (xa)^2 = xa = xax$. That is $x(ax)^2 = xax$. Since the left cancellation law is valid in $N$, $(ax)^2 = ax$. Thus $ax \in E$. 

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Proposition 3.3

Let $N$ be a $R$-near ring without nonzero zero divisors. If $N$ is commutative then $N$ is Boolean.

Proof:

Let $a \in N$. Since $N$ is an $R$-near ring, there exists $x \in N$ such that $xax = xa$. Since $N$ is commutative, $x( xa) = xa \Rightarrow x^2a = xa \Rightarrow (x^2 - x)a = 0$. Since $N$ has no nonzero zero divisors, $x^2 - x = 0$. Consequently, $N$ is Boolean.

Proposition 3.4

Let $N$ be a $R$-near ring. If $N$ has no zero divisors, then every left $N$-subgroup and every ideal of $N$ is a $R$-near ring in its own right.

Proof:

Let $M$ be an an left $N$-subgroup of $N$ and let $a \in M$. Since $N$ is a $R$-near ring, there exists $x \in N$ such that $xax = xa$. Take $m = xa \in NM$. Since $N$ is a left $N$-subgroup of $N$, $NM \subseteq M$. Therefore $m \in M$. Since $N$ has no zero divisors, $m \neq 0$. Now $xm = x(xa)x = x(xax) = x(xa) = xm$. Thus $M$ is a $R$-near ring.

Now, let $I$ be an ideal of $N$ and let $a \in I$. Since $N$ is a $R$-near ring, there exists $x \in N$ such that $xax = xa$. Take $i = xa \in NI$. Since $I$ is an ideal of $N$, $NI \subseteq I$. Therefore $i \in I$. Since $N$ has no zero divisors, $m \neq 0$. Now $xix = x(xa)x = x(xax) = x(xa) = xi$. Thus $I$ is a $R$-near ring.

Theorem 3.5

Let $N$ be a nil near ring. Then $N$ is a $R$-near ring if and only if $N$ is zero symmetric.

Proof:

Take $a \in N$. Since $N$ is a $R$-near ring, there exists $x \in N$ such that $xax = xa \Rightarrow (1)$. We shall prove that $xax = xa$ (2) for all positive integer $k$, we use induction on $k$. Equation (2) is true for $k = 1$. Assume that the result is true for $k = s - 1$. If $k = s$, then $xasx = (xax)(as^{-1}x) = (xax)as^{-1}x = x(xas^{-1}) = (xa)a^{-1} = xa^s$. Thus $xax = xa^k$ for all positive integers $k$.

Since $N$ is nil, $a^t = 0$ for some positive integer $t$. Since $xatx = xa^t$, $x0x = x0 \Rightarrow x0 = 0$. Thus $N$ is zero symmetric. Conversely, let $x \in N$. Since $N$ is nil, there exists a positive integer such that $x^k = 0$. This implies $xa = 0$ where $a = x^{k-1}$. Therefore $xax = x(ax) = x(x^{k-1}x) = xx^k = x0 = 0 = xa$. Thus $N$ is a $R$-near ring.

Theorem 3.6

Let $N$ be a Boolean near ring. Each of the following statement implies that $N$ is a $R$-near ring. (i) $N$ is an IFP near ring with identity. (ii) $xN = xNx$ for all $x \in N$ ( $N$ is a $P_1$ near ring) (iii) $N$ is subcommutative.

Proof:

(i) Let $N$ be an IFP near ring with identity 1 and let $x \in N$. Since $N$ is Boolean, $x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0$. Since $N$ has IFP, $xa(x - 1) = 0$ for all $a \in N$. This implies $xax = xa = 0 \Rightarrow xax = xa$.

(ii) Let $x \in N$. Since $xN = xNx$, for any $a \in N$, there exists $y \in N$ such that $xa = xyy$. Therefore $xax = (xa)x = (xy)x = xyx^2 = xyx = xa$. Thus $xax = xa$. Hence $N$ is a $R$-near ring. (iii) Let $x \in N$. Since $N$
is subcommutative, \( xN = Nx \). Therefore for any \( a \in N \), there exists \( y \in N \) such that \( xa = yx \). Therefore \( xax = (xa)x = (yx)x = yx^2 = yx = xa \). That is \( xax = xa \) for all \( a \in N \). Thus \( N \) is a \( R \)-near ring.

**Theorem 3.7**

A commutative zero symmetric Boolean near ring is always a \( R \)-near ring.

**Proof:**

(i) Let \( N \) be a zero symmetric near ring. Let \( x \in N \). If \( x \neq 0 \), we take \( a = x \). Then \( xax = x^2x = xx = xa \). That is \( xax = xa \). If \( x = 0 \), then for any \( a \in N \), \( xax = 0 \). Since \( N \) is commutative \( xa = ax = 0 \). Thus \( N \) is a \( R \)-near ring.

**Theorem 3.8**

Let \( N \) be a commutative near ring without nonzero zero divisors. Then \( N \) is a \( R \)-near ring if and only if \( N \) is Boolean.

**Proof:**

Let \( a \in N \). For any \( x \in N \), \( xax = xa = (xa)x = xa = ax^2 = ax = a(x^2 - x) = ax = 0 \). That is \( xax = xa = 0 \) \( \Rightarrow \) \( xax = xa \). Therefore \( N \) is a \( R \)-near ring. Conversely, Let \( N \) be Boolean. This implies \( x^2 = x \) for all \( x \in N \Rightarrow x^2 - x = 0 \). Since \( N \) has no zero divisors, \( (x^2 - x)a = 0 \) for all \( a \in N \). \( \Rightarrow x^2a - xa = 0 \Rightarrow x(axa) - xa = 0 \Rightarrow x(ax) - xa = 0 \Rightarrow xax = xa \). Thus \( N \) is a \( R \)-near ring.

**Theorem 3.9**

Let \( N \) be a nil near ring and \( I \) a nonzero ideal of \( N \). Then \( N \) is a \( R \)-near ring if and only if \( I \) and \( N/I \) are \( R \)-near rings.

**Proof:**

Let \( I \) be a nonzero ideal of \( N \). Since \( N \) is nil, by theorem 2.15 we get, \( I \) and \( N/I \) are nil. Let \( N \) be a \( R \)-near ring. Since \( N \) is nil, \( N \) is zero symmetric by theorem 3.5. Therefore by theorem 2.16, we get \( I \) and \( N/I \) are zero symmetric. By theorem 3.5, we get \( I \) and \( N/I \) are \( R \)-near rings. Conversely, let us assume that \( I \) and \( N/I \) are \( R \)-near rings. Therefore \( I \) and \( N/I \) are zero symmetric by theorem 3.5 and by theorem 2.16 \( N \) is zero symmetric. Again by theorem 3.5 \( N \) is a \( R \)-near ring.

**Bibliography**


10) D. Radha and P. Meenakshi, Some Structures of Idempotent Commutative Semigroup, International of Science, Engineering and Management (IJSEM), Vol 2, Issue 12, December 2017, ISSN (Online) : 2456-1304.


14) D. Radha and S. Suguna, Normality in Idempotent Commutative $\Gamma$ - Semigroup, International Journal of Science, Engineering and Management (IJSEM) Vol 3, Issue 4, April 2018, ISSN (Online) : 2456-1304.

15) D. Radha and M. Parvathi Banu, Left Singularity and Left Regularity in Near Idempotent $\Gamma$ - Semigroup, International Journal of Science, Engineering and Management (IJSEM) Vol 3, Issue 4, April 2018, ISSN (Online) : 2456-1304.


