EXISTENCE OF SOLUTION FOR QUADRATIC ABSTRACT MEASURE INTEGRO-DIFFERENTIAL EQUATIONS USING HYBRID FIXED POINT THEOREM

[1]P. Ram Mohan Reddy and [2]Sidheshwar S. Bellale
1. Dept. of Mathematics, Government Degree College, Dharpally, Telangana, India.
Email:-rammohan09reddy@gmail.com
2. Dept. of Mathematics Dayanand Science College, Latur, Maharashtra ,India
Email: sidhesh.bellale@gmail.com

Abstract

In this paper, an existence theorem for a nonlinear abstract measure quadratic Integro-differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous case of the nonlinearities involved in the equations.

Key Words and Phrases: Measure Integro-differential equation, Existence theorem, Extremal solutions.

1.1 INTRODUCTION

As a generalization of ordinary integro-differential equations, there is a series of papers dealing with the abstract measure integro-differential equations in which ordinary derivative is replaced by the derivative of set functions, namely, the Radon-Nokodym derivative of a measure with respect to another measure. See Dhage [4, 5], Dhage and Bellale [8] and the references therein. The above mentioned papers also include some already known abstract measure differential equations those considered in Sharma [15, 16] and Shendge and Joshi [14] as special cases.

The origin of the quadratic integral equations appears in the works of Chandrasekhar's H-equation in radioactive heat transfer, but the study of nonlinear integral equations via operator theoretic techniques seems to have been started by Dhage [3] in the year 1988. Similarly, the study of nonlinear quadratic differential equations is relatively new and initiated by Dhage and O'Regan [11] in the year 2000. In the beginning, the development of the subject was slow, but recently this topic has gained momentum and growing very rapidly. Several fixed point principles have been formulated for this purpose in Banach algebras by Dhage et al. [6]and Dhage and O'Regan [11]. Since then, several nonlinear quadratic differential and integral equations have been studied in the literature. The following nonlinear differential equation appear in Dhage et al. [6].

For a given closed and bounded interval *J = [0; a]* in ℝ, the set of real numbers, consider the following integro-differential equation (in short IGDE)

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{x(t)}{f\left(t,x(t)\right)}\right) &= g\left(t,x(t),\int_0^t k\left(s,x(s)\right)ds\right) \text{ a.e. } t\in J \\ x(0) &= x_0 \end{aligned}\right\} \qquad (1.1)$$

where $f : J\times R \to R-\{0\}$ is continuous, $g : J\times R\times R \to R$ and $k : J\times R \to R$.

The existence of the solutions to IGDE (1.1) is proved in Dhage et al. [10] by using a new nonlinear alternative of Leray-Schauder type developed in same paper. In this paper we apply a nonlinear alternative of Leray-Schauder type due to Dhage and Bellale [8] involving the product of two operators in a Banach algebra under some weaker conditions than that given in Dhage and O'Regan [11] to a quadratic abstract measure differential equation related to IGDE (1.1) for proving the existence results. The existence of extremal solutions is also proved using a fixed point theorem of Dhage in ordered Banach algebras.

1.2 QUADRATIC INTEGRO-DIFFERENTIAL EQUATIONS

Let *X* be a real Banach space with a convenient norm $\|\cdot\|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in *X* is defined by

. (2.1)

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S_x}$ in *X* by

$$\left.\begin{aligned} S_x &= \{rx \mid -\infty < r < 1\}, \\ \overline{S_x} &= \{rx \mid -\infty < r \le 1\} \end{aligned}\right\} \tag{2.2}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x1} \subset S_{x2}$, or, equivalently $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let *M* denote the σ-algebra of all subsets of *X* such that (*X*, *M*) is a measurable space. Let ca(*X*, *M*) be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on ca(*X*, *M*) by

$$\| p \| = | p |(X) , \tag{2.3}$$

where, $|p|$ is a total variation measure of *p* is given by

$$| p |(X) = \sup \sum_{i=1}^{\infty} | p(E_i) |, E_i \subset X, \tag{2.4}$$

where supremum is taken over all possible partition $\{E_i : i \in N\}$ of *X*. It is known that *ca*(*X*, *M*) is a Banach space with respect to the norm $\|\cdot\|$, given by (2.3). For any nonempty subset *S* of *X*, let $L^1_\mu(S, R)$ denote the space of μ-integrable real-valued functions on *S* which is equipped with the norm $\|\cdot\|_{L^1_\mu}$ given by

$$\| \phi \|_{L^1_\mu} = \int_S | \phi(x) | \, d\mu \cdot$$

for $\phi \in L^1_\mu(S, M)$. Let $p_1, p_2 \in \mathrm{AC}(X, M)$ and define a multiplication composition $\circ$ in ca(*X*, *M*) by

$$(p_1 \circ p_2)(E) = p_1(E) p_2(E)$$

for all $E \in M$. Then we have:

Let μ be a σ-finite positive real measure on *X*, and let $p \in$ AC(*X*, *M*). We say *p* is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies *p*(*E*) = 0 for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ-algebra on S_{xo}. Let $z \in X$ such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form $\overline{S_x}, x \in \overline{x_0 z}$.

Given a $p \in \mathrm{ac}(X, M)$ with $p << \mu$, we consider the abstract measure integro-differential equation (AMIGDE) of the form

$$\frac{d}{d\mu}\left(\frac{p(\overline{S_x})}{f\left(x, p(\overline{S_x})\right)} \right) = g\left(x, p(\overline{S_x}), \int_{\overline{S_x}} k\left(t, p(\overline{S_t})\right) d\mu \right) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z} \tag{2.5}$$

and

$$p(E) = q(E), \quad E \in M_0, \tag{2.6}$$

where *q* is a given known vector measure, $\lambda(\overline{S_x}) = \dfrac{p(\overline{S_x})}{f\left(x, p(\overline{S_x})\right)}$ is a signed measure such that $\lambda << \mu, \dfrac{d\lambda}{d\mu}$ is a Radon-Nikodym derivative of λ with respect to μ, $f : S_z \times R \to R - \{0\}$, $g : S_z \times R \times R \to R$, $k : S_z \times R \to R$ and the map

$$x \mapsto g\left(x, p(\overline{S_x}), \int_{\overline{S_x}} k\left(t, p(\overline{S_t})\right) d\mu \right)$$

is σ-integrable for each $p \in \mathrm{AC}(X, M_z)$.

Definition 2.1. Given an initial real measure *q* on M_0, a vector measure $p \in \mathrm{ca}(S_z, M_z)(z > x_0)$ is said to be a solution of AMIGDE (2.5)-(2.6), if

(i) $p(E) = q(E), E \in M_0$,

(ii) $p << \mu$ on $\overline{x_0 z}$, and

(iii) p satisfies (2.5) a.e. [μ] on $\overline{x_0 z}$.

Remark 2.1. The AMIGDE (2.5)-(2.6) is equivalent to the abstract measure integral equation (in short AMIE)

$$p(E) = [\, f(x, p(E_1))]\left(\int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_x)) d\mu\right)\right),$$

$$\text{if } E \in M_z, E \subset \overline{x_0 z}, \tag{2.7}$$

and

$$p(E) = q(E) \text{ if } E \in M_0 \tag{2.8}$$

A solution p of abstract measure AMIGDE (2.5)-(2.6) in $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x0}, q)$.

Note that our AMIGDE (2.5)-(2.6) includes an abstract measure differential equation considered in Dhage and Bellale [7] as a special case. To see this, define $f(x, y) = 1$ for all $x \in \overline{x_0 z}$, and $y \in R$ then AMIGDE (2.5)-(2.6) reduces to

$$\frac{dp}{d\mu} = g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t) d\mu \right) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}, \tag{2.9}$$

and

$$p(E) = q(E), E \in M_0. \tag{2.10}$$

Thus, our AMIGDE (2.5)-(2.6) is more general and we claim that it is a new to the literature on measure differential equations. Therefore, the results of the present study are new and original contribution to the theory of nonlinear differential equations and include some of earlier results as special cases. In the following section we shall prove some auxiliary results which will be needed in to each sequel.

1.3 AUXILIARY RESULTS

Let E be a Banach algebra and let $T : X \to X$. T is called compact if $\overline{T(X)}$ is a compact subset of X. T is called totally bounded if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. T is called completely continuous if T is continuous and totally bounded on X. Every compact operator is totally bounded, but the converse may not be true, however, the two notions are equivalent on a bounded subset of X.

An operator $T : X \to Y$ is called D-Lipschitz if there exists a continuous and nonde-creasing function $\psi : R^+ \to R^+$ such that

$$\| Tx - Ty \| \le \psi(\| x - y \|) \tag{3.1}$$

for all $x, y \in X$, where $\psi(0) = 0$. The function ψ is called a D-function of T on X. In particular, if $\psi(r) = \alpha r, \alpha > 0, T$ is called a Lipschitz with the Lipschitz constant f. Further if α < 1, then T is called a contraction with contraction constant α. Again if $\psi(r) < r$ for $r > 0$, then T is called a nonlinear contraction on X with D-function ψ.

Now we are ready to state a fixed point theorem which will be employed in the subsequent of the chapter.

Theorem 3.1 (Dhage [3]). *Let U and $\overline{U}$ denote respectively the open and closed bounded subset of a Banach algebra X such that $0 \in U$. Let $A, B : \overline{U} \to X$ be two operators such that*

(a) A is D-Lipschitz,

(b) B is completely continuous, and

(c) $M\phi(r) < r, r > 0,$ where $M = \| B(\overline{U}) \|$

Then, either

(i) the equation $AxBx = x$ has a solution in $\overline{U}$, or

(ii) there is a point $u \in \partial U$ such that $u = \lambda AuBu$ for some 0 < λ < l, where ∂U is a boundary of U in X.

An interesting corollary to Theorem 3.1 in the applicable form is

Corollary 3.1. *Let $B_r(0)$ and $\bar{B}_r(0)$ denote respectively the open and closed balls in a Banach algebra centered at origin 0 of radius r for some real number r > 0. Let A, B : $\bar{B}_r(0) \to X$ be two operators such that*

(a) A is D-Lipschitz with Lipschitz constant α,

(b) B is compact and continuous, and

(c) $\alpha M < 1$, *where* $M = \| B(\bar{B}_r(0)) \|$.

Then either

(i) the operator equation $AxBx = x$ *has a solution x in X with* $\| x \| \le r$, *or*

(ii) there is an $u \in X$ *with* $\| u \| = r$ *such that* $\lambda AuBu = u$ *for some* $0 < \lambda < 1$.

We define an order relation $\le$ in $ca(S_z, M_z)$ with the help of the cone *K* in $ca(S_z, M_z)$ given by

$$K = \{ p \in ca(S_z, M_z) \mid p(E) \ge 0 \text{ for all } E \in M_z \}. \tag{3.2}$$

Thus, for any $p_1, p_2 \in ca(s_z, M_z)$ we have

$$p_1 \le p_2 \text{ if and only if } p_2 - p_1 \in K \tag{3.3}$$

or, equivalently

$$p_1 \le p_2 \Leftrightarrow p_1(E) \le p_2(E) \tag{3.4}$$

for all $E \in M_z$.

Obviously the cone K is positive in $ca(S_z, M_z)$. To see this, let $p_1, p_2 \in K$. Then $p_1(E) \ge 0$ and $p_2(E) \ge 0$ for all $E \in M_z$. By the multiplication composition,

$$(p_1 \circ p_2)(E) = p_1(E) p_2(E) \ge 0$$

for all $E \in M_z$. As a result $p_1 \circ p_2 \in K$, and so *K* is a positive cone in $ca(S_z, M_z)$ The following lemmas follow immediately from the definition of the positive cone K in $ca(S_z, M_z)$.

1.4 EXISTENCE RESULTS

We need the following definition in the sequel.

Definition 4.1. A function $\beta : S_z \times R \times R \to R$ is called **Caratheodory** if

(i) $x \to \beta(x, y_1, y_2)$ is μ-measurable for each $y_1, y_2 \in R$, and

(ii) the function $(y_1, y_2) \mapsto \beta(x, y_1, y_2)$ is continuous almost everywhere [μ] on $\overline{x_0 z}$.

A Carathreodory function β on $S_z \times R \times R$ is called L^1_μ-Carathreodory if

(ii) for each real number *r* > 0 there exists a function $h_r \in L^1_\mu(S_z, R_+)$ such that

$$| \beta(x, y_1, y_2) | \le h_r(x) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}.$$

for all $y_1, y_2 \in R$ with $| y_1 | \le r$ and $| y_2 | \le r$.

A function $\psi : R_+ \to R_+$ is called submultiplicative if $\psi(\lambda r) \le \lambda \psi(r)$ for all real number $\lambda > 0$. Let Ψ denote the class of functions $\psi : R_+ \to R_+$ satisfying the following properties:

(i) ψ is continuous,

(ii) ψ is nondecreasing, and

(iii) ψ is submultiplicative.

A member $\psi \in \Psi$ is called a D-function on R_+. There do exist D-functions, in fact, the function $\psi : R_+ \to R_+$ defined $\psi(\lambda r) \le \lambda \psi(r)$ is a D-function on R_+:

We consider the following set of assumptions:

(A_0) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the pseudo-metric *d* defined on M_z by $\mu(E_1, E_2) = \mu(E_1 \Delta E_2), E_1, E_2 \in M_z$

(A_1) The function $x \mapsto | f(x, 0) |$ is bounded with $F_0 = \sup_{z \in S} | f(x, 0) |$.

(A_2) The function *f* is continuous and there exists a bounded function $\alpha : S_z \to R^+$ with bound $\| \alpha \|$ such that

$$| f(x, y_1) - f(x, y_2) | \le \alpha(x) | y_1 - y_2 | \text{ a.e. } [\mu], x \in \overline{x_0 z},$$

for all $y_1, y_2 \in R$.

(B_0) q is continuous on M_z with respect to the pseudo-metric d defined in (A_1).

(B_1) The function $x \mapsto k(x, p(\overline{S}_x))$ is μ -integrable and there is a function $\gamma \in S_x^1(S_z, \mathbb{R}_+)$ satisfies

$$|k(x,y)| \le \gamma(x)|y| \quad a.e.[\mu] \text{ om } \overline{x_0 z}$$

for all $y \in R$.

(B_2) The function $g(x, y_1, y_2)$ is Carathreodory.

(B_3) There exists a function $\phi \in L_\mu^1(S_z, R^+)$ such that $\phi(x) > 0$ a.e. $[\mu]$ on $\overline{x_0 z}$ and a D-function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|g(x, y_1, y_2)| \le \phi(x)\psi(|y_1| + |y_2|) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z}$$

for all $y_1, y_2 \in R$.

We frequently use the following estimate of the function g in the subsequent part of the paper. For any $p \in ca(S_z, M_z)$, one has

$$\begin{aligned}
&\left| g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\left(t, p(\overline{S}_t)\right) d\mu\right)\right| \\
&\le \phi(x)\psi\left(|p(S_x)| + \int_{\overline{S}_x} |k(t, p(\overline{S}_t))|\, d\mu\right) \\
&\le \phi(x)\psi\left(|p(S_z)| + \int_{\overline{S}_z} \gamma(s)\, p(\overline{S}_t)\,|\, d\mu\right) \\
&\le \phi(x)\psi\left(\|p\| + \int_{\overline{S}_x} \gamma(t)\|p\|\, d\mu\right) \\
&\le \phi(x)\psi\left(\|p\| + \|\gamma\|_{L_\mu^1}\|p\|\right) \\
&\le \phi(x)\left(1 + \|\gamma\|_{L_\mu^1}\right)\psi(\|p\|).
\end{aligned}$$

Theorem 4.1. *Suppose that the assumptions (A_0)-(A_2) and (B_0)-(B_3) hold. Suppose that there exists a real number r > 0 such that*

$$r > \frac{F_0\left[\|q\| + \|\phi\|_{L_\mu^1}(1 + \|\gamma\|_{L_\mu^1})\psi(r)\right]}{1 - \|\alpha\|\left[\||q\| + \|\phi\|_{L_\mu^1}(1 + \|\gamma\|_{L_\mu^1})\psi(r)\right]}, \tag{4.1}$$

where $\|\alpha\|\left[\|q\| + \|\phi\|_{L_\mu^1}\left(1 + \|\gamma\|_{L_\mu^1}\right)\psi(r)\right] < 1$ *and* $F_0 = \sup_{x \in S_z}|f(x,0)|$ *then the AMIGDE (2.5) –(2.6) has a solution on* $\overline{x_0 z}$.

Proof. Consider an open ball $B_r(0)$ in ca(S_z, M_z) centered at the origin 0 and of radius r, where r satisfies the inequalities in (4.1). Define two operators

$$A, B : \overline{B}_r(0) \to ca(S_z, M_z)$$

By

$$Ap(E) = \begin{cases} 1, & \text{if } E \in M_0, \\ f(x, p(E)), & \text{if } E \in M_z, E \subset \overline{x_0 z} \end{cases} \tag{4.2}$$

and

$$Bp(E) = \begin{cases} q(E), & \text{if } E \in M_0, \\ \int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) d\mu, & \text{if } E \in M_z, E \subset \overline{x_0 z} \end{cases} \tag{4.3}$$

We shall show that the operators A and B satisfy all the conditions of Corollary 3.1 on $\overline{B}_r(0)$.

Step I : First, we show that A is a Lipschitz on $\overline{B}_r(0)$. Let $p_1, p_2 \in \overline{B}_r(0)$ be arbitrary. Then by assumption (A_2),

$$
\begin{aligned}
|Ap_1(E) - Ap_2(E)| &= |f(x, p_1(E)) - f(x, p_2(E))| \\
&\le \alpha(x)\,|p_1(E) - p_2(E)| \\
&\le \|\alpha\|\,|p_1 - p_2|(E)
\end{aligned}
$$

for all $E \in M_2$. Hence by definition of the norm in $ca(S_z, M_z)$ one has

$$\|Ap_1 - Ap_2\| \le \|\alpha\|\,\|p_1 - p_2\|$$

for all $p_1, p_2 \in ca(S_z, M_z)$. As a result *A* is a Lipschitz operator on $\bar{B}_r(0)$ with the Lipschitz constant $\|\alpha\|$.

Step II : We show that *B* is continuous on $\bar{B}_r(0)$. Let $\{p_n\}$ be a sequence of vector measure of vector measures in $\bar{B}_r(0)$ converging to a vector measure *p*. Then by dominated convergence theorem,

$$
\begin{aligned}
\lim_{n\to\infty} \bar{B}p_n(E) &= \lim_{n\to\infty} \int_E g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)\right) d\mu\right) d\mu \\
&= \int_E g\left(x, p(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p(\bar{S}_t)\right) d\mu\right) d\mu \\
&= \bar{B}_p(E)
\end{aligned}
$$

for all $E \in M_z, E \subset \overline{x_0 z}$. Similarly, if $E \in M_0$, then

$$\lim_{n\to\infty} \bar{B}_{pn}(E) = q(E) = Bp(E),$$

and so *B* is a continuous operator on $\bar{B}_r(0)$.

Step III : Next, we show that B is a totally bounded operator on $\bar{B}_r(0)$. Let $\{p_n\}$ be a sequence in $\bar{B}_r(0)$. Then we have $\|p_n\| \le r$ for all $n \in N$. We shall show that the set $\{Bp_n : n \in \mathbf{N}\}$ is uniformly bounded and equi-continuous set in ca(S_z, M_z). In this step, we first show that $\{Bp_n\}$ is uniformly bounded.

Let $E \in M_z$. Then there exists two subsets $F \in M_0$ and $G \in M_z, G \subset \overline{x_0 z}$ such that

$$E = F \cup G \text{ and } F \cap G = \emptyset.$$

Hence by definition of *B*,

$$
\begin{aligned}
|B_{Pn}(E)| &\le |q(F)| + \int_G \left| g\left(s, p_n(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)\right) d\mu\right)\right| d\mu \\
&\le \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|) d\mu \\
&\le \|q\| + \int_E \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi)(\|p_n\|) d\mu \\
&\le \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)
\end{aligned}
$$

for all $E \in M_z$. From (3.3) it follows that

$$\begin{aligned}\|Bp_n\| &= |Bp_n|(S_z)\\ &= \sup_{\sigma}\sum_{i=1}^{\infty}|Bp_n(E_i)|\\ &= \|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|P_n\|)\\ &= \|q\|+\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(r)\end{aligned}$$

for all $n\in\mathbf{N}$. Hence the sequence $\{Bp_n\}$ is uniformly bounded in $B(\bar{\mathrm{B}}_r(0))$.

Step IV: Next we show that $\{Bp_n : n\in\mathbf{N}\}$ is a equi- continuous set in ca(S_z, M_z). Let E_1, E_2 $\in$ M_z. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_Z, G_1 \subset \overline{x_0 z}, G_2 \subset \overline{x_0 z}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

$$\left.\begin{aligned}G_1 &= (G_1 - G_2)\cup(G_2\cap G_1),\\ G_2 &= (G_2 - G_1)\cup(G_1\cap G_2).\end{aligned}\right\} \tag{4.4}$$

Therefore, we have

$$Bp_n(E_1) - Bp_n(E_2) \le q(F_1) - q(F_2) + \int_{G_1-G_2} g\left(x, p_n(\bar{S}_z), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)d\mu\right)\right)d\mu$$

$$-\int_{G_2-G_1} g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)\right)d\mu\right)d\mu$$

Since g is Caratheodory and satisfies (B_3), we have that

$$|Bp_n(E_1) - Bp_n(E_2) \le |q(F_1) - q(F_2)| + \int_{G_1\Delta G_2}\left|g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)\right)d\mu\right)\right|d\mu,$$

$$\le |q(F_1) - q(F_2)| + \int_{G_1\Delta G_2}\phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|p_n\|)d\mu.$$

Assume that

$$d(E_1, E_2) = |\mu|(E_1\Delta E_2)\to 0.$$

Then we have $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1\Delta G_2)\to 0$. As q is continuous on compact M_z, it is uniformly continuous and so

$$|Bp_n(E_1) - Bp_n(E_2) \le |q(F_1) - q(F_2)| + \int_{G_1\Delta G_2}\left|g\left(x, p_n(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p_n(\bar{S}_t)\right)d\mu\right)\right|d\mu$$

$$\to\ 0 \text{ as } E_1 \to E_2$$

uniformly for all $E_1, E_2 \in M_z$ and $n\in\mathbf{N}$,

This shows that $\{Bp_n : n\in\mathbf{N})$ is a equi-continuous set in ca(S_z, M_z). Now an application of the Arzela-Ascolli theorem yields that B is a totally bounded operator on $\bar{\mathrm{B}}_r(0)$. Now B is continuous and totally bounded operator on $\bar{\mathrm{B}}_r(0)$, it is completely continuous operator on $\bar{\mathrm{B}}_r(0)$.

Step V: Finally, we show that hypothesis (c) of Corollary 3.1. is satisfied . The Lipschitz constant of A is $\|\alpha\|$. Here, the number M in the hypothesis (c) is given by

$$M = \|B(\bar{\mathrm{B}}_r(0))\|$$
$$= \sup\{\|Bp\| : p \in \bar{\mathrm{B}}_r(0)\}$$
$$= \sup\{|Bp|(S_z) : p \in \bar{\mathrm{B}}_r(0)\}. \tag{4.5}$$

Now, let $E \in M_z$. Then there are sets $F \in M_0$ and $G \in M_z, G \subset \overline{x_0 z}$ such that

$$E = F \cup G \text{ and } F \cap G = \emptyset .$$

From the definition of *B* it follows that

$$Bp(E) = q(F) + \int_G g\left(x, p(\bar{S}_x), \int_{\bar{S}_x} k\left(t, p(\bar{S}_t)\right) d\mu\right) d\mu$$

Therefore ,

$$|Bp(E)| \le |q(F)| + \int_G \left| g\left(x, p(S_x), \int_{S_x} k\left(t, p(S_t)\right) d\mu\right)\right| d\mu$$

$$\le \|q\| + \int_G \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|) d\mu$$

$$\le \|q\| + \int_{\overline{x_0 z}} \phi(x)\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|) d\mu$$

$$= \|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(\|p\|)$$

Hence, from (4.6) it follows that

$$\|Bp\| \le \|q\| + \|\phi\|_{L^1_\mu}(1 + \|\gamma\|_{L^1_\mu})\psi(\|p\|)$$

for all $p \in \bar{\mathrm{B}}_r(0)$. As a result we have

$$M = \|B(\bar{\mathrm{B}}_r(0)\| Bp\| \le \|q\| + \|\phi\|_{L^1_\mu}(1 + \|\gamma\|_{L^1_\mu})\psi(\|p\|) .$$

Now,

$$\alpha M \le \|\alpha\|\left[\|q\| + \|\phi\|_{L^1_\mu}\left(1 + \|\gamma\|_{L^1_\mu}\right)\psi(r)\right] < 1$$

and so, hypothesis (c) of Corollary 3.1 is satisfied.

Now an application of Corollary 3.1 yields that either the operator $AxBx = x$ has a solution, or there is a $u \in ca(S_z, M_z)$ such that $\|u\| = r$ satisfying $u = \lambda AxBx$ for some 0 < λ < 1. We show that this latter assertion does not hold. Assume the contrary. Then we have

$$u(E) = \left\{\lambda[f(x, u(E))]\left(\int_E g\left(x, u(\bar{S}_x), \int_{\bar{S}_x} k\left(t, u(\bar{S}_t)\right) d\mu\right) d\mu\right),\right.$$

$$\text{if } E \in M_z, E \subset \overline{x_0 z},$$

and $u(E) = q(E)$, if $E \in M_0$

for some 0 < λ < 1.

If $E \in M_z$, then there sets $F \in M_0$ and $G \in \overline{M_z}, G \subset \overline{x_0 z}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then, we have

$$u(E)=\lambda Au(E)Bu(E)$$

$$=q(F)+\lambda[f(x,u(G))]\left(\int_G g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)d\mu\right)$$

$$=|q(F)|+\left(\int_G g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)d\mu\right)$$

$$+|f(x,0)|\left(\int_G\left|g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)\right|d\mu\right)$$

$$=q(F)+\lambda[f(x,u(G))-f(x,0)]\left(\int_G g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)d\mu\right)$$

Hence

$$+\lambda f(x,0)\left(\int_G g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)d\mu\right)$$

$$|u(E)|\le|q(F)|+\lambda(|f(x,u(G))-f(x,0)|)\left(\int_G g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)d\mu\right)$$

$$+|f(x,0)|\left(\int_G\left|g\left(x,u(\bar{S}_x),\int_{\bar{S}_x} k\left(t,u(\bar{S}_t)\right)d\mu\right)\right|d\mu\right)$$

$$\le\|q\|+\lambda(\alpha(x)|u(G)|+F_0)\left(\int_G \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|d\mu\right)$$

$$\le\|q\|+[\|\alpha\|\|u|(E)|+F_0]\left(\int_{x_0 z} \phi(x)\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)d\mu\right)$$

$$\le\|q\|+[\|\alpha\|\|u|(E)|+F_0]\left[\|\phi\|_{L^1_\mu}(1+\|\gamma\|_{L^1_\mu})\psi(\|u\|)\right],$$

which further implies that

$$\|u\|\le\|q\|+\left(\|\alpha\|\|\|u\|\left[\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\right]\right)$$

$$+F_0\left[\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\right]$$

$$\le\frac{\|q\|+F_0\left[\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\right]}{1-\|\alpha\|\left[\|\phi\|_{L^1_\mu}\left(1+\|\gamma\|_{L^1_\mu}\right)\psi(\|u\|)\right]}$$

Substituting $\|u\|=r$ in the above inequality yields

$$r \leq \frac{\| q \| + F_0\left[\| \phi \|_{L^1_\mu}\left(1+\| \gamma \|_{L^1_\mu}\right)\psi(\| u \|)\right]}{1-\| \alpha \|\left[\| \phi \|_{L^1_\mu}\left(1+\| \gamma \|_{L^1_\mu}\right)\psi(\| u \|)\right]} \tag{4.6}$$

which is a contradiction to the first inequality. In consequence, the operator equation *p(E) = Ap(E)Bp(E)* has a solution $u(\bar{S}_{x0}, q)$ in ca(S_z, M_z) with $\| u \| \leq r$. This further implies that the AMIGDE (2.5)-(2.6) has a solution on $\overline{x_0 z}$ This completes the proof.

<u>Example</u> 4.1. Given $p \in ca(S_z, M_z)$ with p << μ, consider the AMIGDE.

$$\frac{d}{d\mu}\left(\frac{p(\bar{S}_x)}{1+|p(\bar{S}_x)|}\right) = \frac{\phi(x)p(\bar{S}_x)}{1+p^2(\bar{S}_x)} a.e.[\mu] \text{ on } \overline{x_0 z}\ , \tag{4.8}$$

$$p(\bar{S}_{x_0}) = q \in R\, , \tag{4.8}$$

where $\phi : \overline{x_0 z} \to R^+$ is μ-integrable.

Define the functions $f : S_z \times R \to R - \{0\}$ and $g : S_z \times R \to R$ by $f(x, y) = 1 + |y|$ and $g(x, y) = \dfrac{\phi(x)y}{1+y^2}$

respectively. Below we shall show that the functions *f* and *g* satisfy all the conditions of Theorem 4.1. Obviously *f* is continuous on the domain of its definition. Let $y_1, y_2 \in R$. Then we have

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \ |1+|y_1| - 1 - |y_2|| \\ &= \ ||y_1| - |y_2|| \\ &\leq \ |y_1 - y_2| \end{aligned}$$

which shows that *f*(*x*, *y*)satisfies the Lipschitz condition in y with the Lipschitz constant α = 1. Obviously the function *g*(*x*, *y*) is Caratheodory on $\overline{x_0 z}$. To see this, note that the function is obvious $x \to \dfrac{\phi(x)y}{1+y^2}$ sly μ-measurable for all $y \in R$ and the function $y \to \dfrac{\phi(x)y}{1+y^2}$ is continuous for all $x \in \overline{x_0 z}$. Again,

$$g(x, y_1, y_2) = g(x, y_1) = \left|\frac{\phi(x)y_1}{1+y_1^2}\right| \leq |\phi(x)| = \phi(x)\psi(|y_1|),$$

where $\psi : R^+ \to R^+$ is defined by $\psi(r) = 1$.

Thus, if $\| q \| + \| \phi \|_{L^1} < 1$, then all the assumptions (A_0)-(A_2) and (B_0)-(B_3) of Theorem 4.1 are satisfied. Hence, the AMIGDE (4.7) has a solution $p(\bar{S}_{x0}, q)$ defined on $\overline{x_0 z}$.

References,

[1] **J. Banas , K. Goebel**, Measures of non compactness in Banach space, *in: Lecture Notes in pure and Applied Mathematics , Vol* ***60*** *, Dekker, New York, 1980.*

[2] **P. C. Das and R. R. Sharma**, Existence and stability of measure differential equations, *Zech. Math. J.,* ***22*** *(1972), 145 - 158.*

[3] **B. C. Dhage**, On α-condensing mappings in Banach algebras,*Math. Student 63 (1994), 146-152.*

[4] **B. C. Dhage**, On abstract measure Integro-differental equations, *J. Math. Phy. Sci. 20*(1986), 367- 380.

[5] **B. C. Dhage**, On system of abstract measure Integro-differential inequalities and applications, *Bull. Inst. Math. Acad. Sinica 18 (1989), 65-75.*

[6] **B. C. Dhage, D. N. Chate and S. K. Ntouyas,** Abstract measure differential equations, *Dynamic Systems & Appl. 13 (2004), 105-108.*

[7] **B. C. Dhage and S. S. Bellale**, Abstract measure Integro-differential equations, *Global Jour. Math. Anal. 1 (1-2) (2007), 91-108.*

[8] **B. C. Dhage and S. S. Bellale**, Existence theorem for perturbed Abstract measure ______differential equations Nonlinear Analysis, *Theory; methods and Applications"(2008) doi:1016j.na.2008.11.057.*

[9] **B. C. Dhage and S. S. Bellale,** Local asymptotic stability for Nonlinear quadratic functional equations, *Electronic Journal of Qualitative Theory of differential Equations, 2008, No. 1-13."*

[10] **B. C. Dhage, U. P. Dolhare and S. K. Ntouyas**, Existence theorems for nonlinear first order functional differential equations in Banach algebras, *Comm. Appl. Nonlinear Anal.10 (4) (2003), 59-69.*

[11] **B. C. Dhage and D. O'Regan**, A fixed point theorem in Banach algebras with Applications to nonlinear integral equation, *Functional Diffrential Equations 7(3-4) (2000), 259-267.*

[12] **J. Dugundji and A. Granas**, Fixed Point Theory, *Monograhie Math. PNW, Warsaw,1982.*

[13] **S. Hekkila and V. Lakshikantham**, Monotone Iterative Technique for Discontinuous Nonlinear Defferential Equations, *Marcel Dekker Inc., New York 1994.*19

[14] **S. R. Joshi**, A system of abstract measure delay differential equations, *J.Math. Phy. Sci.13 (1979), 497-506.*

[15] **R. R. Sharma**, An abstract measure di_erential equation,*Proc. Amer. Math. Soc. 32 (1972), 503-510.*

[16] **R. R. Sharma**, A measure di_erential inequality with applications, *Proc. Amer. Math.Soc. 48 (1975), 87-97.*

[17] **G. R. Shendge and S. R. Joshi**, Abstract measure di_erential inequalities and applications, *Acta Math.Hung. 41(1983), 53-54.*