THE SPACE OF MINIMAL PRIME S-IDEALS IN 0-DISTRIBUTIVEALMOST SEMILATTICES

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Abstract : Let L be a 0-distributive almost semilattice (0-distributive ASL) and $\mathfrak{M}$ be its minimal spectrum. It is shown that $\mathfrak{M}$ is Hausdorff. The compactness of $\mathfrak{M}$ has been characterized in several ways.

Key Words: S-ideal, prime S-ideal, 0-distributive ASL, minimal prime S-ideal, hull-kernel topology, dual hull kernel topology.

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1 INTRODUCTION

Henriksen and Jerison [1] investigated the space of minimal prime ideals of a commutative ring extending the considerations of Kist [2]. They succeeded in obtaining sufficient conditions for their respective spaces to be compact. This inspired Speed [9] [10] to investigate minimal prime ideals of a distributive lattice with 0. Fortunately, the lattice theoretic situation enabled Speed [9] to obtain much deeper results; so much so, he could characterize the compact-
-ness of the space of minimal prime ideals of a distributive lattice with 0 in a much more elegant manner. Later Pawar and Thakare [8] studied the space of minimal prime ideals when it was carried the hull-kernel topology.

In this paper, we shall mainly be concerned here with the space of minimal prime S-ideals when it carries the hull-kernel topology.

2 Preliminaries

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2. 1 : An ASL with 0 is an algebra $(L, o, 0)$ of type (2, 0) satisfies the following conditions:

1) $(x\ o\ y)\ o\ z = x\ o\ (y\ o\ z)$
2) $(x\ o\ y)\ o\ z = (y\ o\ x)\ o\ z$
3) $x\ o\ x = x$
4) $0\ o\ x = 0, for\ all\ x, y, z \in$ L.

Definition 2. 2 : Let L be an ASL. A nonempty subset I of L is said to be an S-ideal if it satisfies the following conditions:

1) If $x \in I$ and $a \in L$, then $x\ o\ a \in I$.
2) If $x, y \in I$, then there exists $d \in I$ such that $d\ o\ x = x, d\ o\ y = y$.

Definition 2. 3 : A nonempty subset F of an ASL L is said to be a filter if F satisfies the following conditions:

1) $x, y \in F\ implies\ x\ o\ y \in F$.
2) If $x \in F$ and $a \in L$ such that $a\ o\ x = x$, then $a \in F$.

Definition 2. 4 : A proper S-ideal P of an ASL L is said to be a prime if for any $x, y \epsilon L$, $x\ o\ y \in P\ imply\ x \epsilon P\ or\ y \epsilon P$.

Definition 2. 5 : Let L be an ASL with unimaximal element. Then a proper filter F of L is said to be a prime filter if for any filters $F_1\ and\ F_2\ of\ L, F_1 \cap F_2 \subseteq F$ implies that either $F_1 \subseteq F\ or\ F_2 \subseteq F$.

Definition 2. 6 : A proper filter F of L is said to be maximal if for any filter G of L such that $F \subseteq G \subseteq L$, then either $F = G\ or\ G = L$.

Definition 2.7 : Let L be an ASL with 0. Then L is said to be 0-distributive if for any $x, y, z \in L, x\ o\ y = 0\ and\ x\ o\ z = 0$ then there exists $d \epsilon L$ such that $d\ o\ y = y, d\ o\ z = z\ and\ d\ o\ x = 0$.

Definition 2. 8 : Let L be an ASL with 0. Then for any non empty subset A of L, $A^* = \{x \epsilon L : x\ o\ a = 0\ for\ all\ a \in A\}$ is called the annihilator of A.

Note that if $A = \{a\}$, then we denote $A^* = \{a\}^*\ by\ [a]^*$.

Theorem 2. 9 : Let L be an ASL with 0. Then for any ideals $I, J\ of\ L,$ we have the following.

(1) $I^* = \bigcap_{a \in I} [\,a\,]^*$
(2) $(\,I \cap J\,)^* = (\,J \cap I\,)^*$
(3) $I \subseteq J \Rightarrow J^* \subseteq I^*$
(4) $I^* \cap J^* \subseteq (\,I \cap J\,)^*$
(5) $(\,I \cap J\,)^{**} = I^{**} \cap J^{**}$
(6) $I \subseteq I^{**}$
(7) $I^{***} = I^*$
(8) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq J^{**}$
(9) $I \cap J = (\,0\,] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$
(10) $(\,I \cup J\,)^* = I^* \cap J^*$

Theorem 2. 10 : Let L be an ASL with 0. Then for any $x, y \in L,$ we have the following.

(1) $x \leq y \Rightarrow [\,y\,]^* \subseteq [\,x\,]^*$
(2) $[\,x\,]^* \subseteq [\,y\,]^* \Rightarrow [\,y\,]^{**} \subseteq [\,x\,]^{**}$
(3) $x \in [\,x\,]^{**}$
(4) $(\,x\,]^* = [\,x\,]^*$
(5) $(\,x\,] \cap [\,x\,]^* = \{0\}$
(6) $[\,x\,o\,y\,]^* = [\,y\,o\,x\,]^*$
(7) $[\,x\,]^* \cap [\,y\,]^* \subseteq [\,x\,o\,y\,]^*$
(8) $[\,x\,o\,y\,]^{**} = [\,x\,]^{**} \cap [\,y\,]^{**}$
(9) $[\,x\,]^{***} = [\,x\,]^*$
(10) $[\,x\,]^* \subseteq [\,y\,]^*$ *if and only if* $[\,y\,]^{**} \subseteq [\,x\,]^{**}$

Theorem 2. 11 : Let L be an ASL with 0, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent:

(1) L is 0-distributive ASL
(2) A^* is an S-ideal, for all $A\,(\neq \emptyset) \subseteq L$.
(3) $SI(L)$ pseudo- complemented semilattice.
(4) $SI(L)$ is 0-distributive semilattice.
(5) $PSI(L)$ is 0-distributive semilattice.

Theorem 2. 12 : Let L be an ASL and P be a proper S-ideal of L. Then P is prime S-ideal if and only if for any S-ideals $I\ and\ J\ of\ L,\ I \cap J \subseteq P\ imply\ I \subseteq P\ or\ J \subseteq P.$

Theorem 2. 13 : Every proper filter in ASL L is contained in a maximal filter.

Theorem 2. 14 : Let L be a 0-distributive ASL. Then every maximal filter of L is a prime filter.

Theorem 2. 15 : Let be L an ASL. Then a subset P of L is a prime S-ideal if and only if $L - P$ is a prime filter.

Theorem 2. 16 : Let L be a 0-distributive ASL. Then a subset M of L is a minimal prime S-ideal if and only if $L - M$ is a maximal filter.

Theorem 2. 17 : Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if $[\,x\,]^* - M \neq \emptyset$ for any $x \in M$.

Corollary 2. 18 : Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if it contains precisely one of $\{\,x\,\},\ [\,x\,]^*\ for\ every\ x \in L.$

Corollary 2.19 : Let L be a 0-distributive ASL and let P be a minimal prime S-ideal in L. Then for every $x \in L,\ [\,x\,]^{**} \nsubseteq P$ if and only if $[\,x\,]^* \subseteq P$.

Theorem 2. 20 : Let L be a 0-distributive ASL in which intersection of any family of S-ideals is again an S-ideal. Then for any S-ideal I of L, I^* is the intersection of all minimal prime S-ideals not containing I.

Corollary 2. 21 : The intersection of all minimal prime S-ideals of a 0-distributive ASL is $\{0\}$.

Lemma 2. 22 : Let L be a 0-distributive ASL and let $a\,(\neq 0) \in L$. Then there exists a minimal prime S-ideal not containing a.

Lemma 2. 23 : Let L be a 0-distributive ASL. Then for any $x \in L, [x]^* = \cap\{\,M \in \mathfrak{M} : x \notin M\,\}$.

3. Notation and Theorem

Let L be a 0-distributive ASL. As usual, for a subset $\mathcal{R}$ of $\mathfrak{M}$ (the set of all minimal prime S-ideals in L), we write the kernel of $\mathcal{R}$, denoted by $K(\mathcal{R})$, the set given by $\cap\{\,P : P \in \mathcal{R}\,\}$. For a nonempty subset A of L, the hull of A, denoted by $h(A)$, is the set $\{\,P \in \mathfrak{M} : A \subseteq P\,\}$. Let us also adopt the notation M_x to denote the set $\{\,P \in \mathfrak{M} : x \notin P\,\}$. The hull-kernel topology on $\mathfrak{M}$ is obtained by taking the family $\{\,M_x : x \in L\,\}$ as the base for open sets. $\mathfrak{M}$ together with this topology is called the minimal spectrum of L; and we shall continue to designate it by $\mathfrak{M}$. Also, it can be easily seen that in the hull-kernel topology on $\mathfrak{M}$ open sets are of the form M_I, where $M_I = \{\,P \in \mathfrak{M} : I \nsubseteq P\,\}$ and $h(I) = \mathfrak{M} - M_I$.

In this section, we derive a set of identities for $\mathfrak{M}$ to be compact in its hull-kernel topology. For this, first we need the following.

Theorem 3. 1 : Let L be an ASL and let P be a prime S-ideal of L. Then P is a minimal prime S-ideal if and only if $L - P$ is a maximal prime filter.

Proof : Suppose P is a minimal prime S-ideal of L. Now, we shall prove that $L - P$ is a maximal prime filter. Then by lemma 2.15, $L - P$ is a prime filter. Suppose Q is a prime filter of L such that $L - P \subseteq Q$. Then $L - Q \subseteq P$ and

$L-Q$ is a prime S-ideal. Therefore $L-Q=P$ since P is minimal. It follows that $Q=L-P$. Therefore $L-P$ is a maximal prime filter. Conversely, suppose $L-P$ is a maximal prime filter. Then by lemma 2.15, we get P is a prime S-ideal. Suppose Q is a prime S-ideal of L such that $Q\subseteq P$. Then $L-P\subseteq L-Q$ and $L-Q$ is a prime filter. Therefore $L-P=L-Q$. Hence $P=Q$. Therefore P is a minimal prime S-ideal.

Corollary 3.2 : Let L be a 0-distributive ASL. Then a filter Q of L is maximal if and only if Q is a maximal prime filter.

Now, we improve some important relations between the S-ideals of 0-distributive ASL and the corresponding open sets in the hull-kernel topology on $\mathfrak{M}$.

Lemma 3.3 : Let L be an 0-distributive ASL. Then for any $I, J \in SI(L)$, we have the following.

1) $I\subseteq J \Rightarrow M_I \subseteq M_J$
2) $I\subseteq J \Rightarrow h(J)\subseteq h(I)$
3) $M_I \cap M_J = M_{I\cap J}$
4) $h(I)\cup h(J) = h(I\cap J)$

Proof : 1. Suppose $I\subseteq J$ and suppose $P\in M_I$. Then $I\nsubseteq P$. Therefore $J\nsubseteq P$. Hence $P\in M_J$. Thus $M_I\subseteq M_J$.

2. Suppose $I\subseteq J$ and suppose $P\in h(J)$. Then $J\subseteq P$. Therefore $I\subseteq P$. Hence $P\in h(I)$. Thus $h(J)\subseteq h(I)$.

3. Clearly, $M_{I\cap J}\subseteq M_I\cap M_J$. Conversely, suppose $P\in M_I\cap M_J$. Then $P\in M_I$ and $P\in M_J$. Therefore $I\nsubseteq P$ and $J\nsubseteq P$. It follows that $I\cap J\nsubseteq P$. Therefore $P\in M_{I\cap J}$. Thus $M_I\cap M_J\subseteq M_{I\cap J}$. Therefore $M_{I\cap J}=M_I\cap M_J$.

4. $h(I)\cup h(J) = M_I^c \cup M_J^c = \left(M_I\cap M_J\right)^c = \left(M_{I\cap J}\right)^c = h(I\cap J)$.

Corollary 3.4 : Let L be an 0-distributive ASL and $x, y\in L$. Then we have the following.

1) $x\le y \Rightarrow M_x\subseteq M_y$
2) $x\le y \Rightarrow h(y)\subseteq h(x)$
3) $M_x\cap M_y = M_{x\,o\,y}$
4) $h(x)\cup h(y) = h(x\,o\,y)$

Proof : 1. Suppose $x\le y$ and suppose $P\in M_x$. Then $x\notin P$. Therefore $y\notin P$. Hence $P\in M_y$. Thus $M_x\subseteq M_y$. 2. Suppose $x\le y$ and suppose $P\in h(y)$. Then $y\in P$. Therefore $x\in P$. Hence $P\in h(x)$. Thus $h(y)\subseteq h(x)$.

3. We have $\in M_x\cap M_y \Leftrightarrow P\in M_x$ and $P\in M_y \Leftrightarrow x\notin P$ and $y\notin P \Leftrightarrow x\,o\,y\notin P \Leftrightarrow P\in M_{x\,o\,y}$.

Therefore $M_x\cap M_y = M_{x\,o\,y}$.

4. We have $P\in h(x)\cup h(y) \Leftrightarrow P\in h(x)$ or $P\in h(y) \Leftrightarrow x\in P$ and $y\in P \Leftrightarrow x\,o\,y\in P \Leftrightarrow P\in h(x\,o\,y)$. Therefore $h(x)\cup h(y)=h(x\,o\,y)$.

The following lemma exhibits the relation between the annihilator S-ideal of 0-distributive ASL L and the basic open sets, basic closed sets of $\mathfrak{M}$ in the hull-kernel topology.

Theorem 3.5 : Let L be a 0-distributive ASL and $x, y\in L$. Then we have the following.

1) $M_x = h([x]^*)$
2) $h(x) = h([x]^{**})$
3) $[x]^*\subseteq [y]^* \Leftrightarrow h(x)\subseteq h(y)$
4) $M_y\subseteq M_x \Leftrightarrow [x]^*\subseteq [y]^* \Leftrightarrow [y]^{**}\subseteq [x]^{**}$
5) $[x]^{**} = [y]^* \Leftrightarrow h(x) = h([y]^*)$

Proof : Proofs of conditions (1) and (2) follows by corollary 2.18 and 2.19.

3. Suppose $[x]^*\subseteq [y]^*$ and suppose $P\in h(x)$. Then $x\in P$. Hence by corollary 2.18, we get $[x]^*\nsubseteq P$. Therefore $[y]^*\nsubseteq P$. Hence $y\in P$. Therefore $P\in h(y)$. Thus $h(x)\subseteq h(y)$. Conversely, suppose $h(x)\subseteq h(y)$. Let $a\notin [y]^*$. Then $a\,o\,y\neq 0$. Therefore by lemma 2.22, there exists a minimal prime S-ideal (say) P of L such that $a\,o\,y\notin P$. It follows that $a\notin P$ and $y\notin P$. Hence, we get $a\notin P$ and $P\notin h(y)$. This implies $a\notin P$ and $P\notin h(x)$. Hence $a\notin P$ and $x\notin P$. It follows that $a\,o\,x\notin P$, since P is a prime S-ideal of L. Therefore $a\,o\,x\neq 0$, we get $a\notin [x]^*$. Thus $[x]^*\subseteq [y]^*$.

4. We have $[x]^*\subseteq [y]^* \Leftrightarrow h(x)\subseteq h(y) \Leftrightarrow \mathfrak{M}-h(y)\subseteq \mathfrak{M}-h(x) \Leftrightarrow M_y\subseteq M_x$. Therefore $[x]^*\subseteq [y]^* \Leftrightarrow M_y\subseteq M_x$.

5. Suppose $[x]^{**}=[y]^*$. Then $h([x]^{**})=h([y]^*)$. Hence by (2), we get $h(x)=h([y]^*)$. Conversely, suppose $a\notin [x]^{**}$. Then there exists $t\in [x]^*$ such that $a\,o\,t\neq 0$. Therefore by lemma 2.22, there exists a minimal prime S-ideal P of L such that $a\,o\,t\notin P$. Hence $a\notin P$ and $t\notin P$. Since $t\,o\,x=0\in P$, $x\in P$. Therefore $P\in h(x)=h([y]^*)$. Hence $[y]^*\subseteq P$, we get $a\notin [y]^*$. Thus $[y]^*\subseteq [x]^{**}$. Similarly, we get $[x]^{**}\subseteq [y]^*$. Therefore $[x]^{**}=[y]^*$.

In [5], the authors proved that the pseudo-complement of an S-ideal I in a 0-distributive ASL L is the intersection of all minimal prime S-ideals not containing I. In the language that was introduced above one can write this assertion in the following compact and convenient form.

Theorem 3.6 : Let L be a 0-distributive ASL. Then for any S-ideal I of L, $I^* = K\left(\mathfrak{M}-h(I)\right)$.

Proof : We have $I^* = \cap\{P\in\mathfrak{M} : I\nsubseteq P\} = \cap\{P\in\mathfrak{M} : P\in M_I\} = \cap\{P\in\mathfrak{M} : P\notin h(I)\} = \cap\{P\in\mathfrak{M} : P\in\mathfrak{M}-h(I)\} = K\left(\mathfrak{M}-h(I)\right)$. Therefore $I^* = K\left(\mathfrak{M}-h(I)\right)$.

Recall that in a 0-distributive ASL L for any $x \in L$, we have $[x]^* = \cap\{P \in \mathfrak{M} : x \notin P\}$ and hence $[x]^* = K(M_x)$. Hence we have the following.

Corollary 3.7 : Let L be a 0-distributive ASL. Then for every $x \in L, h(K(M_x)) = M_x = h([x]^*)$. In particular, $h(x)$ *and* $h([x]^*)$ are clopen sets in $\mathfrak{M}$ that are disjoint.

Proof : Let $x \in L$. Then we have $h(K(M_x)) = h([x]^*) = \{P \in \mathfrak{M} : [x]^* \subseteq P\} = \{P \in \mathfrak{M} : x \notin P\} = M_x$. Therefore $h(K(M_x)) = h([x]^*) = M_x$.

By theorem 3.6, it can be easily seen that if I is an S-ideal in 0-distributive ASL L, then $I^* = K(\mathfrak{M} - h(I)) = K(M_I)$ and M_I is an open subset of $\mathfrak{M}$. Therefore we have the following.

Corollary 3.8 : A subset Y of L is the disjoint complement of I^*, for some S-ideal I of L if and only if Y is the kernel of some open subset of $\mathfrak{M}$.

We see that theorem 3.6, above states a property of the disjoint complement I^* of an S-ideal. But, we see, an account of theorem 3.1, much more is true.

Theorem 3.9 : Let L be a 0-distributive ASL. Then for any nonempty subset $A(\neq \{0\})$ of $L, A^* = K(h(A^*))$.

Proof : Suppose $A(\neq \{0\}) \subseteq L$. Now, we shall prove that $A^* = K(h(A^*))$. Suppose $t \in L$ such that $t \notin A^*$. Then $x\, o\, t \neq 0$, *for some* $x \in A$. Therefore there exists a maximal filter (say) F of L such that $x\, o\, t \in F$. Now, since F is a maximal filter, $L - F$ is a minimal prime S-ideal. Let $z \in A^*$. Then we have $z\, o\, x = 0 \in L - F$. Since $L - F$ is prime, either $z \in L - F$ or $x \in L - F$. It follows that $z \in L - F$ since $x\, o\, t \in F$ and hence $x \in F$. Therefore $A^* \subseteq L - F$. Hence $L - F \in h(A^*)$. Again, since $x\, o\, t \in F$ *and* $x\, o\, t \leq t$, $t \in F$. This implies $t \notin L - F$. It follows that $t \notin K(h(A^*))$. Hence $K(h(A^*)) \subseteq A^*$. Conversely, suppose $t \notin K(h(A^*))$. Then $t \notin P$, *for some* $P \in \mathfrak{M}$ *such that* $A^* \subseteq P$. It follows that $t \notin A^*$. Thus $A^* \subseteq K(h(A^*))$. Therefore $A^* = K(h(A^*))$.

As for any two minimal prime S-ideal none of them is contained in the other we see that any two points of $\mathfrak{M}$ are T_1 − separated. Thus, we have the following.

Theorem 3. 10 : The hull-kernel topology on $\mathfrak{M}$ is Hausdorff.

Proof : Suppose $P, Q \in \mathfrak{M}$ such that $P \neq Q$. Then there exists $x \notin P$ such that $x \in Q$. Therefore $P \in M_x$ and $Q \in h(x) = \mathfrak{M} - M_x$ and also $M_x \cap h(x) = M_x \cap (\mathfrak{M} - M_x) = \emptyset$. Therefore $\mathfrak{M}$ is Hausdorff.

One more property of the set $\{M_x : x \in L\}$ is stated in the following. For, this we need the following lemmas.

Lemma 3.11 : Let L be a 0-distributive ASL and let $x \in L$. Then $M_x = \emptyset$ if and only if $x = 0$.

Proof : Suppose $M_x \neq \emptyset$. Then there exists $P \in \mathfrak{M}$ such that $P \in M_x$. Therefore $x \notin P$ and hence $x \neq 0$. Conversely, suppose $x \neq 0$. Then by lemma 2.22, there exists $P \in \mathfrak{M}$ such that $x \notin P$. Therefore $P \in M_x$. Thus $M_x \neq \emptyset$.

Lemma 3.12 : Let L be an ASL and let S be a nonempty subset of L. Then $[S) = \{x \in L : x\, o\, \mathrm{O}_{i=1}^{n} s_i = \mathrm{O}_{i=1}^{n} s_i, s_i \in S, 1 \leq i \leq n, n \textit{ is } + ve \textit{ integer}\}$ is the smallest filter containing S.

Proof : Suppose S is a nonempty subset of L. Then for any $s \in S$, we have $s = s\, o\, s$ and hence $s \in [S)$. Thus $[S)$ is nonempty. Now, we shall prove that $[S)$ is a filter. Let $x, y \in [S)$. Then $x\, o\,(\mathrm{O}_{i=1}^{n} s_i) = \mathrm{O}_{i=1}^{n} s_i$ and $y\, o\,(\mathrm{O}_{i=1}^{m} t_i) = \mathrm{O}_{i=1}^{m} t_i$, where $s_i, t_i \in S, 1 \leq i \leq n$ *and* $1 \leq i \leq m$. Therefore $(x\, o\,(\mathrm{O}_{i=1}^{n} s_i))\, o\,(y\, o\,(\mathrm{O}_{i=1}^{m} t_i)) = \mathrm{O}_{i=1}^{n} s_i\, o\, \mathrm{O}_{i=1}^{m} t_i$. It follows that $(x\, o\, y)\, o\, \mathrm{O}_{j=1}^{n+m} z_j = \mathrm{O}_{j=1}^{n+m} z_j$, $z_j \in S$. Hence $x\, o\, y \in [S)$. Again, let $x \in [S)$ *and* $t \in L$ *such that* $t\, o\, x = x$. Then $x\, o\,(\mathrm{O}_{i=1}^{n} s_i) = \mathrm{O}_{i=1}^{n} s_i$, $s_i \in S, 1 \leq i \leq n$. Therefore $t\, o\,(x\, o\,(\mathrm{O}_{i=1}^{n} s_i)) = t\, o\,(\mathrm{O}_{i=1}^{n} s_i)$. Hence $(t\, o\, x)\, o\,(\mathrm{O}_{i=1}^{n} s_i) = t\, o\,(\mathrm{O}_{i=1}^{n} s_i)$. It follows that $x\, o\,(\mathrm{O}_{i=1}^{n} s_i) = t\, o\,(\mathrm{O}_{i=1}^{n} s_i)$. Hence $\mathrm{O}_{i=1}^{n} s_i = t\, o\,(\mathrm{O}_{i=1}^{n} s_i)$. Therefore $t \in [S)$. Thus $[S)$ is a filter. Now, it remains to prove that $[S)$ is the smallest filter of L containing S. Suppose F is a filter of L such that $S \subseteq F$. Then for any $x \in [S)$, we have $x\, o\,(\mathrm{O}_{i=1}^{n} s_i) = \mathrm{O}_{i=1}^{n} s_i$, $s_i \in S, 1 \leq i \leq n$. Since $S \subseteq F$, $s_i \in F$. It follows that $x \in F$. Hence $[S) \subseteq F$. Therefore $[S)$ is the smallest filter containing S.

Theorem 3. 13 : Let Σ be any indexing set and let $\{x_r : r \in \Sigma\}$ be a subset of 0-distributive ASL L such that the collection$\{M_{x_r}\}$ has the finite intersection property. Then the intersection of all $\{M_{x_r}\}$, $r \in \Sigma$ is nonempty.

Proof : Suppose $\{M_{x_r} : r \in \Sigma\}$ is a collection of sets in $\mathfrak{M}$ with finite intersection property. Now, put $\Delta = \{x_r : r \in \Sigma\}$ and put $F = [\Delta)$. Suppose $F = L$. Then $0 \in L = F$. Therefore $\mathrm{O}_{i=1}^{n} x_r = 0\, o\, \mathrm{O}_{i=1}^{n} x_r$, $x_r \in \Delta$, $1 \leq r \leq n$. This implies $\mathrm{O}_{i=1}^{n} x_r = 0$. It follows that $M_{\mathrm{O}_{i=1}^{n} x_r} = M_0 = \emptyset$. Therefore $\cap_{i=1}^{n} M_{x_r} = \emptyset$, a contradiction to finite intersection property. Therefore $F \neq L$. Hence F is a proper filter. It follows that there exists a maximal filter (say) H of L such that $F \subseteq H$. Again, it follows that $L - H$ is a minimal prime S-ideal. Now, let $x_r \in \Delta$. Then $x_r \in H$. Therefore $x_r \notin L - H$. Hence $L - H \in M_x$. Therefore $L - H \in \cap_{x_r \in \Delta} M_{x_r}$. Thus $\cap_{x_r \in \Delta} M_{x_r} \neq \emptyset$.

We consider the family $\{h(x) : x \in L\}$ to be a subbase for $\mathfrak{M}$. Then the resulting topology is called the dual hull-kernel topology. We denote by τ_h the hull-kernel topology and by τ_d the dual hull-kernel topology on $\mathfrak{M}$. Now, we prove the following.

Theorem 3. 14 : The hull-kernel topology on $\mathfrak{M}$ is finer than the dual hull-kernel topology.

Proof : Clearly, $h(x) = \mathfrak{M} - M_x$, for any $x \in L$. Also, by theorem 3.10, M_x is closed in $\mathfrak{M}$. Therefore $h(x)$ is open in $\mathfrak{M}$. Hence $h(x)$ is open in the hull-kernel topology; τ_h is finer than τ_d.

A sufficient condition of the equality of these two topologies $\tau_h$and τ_d on $\mathfrak{M}$ is stated in the following.

Theorem 3. 15 : Let L be a 0-distributive ASL. If for every $x \in L$, there exists $x' \in L$ such that $[x']^* = [x]^{**}$ then $\tau_h = \tau_d$.

Proof : Clearly, $\tau_d \subseteq \tau_h$. Now, we shall prove that $\tau_h \subseteq \tau_d$. Let $M_x \in \tau_h$. Then $x \in L$. Therefore there exists $x' \in L$ such that $[\,x'\,]^* = [\,x\,]^{**}$. Now, $M_x = h\,(\,[\,x\,]^*\,) = h\,(\,[\,x'\,]^{**}\,) = h\,(\,x'\,)$. It follows that every basic open set in τ_h is open in τ_d . Thus $\tau_h \subseteq \tau_d$. Therefore $\tau_h = \tau_d$.

We now state our main result that provides us with necessary and sufficient conditions for $\mathfrak{M}$ to be compact in its hull-kernel topology.

Theorem 3.10 : Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent.

1) $\mathfrak{M}$ is compact.
2) Finite unions of $\{\,M_x : x \in L\,\}$ form a Boolean lattice.
3) For $x \in L$, there exist $t_i \in L, i = 1, 2, \ldots\ldots, n$ such that $t_i \in [\,x\,]^*$ and $[\,x\,]^* \cap \bigcap_{i=1}^{n}[\,t_i\,]^* = \{\,0\,\}$.
4) For $x \in L$, there exist $t_i \in L, i = 1, 2, \ldots\ldots, n$ such that $[\,x\,]^{**} = \bigcap_{i=1}^{n}[\,t_i\,]^*$
5) $\tau_h = \tau_d$.
6) $\{\,h\,(\,x\,) : x \in L\,\}$ is a subbasis for the open sets of $(\,\mathfrak{M}, \tau_d\,)$.
7) $\{\,M_x : x \in L\,\}$ is a subbasis for the open sets of $(\,\mathfrak{M}, \tau_h\,)$.

Proof : (1) $\Longrightarrow$ (2) : Suppose $\mathfrak{M}$ is compact in the hull-kernel topology. Now, put $B_h = \{\,M_x : x \in L\,\}$ and put B is the set of all finite unions of elements in B_h . Now, we shall prove that B is a Boolean lattice. Now, we have $h\,(\,[\,x\,]^*\,) = \bigcap_{t \in [\,x\,]^*} h\,(\,t\,)$. Therefore $h\,(\,x\,) \cap h\,(\,[\,x\,]^*) = h\,(\,x\,) \cap \bigcap_{t \in [\,x\,]^*} h\,(\,t\,) = \emptyset$ and $\{\,h\,(\,x\,) \cap h\,(\,t\,) : t \in [\,x\,]^*\,\}$ is a class of closed sets in the compact space $h\,(\,x\,)$. Hence there exists $t_1, t_2, \ldots\ldots, t_n \in [\,x\,]^*$ such that $h\,(x\,) \cap h(\,t_1) \cap h\,(\,t_2\,) \cap \ldots . \cap h\,(\,t_n\,) = \emptyset$. This implies $(\,h\,(\,x\,) \cap h\,(t_1\,) \cap h\,(\,t_2\,) \cap \ldots\ldots \cap h\,(\,t_n\,)\,)^c = \emptyset^c$. It follows that $M_x \cup M_{t_1} \cup M_{t_2} \cup \ldots\ldots \cup M_{t_n} = \mathfrak{M}$. Hence $M_x \cup \bigcup_{i=1}^{n} M_{t_i} = \mathfrak{M}$. Now, we shall prove that $M_x \cap \bigcup_{i=1}^{n} M_{t_i} = \emptyset$. Suppose $M_x \cap \bigcup_{i=1}^{n} M_{t_i} \neq \emptyset$. Then there exists $P \in \mathfrak{M}$ such that $P \in M_x$ and $P \in \bigcup_{i=1}^{n} M_{t_i}$. This implies $x \notin P$ and $t_i \notin P$ for some $i, 1 \leq i \leq n$. It follows that $x\,o\,t_i \notin P$. Hence $x\,o\,t_i \neq 0$. Therefore $t_i \notin [\,x\,]^*$, a contradiction to $t_i \in [\,x\,]^*$. Thus $M_x \cap \bigcup_{i=1}^{n} M_{t_i} = \emptyset$. Therefore $\bigcup_{i=1}^{n} M_{t_i}$ is the complement of M_x. Since B_h is a bounded semilattice. It follows from theorem 1[11], B is a Boolean lattice.

(2) $\Longrightarrow$ (3) : Assume (2). Let $x \in L$. Then $M_x \in B$. Since B is a Boolean lattice, M_x has the complement (say) $\bigcup_{i=1}^{n} M_{t_i}$. Therefore $M_x \cap \bigcup_{i=1}^{n} M_{t_i} = \emptyset$ and $M_x \cup \bigcup_{i=1}^{n} M_{t_i} = \mathfrak{M}$. Since $M_x \cap \bigcup_{i=1}^{n} M_{t_i} = \emptyset, (\,M_x \cap M_{t_1}) \cup (\,M_x \cap M_{t_2}\,) \cup \ldots . . \cup (\,M_x \cap M_{t_n}\,) = \emptyset$. Therefore $M_{x\,o\,t_1} \cup M_{x\,o\,t_2} \cup \ldots\ldots \cup M_{x\,o\,t_n} = \emptyset$. Hence $M_{x\,o\,t_i} = \emptyset$ for all $i, 1 \leq i \leq n$. This implies $M_{x\,o\,t_i} = M_0$ for all $i, 1 \leq i \leq n$. It follows that $x\,o\,t_i = 0$ for all $i = 1,2, \ldots . n$. Hence $t_i \in [\,x\,]^*$ for all $i = 1,2, \ldots . n$. Now, $M_x \cup \bigcup_{i=1}^{n} M_{t_i} = \mathfrak{M}$. This implies $K\,(\,M_x \cup \bigcup_{i=1}^{n} M_{t_i}) = K\,(\,\mathfrak{M}\,)$. But, we have $K\,(\,\mathfrak{M}\,) = \{\,0\,\}$. Therefore $K\,(\,M_x \cup \bigcup_{i=1}^{n} M_{t_i}) = \{\,0\,\}$. It follows that $K\,(\,M_x\,) \cap \bigcap_{i=1}^{n} K\,(\,M_{t_i}\,) = \{\,0\,\}$. This implies $[\,x\,]^* \cap \bigcap_{i=1}^{n}[\,t_i\,]^* = \{0\}$.

(3) $\Longrightarrow$ (4) : Assume (3). Let $x \in L$. Then there exists $t_1, t_2, \ldots . , t_n \in L$ such that $t_i \in [\,x\,]^*$ and $[\,x\,]^* \cap \bigcap_{i=1}^{n}[\,t_i\,]^* = \{0\}$. Since $t_i \in [\,x\,]^*$, for all i, it follows that $[\,x\,]^{**} \subseteq [\,t_i\,]^*$ for all i . Hence $[\,x\,]^{**} \subseteq \bigcap_{i=1}^{n}[\,t_i\,]^*$. Suppose $t \in \bigcap_{i=1}^{n}[\,t_i\,]^*$ and $y \in [\,x\,]^*$. Then clearly, $y\,o\,t \in [\,x\,]^* \cap \bigcap_{i=1}^{n}[\,t_i]^*$. Hence $y\,o\,t = 0$. Therefore $t \in [\,x\,]^{**}$. Thus $\bigcap_{i=1}^{n}[\,t_i]^* \subseteq [\,x\,]^{**}$. Therefore $[\,x\,]^{**} = \bigcap_{i=1}^{n}[\,t_i\,]^*$.

(4) $\Longrightarrow$ (5) : Assume (4). Now, we shall prove that the basic open sets $\{\,M_x : x \in L\,\}$ in τ_h are open in τ_d. Let $x \in L$. Then by condition, there exists $t_1, t_2, \ldots . , t_n \in L$ such that $[\,x\,]^{**} = \bigcap_{i=1}^{n}[\,t_i\,]^*$. Hence $h\,(\,[\,x\,]^{**}\,) = h\,(\,\bigcap_{i=1}^{n}[\,t_i\,]^*\,)$. This implies $h\,(\,[\,x\,]^{**}\,) = \bigcup_{i=1}^{n} h\,(\,[\,t_i\,]^*\,)$. Hence $h\,(\,[\,x\,]^{**}) = \bigcup_{i=1}^{n} M_{t_i}$. Therefore $h\,(\,x\,) = \bigcup_{i=1}^{n} M_{t_i}$. Hence we get $M_x = \bigcap_{i=1}^{n} h(\,t_i\,)$, which is finite intersection of open sets in τ_d and hence is open. Thus M_x is open in τ_d.

(5) $\Longrightarrow$ (1) : Suppose $\tau_h = \tau_d$. Then we have M_x is a basic closed set in $\mathfrak{M}$ for any $x \in L$. Now, we shall prove that $\mathfrak{M}$ is compact. Let $\{\,M_x : x \in \Delta\,\}$ be a family of closed sets in $\mathfrak{M}$ with finite intersection property, for some $\Delta \subseteq L$. Now, put $F = [\,\Delta\,)$, filter generated by Δ. Suppose $F = L$. Then we have $0 \in L = F$. It follows that $0\,o\,O_{i=1}^{n}\,x_i = O_{i=1}^{n} x_i, x_i \in \Delta, 1 \leq i \leq n$. This implies $O_{i=1}^{n}\,x_i = 0$. Hence $M_{O_{i=1}^{n} x_i} = M_0 = \emptyset$. It follows that $\bigcap_{i=1}^{n} M_{x_i} = \emptyset$, a contradiction to finite intersection property. Therefore $0 \notin F$. Hence F is a proper filter. Therefore by Zorn's lemma, F is contained in a maximal filter (say) K. Then clearly, $L - K$ is a minimal prime S-ideal. Now, let $x \in \Delta$. Then $x \in F \subseteq K$ and hence $x \notin L - K$. Therefore $L - K \in M_x$. Thus $L - K \in \bigcap_{x \in \Delta} M_x$. Hence $\bigcap_{x \in \Delta} M_x \neq \emptyset$. Therefore $\mathfrak{M}$ is compact in the hull-kernel topology. The equivalence of (5), (6) and (7) is trivial.

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