ON THE SOLUTIONS FOR MHD BOUNDARY LAYER FLOW OF GENERALIZED NON-NEWTONIAN FLUIDS IN A POROUS MEDIUM

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Abstract

The exact analytical and numerical solution for MHD flow of a generalized non-Newtonian fluid filling the porous half-space is investigated. The formulation of the problem is givenusing modified Darcy's law for a present fluid. The fluid is incompressible, electrically conducting and a uniformmagnetic field is applied normal to the flow by neglecting the induced magnetic field.The governing non-linear partial differential equation with auxiliary conditions isderived and it then reduced to by employing reduction and solutions have been developed using the similarity approach. Both analytical and Numerical solutions are derived. The influence ofvarious parameters of interest has been shown and discussed in detail. A comparisonof the present analysis with those available, shows excellent agreement between analytic and numericalsolutions. Besides the importance of exact solutions the non-Newtonian flows in a porousmedium are important in engineering fields such as enhanced oil recovery,paper and textile coating and composite manufacturing processes.

1. Introduction:

There are many different Phenomena that can be encountered while studying the flow of various fluids. Not only is there diversity in the phenomena themselves, but there is considerable diversity in the structure of the fluids being investigated. The structure of a fluid may be described from a continuum or a molecular viewpoint. There exist a number of theories of varying degrees of generalization that are capable of describing most of the phenomena likely to be seen. The shear rate dependence of the viscosity can be described by the modified Newtonian model. This modified Newtonian model is known as non-Newtonian model as it do not follow the Newton's law of viscosity.

The boundary layer behavior over stretching surfaces is important as it occurs in several engineering process, for instance, the aerodynamic extrusion of plastic sheets, glass fiber production, the cooling and drying of paper and textiles and so forth. In view of these applications, Sakiadis [1] initiated a theoretical study for the momentum transfer occurring in the boundary layer adjacent to a continuous surface moving steady through an otherwise quiescent fluid environment. After Sakiadis [1], the boundary layer flow caused by stretching surfaces has drawn the attention of many researchers. The dynamics of the boundary layer flow over stretching surfaces originated from the pioneering work of Crane [2]. He extended the work of Sakiadis by assuming that the velocity of the sheet varies linearly with the distance from the slit and found closed form analytic solution for the self-similar equations.

Thereafter, numerous investigations were made on the boundary layer flows of viscous fluids over stretching surfaces under different physical situations [3, 4, 5]. Andersson et al. [6] studied the magnetohydrodynamic flow of a power law fluid over a stretching surface. Numerous excellent works [7, 8, 9] on boundary layer flows of non-Newtonian fluids are now available. All of the above studies are concerned with the boundary layer flows of Newtonian and/or non-Newtonian fluids. However, to the best of the author's knowledge, no attempts have thus far been communicated with regard to boundary layer flow of a generalized non-Newtonian fluid due to a stretching surface. One such generalized non-Newtonian fluid which is a special case of the modified Newtonian fluids is considered in this work.

The generalized non-Newtonian fluid model proposed here is found to be frequently used in oil engineering. Many real fluids follow this model. Polymeric suspensions such as waterborne coatings are known to be non-Newtonian in nature and are hope to follow the present fluid model [10]. One particular case of the generalized non-Newtonian fluid is known as Sisko fluid model is the most suitable model for the flows of greases [11]. In spite of their wide occurrence in scores of industry setting only few problems involving Sisko fluids are studied by the investigators [12, 13, 14, 15].

Here we follow the convention used in the rheological literature of representing the non-Newtonian viscosity of the fluid by η, which is function of $\left|\frac{du}{dy}\right|$ or $|\tau_{yx}|$. Occasinally we will make use of μ when referring to the non-Newtonian viscosity, such a change of notation will be indicated as needed. The change in viscosity in expected to depend on the magnitude rather than the sign of shear rate (shear stress), hence the use of absolute value signs in the above expression. Various empirical relations can be tried for the function η soas fit non-Newtonian viscosity curves obtained experiments or other measurement.

The constitutive relation given can be readily extended to any arbitrary flow field $v = v(x, y, z, t)$. For an incompressible Newtonian fluid the constitutive relation is

$$\tau = -\mu\dot{\gamma} \tag{1}$$

In which $\dot{\gamma}$ is the rate of deformation tensor.

For a modified Newtonian fluid (or non-Newtonian fluids) the consecutive relation is

$$\tau = -\eta\dot{\gamma}, \tag{2}$$

Here the non-Newtonian viscosity η, a scalar, is a function of $\dot{\gamma}(or\tau)$ as well of tempreture and pressure. Thus the constitutive equation for Newtonian and non Newtonian fluids is different. This confirms that molecular structure of both fluids is totally different.

A number of empirical relations for $\eta(\dot{\gamma})$ have been derived from raw data. It is more convenient and useful to make use of analytical expression of $\eta(\dot{\gamma})$ that has been found to fit the experimental data with sufficient accuracy. One of the most popular and commonly used empirical relation found in the literature is the Ostwald Waele model (power-law model) given by

$$\tau = -\{K|\dot{\gamma}|^{n-1}\}\dot{\gamma},$$

While the modified Newtonian fluid (or non-Newtonian fluids) model is used to some fluids of importance in industry, It must be emphasized that it does have severe limitations.

This fluid model cannot accounts for phenomena involving normal faces, viscoelastic time dependent effects, or flows that are not dominated by steady shear.

We see that although the modified Newtonian fluid (or non-Newtonian fluids) model has its origins in empirical data; it has acquired a legitimate basis from recent continuum mechanics theories.

2. Generalized Non-Newtonian Fluids

It is interesting to observe that the linear combination of constitutive equations of Newtonian fluids [Eq. (1)] and constitutive equations of modified Newtonian fluids [Eq. (2)] give a rise most general constitutive equation:

$$\tau = p(-\mu\dot{\gamma}) + q(-\eta\dot{\gamma}) \tag{3}$$

Where p and q are arbitrary constants,

The equation (3) is now constitutive equation of generalized non-Newtonian fluids. Example of this generalized non-Newtonian fluidcan be foundin literature.Under usual Prandtl boundary layerassumption the stress-strain relationship is given by equation (3) will be:

$$\tau_{yx} = p\frac{\partial u}{\partial y} + q\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{4}$$

Equation (4) usually known as Sisko fluids [11

3. Problem formulation

We consider a Cartesian coordinate system with Y-axis in the vertical upward direction and X axis parallel to the rigid plate at $y = 0$.The flow of an incompressible and electrically conducting generalized non-Newtonian fluid is bounded by an infinite rigid plate. This fluid occupies the porous half-space $y > 0$. The flow is the motion of the plate with the time-dependent velocity $U_0V(t)$.for zero pressure gradients, the resulting problem is,

$$\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left[\left(p + q\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right] - \frac{\phi}{k}\left[p + q\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u - \sigma B_0^2 u, \tag{5}$$

$$u(0,t) = U_0V(t),\ t > 0$$

$$u(\infty,t) = 0,\ t > 0 \tag{6}$$

$$u(y,0) = g(y),\ y > 0$$

In which U_0 is the characteristic velocity. The above equations can be transform into non-dimensional form using the following variables

$$u^* = \frac{u}{U_0},\ y^* = \frac{yU_0}{v},\ t^* = \frac{tU_0^2}{v} \tag{7}$$

$$b^* = \frac{q}{p}\left|\frac{U_0^2}{v}\right|^{n-1},\ \frac{1}{K} = \frac{\phi v^2}{kU_0^2},\ M^2 = \frac{\sigma B_0^2 v}{\rho U_0^2} \tag{8}$$

Accordingly the above boundary value problem after dropping the asterisks becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left[\left(1 + b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right)\frac{\partial u}{\partial y}\right] - \frac{\phi}{k}\left[1 + b\left|\frac{\partial u}{\partial y}\right|^{n-1}\right]u - M^2 u\ , \tag{9}$$

$u(0,t) = V(t), t > 0$

$u(\infty,t) = 0, t > 0$ (10)

$u(y,0) = g(y), y > 0$

Where $h(y) = \frac{g(y)}{U_0}$

4. Exact solutions

In this section, we are interested in obtaining the reductions and solutions of equation (9) and (10). We firstly investigate equation (9) and (10) when the velocity gradient is positive. Then we can write equation (9) as,

$$u_t = u_{yy} + bnu_y^{n-1}u_{yy} - \frac{1}{K} - M^2u - \frac{b}{K}uu_y^{n-1} \quad (11)$$

Where the suffices are refer to partial derivatives.

In the special case of a Newtonian fluid, we have $b = 0$ or $a = 0$ and $n = 1$ in equation (5). In the Newtonian case (11) for $b = 0$reduces to,

$$u_t = u_{yy} - \left(\frac{1}{K} + M^2\right)u. \quad (12)$$

By means of the transformation

$$U(y,t) = u(y,t)\exp\left[\left(\frac{1}{K} + M^2\right)t\right] \quad (13)$$

One can reduce equation (5.23) to the classical heat equation

$$U_t = U_{yy}, \quad (14)$$

With boundary conditions becoming

$U(0,t) = v(t)\exp\left[\left(\frac{1}{K} + M^2\right)t\right]$

$U(\infty,t) = 0,$ (15)

$U(y,0) = h(y).$

A solution of Equation (14) can be found by the reduction

$$U = f\left(yt^{-1/2}\right) \quad (16)$$

The substitution of equation (16) into equation (14) gives the following ordinary differential equation

$$f'' + \frac{1}{2}\lambda f' = 0 \quad (17)$$

Subject to the boundary conditions

$$f(0) = l, f(\infty) = 0 \quad (18)$$

Where l is a constant and prime denotes the derivatives with respect to $\lambda = yt^{-1/2}$.

Now the solution of equations (17) and (18) is simple and given by

$$f(\lambda) = l\left[1 - \operatorname{erf}\left(\frac{\lambda}{2}\right)\right], \tag{19}$$

The erf denotes the error function. Thus in the case of a Newtonian fluid the solution is given by

$$u(y,t) = l\left[1 - \operatorname{erf}\left(\frac{1}{2}yt^{-\frac{1}{2}}\right)\right]\exp\left[-\left(\frac{1}{K} + M^2\right)t\right] \tag{20}$$

With

$$V(t) = l\, exp\left[-\left(\frac{1}{K} + M^2\right)t\right], h(y) = 0. \tag{21}$$

This solution is graphically represented in figures 5.1 to 5.3, for various parameters.

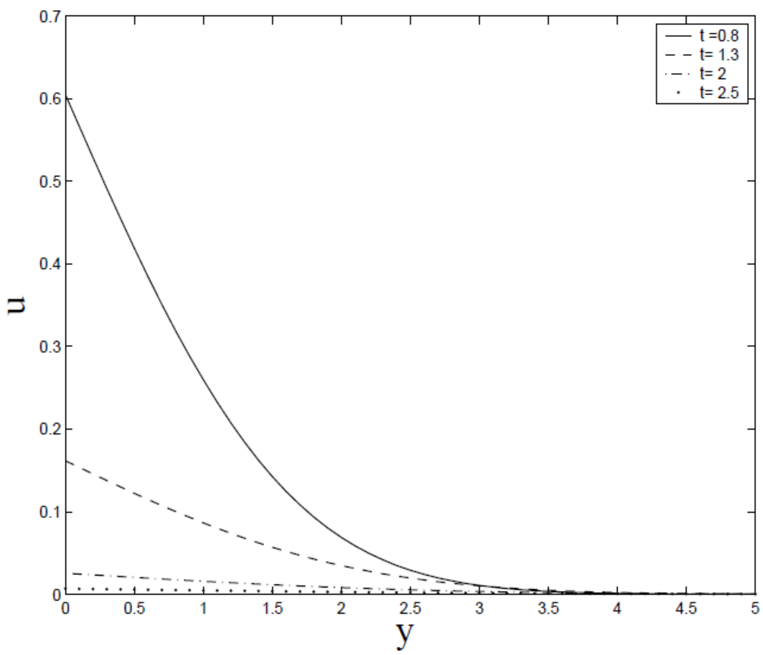

Figure 1: Velocity profile in the case of Newtonian fluid, for various values of time t, when l = 5, M= 0.8 and K = 0.5.

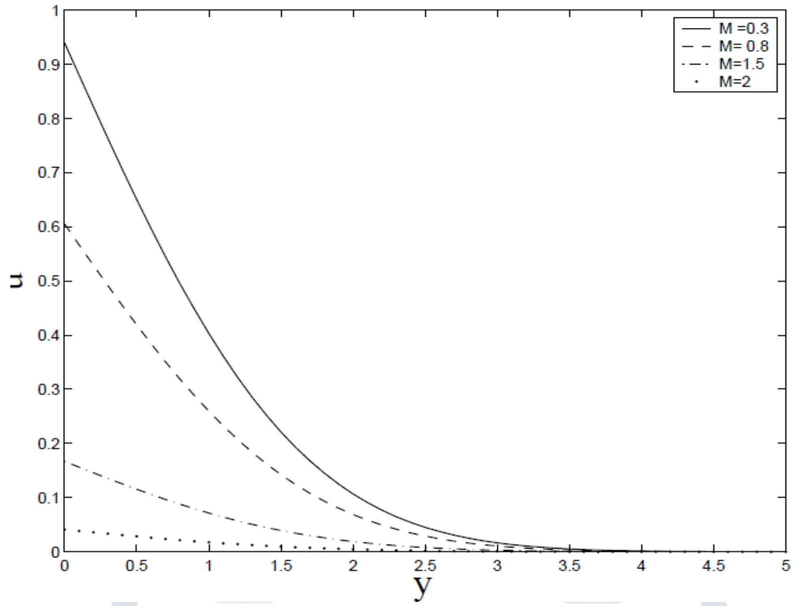

Figure 2: Velocity profile in the case of Newtonian fluid, when varying the Hartmann M, with l =5, t = 0.8 and K = 0.5.

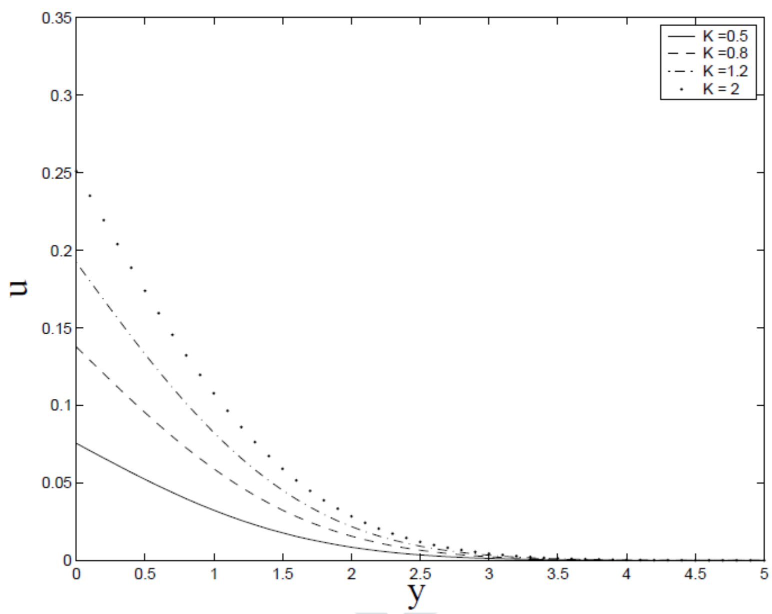

Figure 3: Velocity profile in the case of Newtonian fluid, for various values of the parameter K, when l = 5, t = 0.8 and M = 1.8

We now look at the Non Newtonian case when there is no magnetic field and there is no porous space. Then our problem reduces to

$$u_t = u_{yy} + bnu_y^{n-1}u_{yy} \tag{22}$$

Subject to the boundary conditions (10). travelling wave solutions of constant wave speed c of equation (22) are represented by

$$u(y,t) = f(x_1), x_1 = y - ct \tag{23}$$

Which describes waves travelling away from the boundary. The substitution of equation (23) into equation (22) yields the equations for f

$$-cf^{'} = f^{''} + bn(f^{'})^{n-1}f^{''} \qquad (24)$$

Where prime refers to the total derivatives with respect to x_1.

The equation (24) can be reduced to a first-order equation which is

$$f^{'} + b(f^{'})^{n} + cf + A = 0, \qquad (25)$$

Here A is a constant of integration. The solutions of equation (25) subject to the boundary conditions

$$f(0) = l\,, f(\infty) = 0, \qquad (26)$$

n	Solution $f'(x_1)$
2	$\dfrac{\sqrt{-4b\big(A+cf(x_1)\big)+1}-1}{2b}$
3	$\dfrac{\sqrt[3]{2}\left(-9\big(A+cf(x_1)\big)b^2+\sqrt{3}\sqrt{b^3(27b\big(A+cf(x_1)\big)^2+4}\right)^{\frac{2}{3}}-2\sqrt[3]{3}b}{6^{\frac{2}{3}}b\sqrt[3]{-9\big(A+cf(x_1)\big)b^2+\sqrt{3}\sqrt{b^3(27b\big(A+cf(x_1)\big)^2+4}}}$

With $f(0) = l,\ f(M_1) = 0,\ M_1 > 0.$

Where M_1 sufficiently large equation is can be obtained using Mathematica.

The numerical solutions of equations (24) are plotted in figures 4 and 5 for various values of the emerging parameters in the non-Newtonian case.

We next construct similarity

$$u = t^{\alpha} f(yt^{-\alpha}) \qquad (33)$$

For equation (22), intersection of equation (33) into equation (22) suggests that $\alpha = \frac{1}{2}$. The reduced equation is then

$$f^{''} + bn(f^{'})^{n-1}f^{''} + \tfrac{1}{2}\gamma f^{'} - \tfrac{1}{2}f = 0, \qquad (34)$$

Where $\gamma = yt^{-1/2}$ and the prime denotes differentiation with respect to γ. The boundary conditions (10) become

$$f(0) = l_1, \;\; f(\infty) = 0 \tag{35}$$

Where l_1 is a constant and $v(t) = l_1\sqrt{t}\,, h(y) = 0$

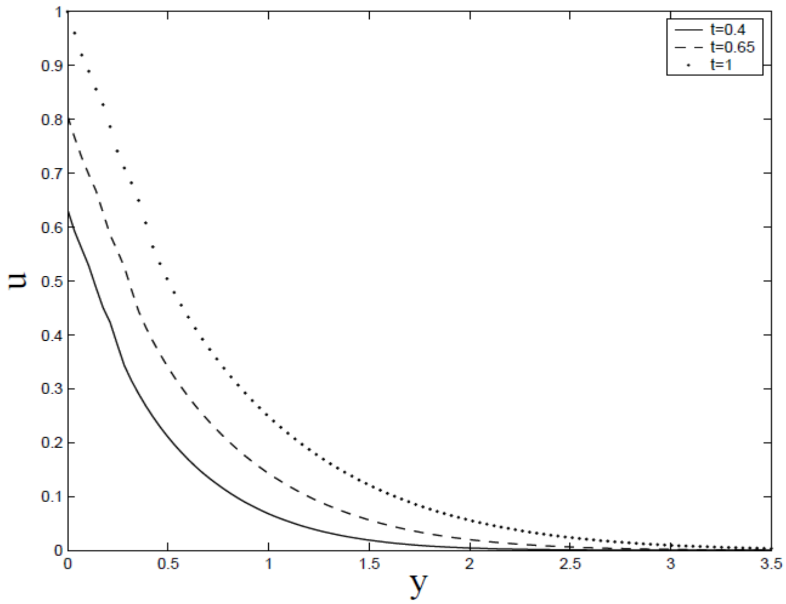

Figure 4: Similarity solution $u = t^{-1/2}f(yt^{-1/2})$ varying time t in the non-Newtonian case whenthere is no magnetic field and no porous space and the flow index is n = 2, with b = 0.5

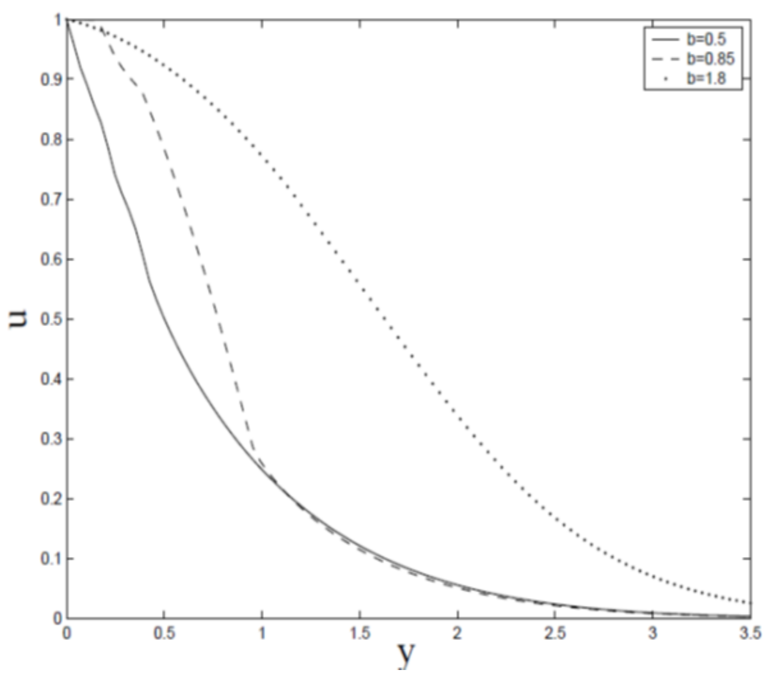

Figure 5: Similarity solution $u = t^{-1/2}f(yt^{-1/2})$ varying the parameter b in the non-Newtonian casewhen there is no magnetic field and no porous space and the flow index is n = 2, with t = 1

5. Exact solutions with magnetic field and porosity

We now look at the case when both magnetic field and porosity effectis taken into accounts, then we need to consider the general equation (11). Time-independent or the steady state solutions of the form,

$$u(y,t) = F(y) \tag{37}$$

Are searched the substitution of equation (37) into equation (11) gives rise to the differential equations for $F(y)$, viz

$$\frac{d}{dy}\left[\left(1 + b\left|\frac{du}{dy}\right|^{n-1}\right)\frac{du}{dy}\right] - \frac{1}{k}\left[1 + b\left|\frac{du}{dy}\right|^{n-1}\right]F - M^2F = 0 \tag{38}$$

Where the relevant condition are

$$F(0) = F_0\,, F(\infty) = 0\,, h(y) = F(y) \tag{39}$$

In which F_0 is a constant .Here $v(t)$ is taken as constant and equation (38) can be reduced to the form

$$\frac{F' + bn(F')^n}{{}^{b}/_{K}(F')^{n-1} + {}^{1}/_{K} + M^2}\, dF' = F dF \tag{40}$$

Where the prime indicates the differential with respect to y. For example, for $n = 2$, we can deduce the first order equation

$$(-3K - 4M^2K^2)\left(\frac{b}{K}F' + \frac{1}{K} + M^2\right) + (1 + 3M^2K + 2M^4K + M^2)$$

$$+K^2\left(\frac{b}{K}F' + \frac{1}{K} + M^2\right)^2 = \frac{b^2}{2K}F^2 + A \tag{41}$$

Where A.is a constant.

We plot the solutions of equations (38) subject to (39) in figures 9, 10 and11 varying b, $k$and M and for fixed n=3

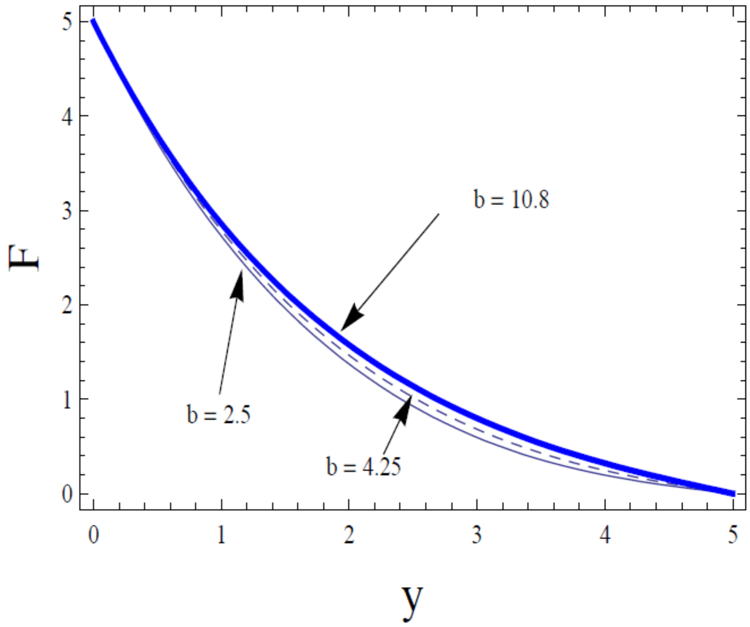

Figure 6: Steady state solution varying the parameter b taking into account the magnetic field and porosity when n = 3, with M = 0.8,K = 1.2

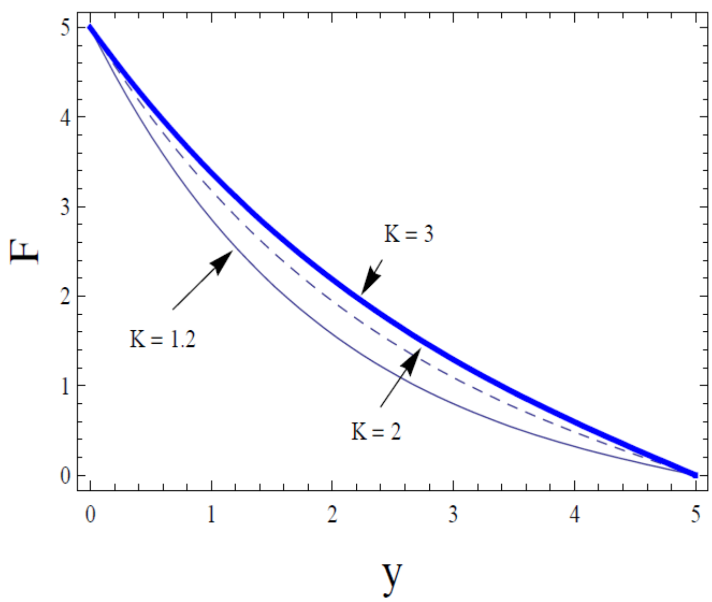

Figure 7: Steady state solution varying the parameter K taking into account the magnetic field and porosity when n = 3, with M = 0.8, b = 10.8

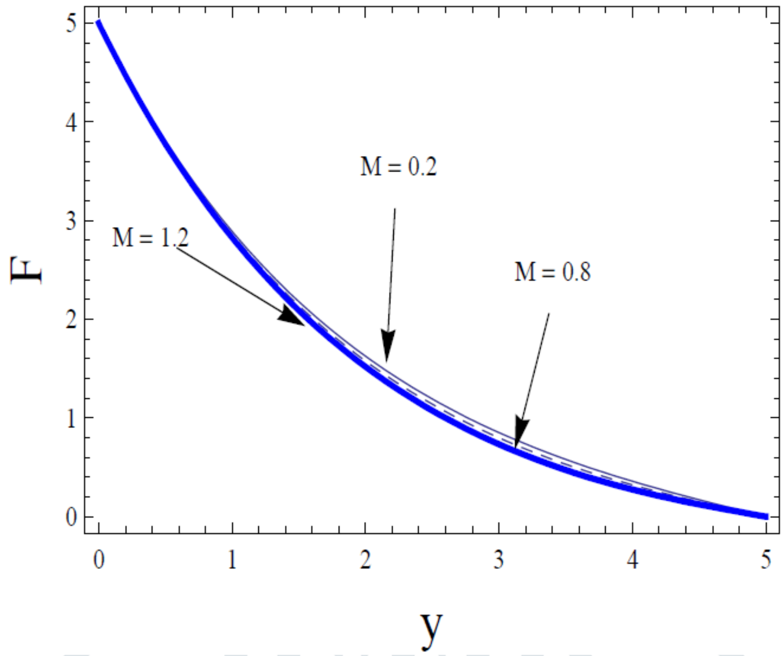

Figure 9: Steady state solution varying the parameter M taking into account the magnetic field and porosity when n = 3, with K = 1.2, b = 10.8

Remark 1: We now make a comment on solutions for which the velocity gradient is negative. For n-1 even in equation (9), one still has the solutions as before. However for n-1 odd, one needs to replace the parameters b by$-b$ in the aforementioned.

Remark 2: The numerical integration of the special case $n = 2$ resulted in some solutions having unusual behavior. This is due to singularity presented by the equations at some point during the integration.

6. Results and discussion

In this paper, we have investigated the unidirectional flow of a generalized non-Newtonian fluid filling the porous filling the porous half-space. Equation (11) has been solved for similarity and numerical solutions. The solutions were first found for the case of a Newtonian fluid, then for the case of non-Newtonian fluids where there is no magnetic field and porosity and finally when both Magnetic field and porosity are taken into account. In figures 1 to 3 we represented the solutions (20) for the Newtonian case for different values of t,$M$and K. It is observed that the velocity is a decreasing function of time and of the parameter M, while it increase by increasing value of b. Since$M$ is the ratio between the magnetic force and the viscous force, an increase in M is interpreted as the magnetic force taking over the viscous force, which results in the decreases of the velocity.

To depict the influence of the parameters on the travelling wave solutions (23), figures 4 and 5 were plotted for the case n=3. It can be seen in both cases that the velocity decreases by increasing the wave speed c, while on the other hand the velocity increases for the large value of b.The study was made on the effect of the time t and of constant parameters

$b$and c. For each varying parameters, we looked at both cases n=2 and n≠2. It is noted that the flow decreases for increasing ty. Figures 6,7 and 8 show that the velocity increases for the large value s of b.although the corresponding variation of the flow is not very significant in the case when n is not equal to two.

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