Satellites: Linearised and Normalised Differential equations of Relative motion of the system

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Abstract

This paper deals with the Linearised and Normalised differential equations of relative motion of the system. We emphasize on comparative study of the Normalised and linearised vector equation of relative motion of the particle of mass \(m_1\) with respect to the center of mass of the system.

Key words: System; Linearised; Normalise; Airdragness, Relative motion.

Introduction:

The effect of air resistance, magnetic force and oblateness of the earth on the motion of the satellites connected by a light, flexible and extensible cable in the central gravitation field of earth. The Lagrange’s equations of motion of the system under the influence of air resistance, magnetic force and oblateness of the earth. The normalized and linearised differential equations of the mass of the particle \(m_1\) of the system have been obtained on the assumption that the length of connecting cable is very small compared to the distance of the satellites from the colure of the earth. Suppose two satellites of the system as the particles of mass \(m_1\) and \(m_2\) having their radius vectors \( \vec{r}_1 \) and \( \vec{r}_2 \) respectively.

Let \(l_0\) the length of the string connecting the two particles of mass \(m_1\) and \(m_2\) suppose \(l'\) be the length of the string at any time.

The constraint of the system

\[
\left| \vec{r}_1 - \vec{r}_2 \right| = l_0^2
\]

\[\text{ .......... (1)}\]

Mathematical Approach:

By using Lagrange’s equation of motion of first kind the equation of motion of the two particles of mass \(m_1\) and \(m_2\) connected by an extensible string of natural length \(l_0\). The influence of air resistance, magnetic force and oblateness of the earth in the form
and

\[
m_2 \ddot{r}_2 + \frac{m_2 \mu \ddot{r}_2}{r_2^3} + \frac{3m_2 \mu k \ddot{r}_2}{r_2^5} - \lambda \left[ \frac{\ddot{r}_1 - \ddot{r}_2}{l_0} \right] (\ddot{r}_1 - \ddot{r}_2) = \vec{F}_a + Q \left( r_2 \times \vec{H} \right)
\]

Where \( \vec{F}_a (i = 1, 2, \ldots) \) is the aerodynamic force

\[
k_2 = e R_e^2 / 3
\]

\[
R = \frac{R_e - R_p}{R_e} = \text{Earth's oblatness}
\]

We come to know that

\[
\vec{\rho}_1 = \frac{m_2}{m_1 + m_2} (\ddot{r}_1 - \ddot{r}_2)
\]

\[
\vec{\rho}_2 = \frac{m_1}{m_1 + m_2} (\ddot{r}_2 - \ddot{r}_1)
\]

Now

\[
m_1 \vec{\rho}_1 + m_2 \vec{\rho}_2 = 0
\]

By neglecting the 2\textsuperscript{nd} and higher order perturbation terms and the equation of the centre of mass of the system as

\[
M \ddot{r} + \frac{2M}{R^3} = 0
\]

The centre of mass of the system can be assumed to move along a keplerian elliptical orbit with the higher degree of accuracy up to and order infinitesimal such as

\[
\frac{\vec{\rho}_1}{R} \text{ and } \frac{\vec{\rho}_2}{R}
\]

In this way, we say that the centre of mass of the system of the two satellites connected by an extensible string in the central gravitation field of two satellites connected by an extensible string in the central gravitational field of attraction moves along a given keplerian elliptical orbit.

\[
\epsilon = \alpha_k - \frac{m}{2}
\]

where

\[
m = \frac{\Omega^2 R_e}{g_e}
\]

\( \Omega = \text{Angular velocity of the earth's rotation} \)

\( R_e = \text{Equatorial radius of the earth} \)

\( R_p = \text{polar radius of the earth} \)

\( g_e = \text{Force of gravity at the equation of the earth} \)

\[
Q_i = \frac{\text{charge on the } i^{\text{th}} \text{ particle}}{\text{Velocity of light } C}; \ (i = 1, 2)
\]
\[ \mathbf{F} = \text{Intensity of the earth's magnetic field for equatorial satellites} \]
\[ \mathbf{F} = -\mathbf{\nabla} \left( \mathbf{m} \cdot \mathbf{r} \right) \]
\[ \mathbf{m} = \text{Magnetic moment of the earth.} \]

Suppose it is assumed that air drag varies as the square of the velocity of the moving particles.
\[ \mathbf{F}_a = -\rho_a C_i \mathbf{\dot{r}}_i \]
\[ \rho_a = \text{density of the air} \]
\[ C_i = \text{Ballistic Coeff}^H \]

Now,
\[ m_1 \ddot{r}_1 + \frac{m_1 \mu r_1}{r_1^3} + 3m_1 \mu k_2 \mathbf{r}_1 = \lambda \left( \frac{(r_1 - r_2) - l_0}{r_1 - r_2} \right) \frac{(r_1 - r_2)}{r_1 - r_2} + \rho_a C_i \mathbf{\dot{r}}_1 + Q_1 (\mathbf{\dot{r}}_1 \times \mathbf{H}) \]

Hence
\[ m_2 \ddot{r}_2 + \frac{m_2 \mu r_2}{r_2^3} + 3m_2 \mu k_2 \mathbf{r}_2 = -\lambda \left( \frac{(r_1 - r_2) - l_0}{r_1 - r_2} \right) \frac{(r_1 - r_2)}{r_1 - r_2} + \rho_a C_i \mathbf{\dot{r}}_2 + Q_2 (\mathbf{\dot{r}}_2 \times \mathbf{H}) \]

We emphasize that the Linearised and Normalised differential Equation of Relative motion of the system.

Now,
\[ \mathbf{\dot{r}}_1 - \mathbf{\dot{r}}_2 + \mu \left( \frac{\mathbf{\dot{r}}_1}{r_1^3} - \frac{\mathbf{\dot{r}}_2}{r_2^3} \right) + \frac{\lambda}{m_1 m_2} \left( \frac{(r_1 - r_2) - l_0}{r_1 - r_2} \right) = \rho_a C_i \mathbf{\dot{r}}_1 - C_5 \mathbf{\dot{r}}_2 + \frac{Q_1}{m_1} (\mathbf{\dot{r}}_1 \times \mathbf{H}) - \frac{Q_2}{m_2} (\mathbf{\dot{r}}_2 \times \mathbf{H}) \]

\[ u \left( \frac{\mathbf{\dot{r}}_1}{r_1^3} - \frac{\mathbf{\dot{r}}_2}{r_2^3} \right) = \frac{\mu}{R^3} (\mathbf{\dot{r}}_1 - \mathbf{\dot{r}}_2) - \frac{3\mu \mathbf{\ddot{R}}}{R^3} \left[ \mathbf{\dot{R}} (\mathbf{\dot{r}}_1 - \mathbf{\dot{r}}_2) \right] \]

\[ 3\mu k_2 \left( \frac{\mathbf{\dot{r}}_1}{r_1^5} - \frac{\mathbf{\dot{r}}_2}{r_2^5} \right) = \frac{3\mu k_2}{R^3} (\mathbf{\dot{r}}_1 - \mathbf{\dot{r}}_2) - \frac{15\mu k_2}{R^7} \left[ \mathbf{\dot{R}} (\mathbf{\dot{r}}_1 - \mathbf{\dot{r}}_2) \right] \]

\[ \rho_a C_i (\mathbf{\dot{r}}_1 - C_5 \mathbf{\dot{r}}_2) = \rho_a \mathbf{\dot{R}} (\mathbf{C}_1 - C_2) + \rho_a \mathbf{R} \]

\[ \mathbf{R} = \frac{\mathbf{\dot{R}} (\mathbf{\dot{R}} \mathbf{\dot{r}}_1 + \mathbf{\dot{r}}_2)}{m_2} + \mathbf{\dot{r}}_2 \left( \frac{C_2 m_2 + C_5 m_1}{m_2} \right) \]

\[ \rho_a [C_1 \mathbf{\dot{r}}_1 - C_2 \mathbf{\dot{r}}_2] = \rho_a \mathbf{\dot{R}} (C_1 - C_2) + \rho_a \mathbf{R} \]
\[
\ddot{H} = -\frac{\vec{\nabla} \vec{M} \cdot \vec{r}}{\eta}\text{ for } (i = 1,2)
\]

\[
\ddot{r}_i - \ddot{r}_2 + \frac{\mu}{R^3} (\vec{r}_i - \vec{r}_2) - \frac{3\mu \vec{R}}{R^5} \left[\vec{R}(\vec{r}_i - \vec{r}_2)\right]
\]

\[
+ \lambda \left(\frac{m_1 + m_2}{m_1 \times m_2} \right) \left[\frac{|\vec{r}_i - \vec{r}_2|}{l_0} \right] (\vec{r}_i - \vec{r}_2)
\]

\[
+ \frac{3\mu \kappa_2}{R^5} (\vec{r}_i - \vec{r}_2) - \frac{15\mu \kappa_2}{R^5} \left[\vec{R}(\vec{r}_i - \vec{r}_2)\right] \vec{R}
\]

\[
+ \rho_s \bar{R} \left( c_1 - c_2 \right) + \rho \alpha \bar{R} \left[\ddot{\bar{R}} \left( \bar{R} \right) + \ddot{\bar{\rho}} \right]
\]

\[
= \frac{c_2 m_1 + c_1 m_2}{m_1 + m_2} = \left(\frac{m_1 + m_2}{m_2}\right) \ddot{\bar{R}}
\]

\[
\text{But, } \vec{r}_1 - \vec{r}_2 = \ddot{\bar{R}} - \ddot{\bar{\rho}} = \left(\frac{m_1 + m_2}{m_2}\right) \ddot{R}
\]

Using (8) in (9) we get on dividing throughout by \(\frac{m_1 + m_2}{m_2}\)

\[
\ddot{\bar{R}} + \frac{\mu \ddot{\bar{R}}}{R^3} + 3\mu \kappa_1 \bar{R} \left( \bar{R} \right) - 3\mu \kappa_2 \bar{R} \left( \bar{R} \right)
\]

\[
+ 15\mu \kappa_2 \bar{R} \left( \bar{R} \right) + \lambda_a \left[1 - \frac{\nu}{\bar{\rho}_1}\right] \ddot{\bar{\rho}}
\]

\[
+ \rho_s \bar{R} \left( c_1 - c_2 \right) \frac{m_2}{m_1 + m_2} + \rho \alpha \bar{R} \left[\ddot{\bar{R}} \left( \bar{R} \right) + \ddot{\bar{\rho}} \right]\frac{c_2 m_1 + c_1 m_2}{m_1 + m_2}
\]

\[
= -\frac{m_2}{m_1 + m_2} \left[\frac{Q_1}{m_1} \vec{r}_1 \times \vec{\epsilon} \left( \vec{M} \vec{r}_1 \right) \frac{Q_2}{m_2} \vec{r}_2 \times \vec{\epsilon} \left( \vec{M} \vec{r}_2 \right)\right]
\]

Where

\[
\lambda_a = \frac{m_1 + m_2}{m_1 m_2} \frac{\dot{\lambda}}{l_0}
\]

\[
\gamma = \frac{m_2 l_0}{m_1 + m_2}
\]
Since
\[
\frac{1}{r^3} = \frac{1}{(r^2)^{\frac{3}{2}}} = \left[\frac{1}{(R + \rho)^{\frac{3}{2}}}\right]^{\frac{3}{2}}; i = 1, 2
\]
\[
= \frac{1}{R^3} - \frac{3R\rho}{R^5}
\]
\[
\frac{\ddot{r}}{r^3} = \left(\ddot{R} + \ddot{\rho}\right) \left[\frac{1}{R^3} - \frac{3R\rho}{R^5}\right]; i = 1, 2
\]
\[
= \frac{\ddot{R} + \ddot{\rho}}{R^3} - \frac{3R\rho}{R^5} \left(\ddot{R} + \ddot{\rho}\right) = 1, 2
\]

\[\text{Hence we have}\]
\[
\frac{Q_1}{m_1} \left\{ \ddot{\vec{r}} \times \vec{\nabla} \left( \frac{M}{r_1^3} \right) \right\} + \frac{Q_2}{m_2} \left\{ \ddot{\vec{r}} \times \vec{\nabla} \left( \frac{M}{r_2^3} \right) \right\}
\]
\[
= \dddot{\vec{R}} \times \vec{\nabla} \left( \frac{M \ddot{R}}{R^3} \right) \left( \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right)
\]

\[\text{We get the linearised vector equation of motion for the particle of mass } m_1 \text{ relative to the}
\]
\[\text{centre of mass the system in the form}\]
\[
\frac{-m_2}{m_1 + m_2} \dddot{\vec{R}} \times \vec{\nabla} \left( \frac{M \ddot{R}}{R^3} \right) \left( \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right)
\]

The condition of constraint reduces to
\[
|\ddot{\rho}|^2 \leq \nu^2
\]

\[\text{The inequality sign holds then the system will be moving with loose string.}\]

The normalize the vector \(\ddot{\rho}\) by introducing.
\[
\ddot{\rho} = \frac{\nu \ddot{\rho}}{l_0}
\]

Then the vector equation of particle of mass \(m_1\) takes the form
\[ \ddot{\rho}_1 + \frac{\mu}{R^3} \dot{\rho}_1 \times \frac{3 \mu R (\hat{R} \cdot \dot{\rho}_1)}{R^5} + a_1 \dot{R} + a_2 \left[ \frac{\dot{R} (\hat{R} \cdot \dot{\rho}_1)}{R^2} + \ddot{\rho}_1 \right] \]

\[ + 3 \mu \kappa_2 \dot{\rho}_1 \times \frac{15 \mu \kappa_2 \dot{R}}{R^3} (\hat{R} \cdot \dot{\rho}_1) + \]

\[ \lambda \left[ 1 - \frac{l_0}{R} \right] \dot{\rho}_1 \times \vec{V} \left( \frac{M \dot{R}}{R^3} \right) \left( \frac{Q_1 - Q_2}{m_1 - m_2} \right) \]

\[ \text{........... (17)} \]

Where

\[ a_1 = \rho a R (c_1 - c_2) \]

\[ a_2 = \rho a \frac{\vec{R}(c_1 m_2 + c_2 m_1)}{m_2} \]

\[ \text{........... (18)} \]

The normalized vector equation of relative motion of the particle of mass \( m_1 \) with respect to the centre of mass of the system

**Conclusion:**

The atmospheric drag consists of two terms, one with coefficient and other with coefficient as the term with coefficient E.g. \( \dot{\rho}_1 \) as a factor which indicates that this is the part of the atmospheric drag arising out of air friction. The coefficient of which is parameter of the aerodynamic force is a small quantity which is multiplied by small quantity \( \dot{\rho}_1 \) and neglect this term as we are considering preservative force of first order only.

**References:**


