α^*-compactness and α^*-connectedness in topological spaces

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Abstract

In this paper, we introduce the new concepts α^*-compactness and α^*-connectedness in topological spaces and obtain some of their properties using α^*-closed sets.

Keywords: α^*-closed sets, α^*-continuous maps, α^*-compactness and α^*-connectedness.

1 Introduction

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers [1-8] have analyzed the basic properties of compactness and connectedness. The notions of compactness and connectedness resulted in motivating mathematicians to generalize these notions further.

K. Vithyasangaran and P. Elango [9] introduced and studied the properties of α^*-closed sets in topological spaces. The aim of this paper is to study α^*-compactness and α^*-connectedness using α^*-closed set and also discuss some of their properties

2 Preliminaries

Throughout this paper (X,τ), (Y,σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of (X,τ), cl(A) and Int(A) denote the closure of A and interior of A respectively.

Definition 2.1. Let (X,τ) be a topological space. Then, a subset A of (X,τ) is called α^*-closed set [9] if int(cl(A))$\subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

The complement of the above mentioned α^*-closed set is α^*-open set.

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Definition 2.2. A function $f:(X,\tau) \to (Y,\sigma)$ is called

(i) α^*-continuous [9] if the inverse image of every closed set in (Y,σ) is α^*-closed in (X,τ).

(ii) α^*-irresolute [9] if the inverse image of every α^*-closed set in (Y,σ) is α^*-closed in (X,τ).

Definition 2.3. A topological space X is said to be T_{α^*}-space [9] if every α^*-closed subset of X is closed subset of X.

3 α^*-compactness

Definition 3.1. A collection $\{A_i : i \in I\}$ of α^*-open sets in a topological space X is called a α^*-open cover of a subset B of X if $B \subset \cup\{Ai : i \in I\}$ holds.

Definition 3.2. A topological space X is α^*-compact if every α^*-open cover of X has a finite subcover.

Definition 3.3. A subset B of a topological space X is said to be α^*-compact relative to X if, for every collection $\{Ai : i \in I\}$ of α^*-open subsets of X such that $B \subset \cup\{Ai : i \in I\}$ there exists a finite subset I_0 of I such that $B \subseteq \cup\{Ai : i \in I_0\}$.

Definition 3.4. A subset B of a topological space X is said to be α^*-compact if B is α^*-compact as a subspace of X.

Theorem 3.1. *Every α^*-closed subset of a α^*-compact space X is α^*-compact relative to X.*

Proof. Let A be α^*-closed subset of α^*-compact space X. Then, A^c is α^*-open in X. Let $M = \{G_\alpha : \alpha \in I\}$ be a cover of A by α^*-open sets in X. Then, $M^* = M \cup A^c$ is a α^*-open cover of X. Since X is α^*-compact, M^* is reducible to a finite subcover of X, say $X = G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha m} \cup A^c$, $G_{\alpha k} \in M$. But, A and A^c are disjoint hence $A \subset G_{\alpha 1} \cup G_{\alpha 2} \cup \ldots \cup G_{\alpha m}$, $G_{\alpha k} \in M$, which implies that any α^*open cover M of A contains a finite subcover. Therefore, A is α^*-compact relative to X. Thus, every α^*-closed subset of α^*-compact space X is α^*-compact. □

Theorem 3.2. *Every α^*-compact space is compact.*

Proof. Let X be a α^*-compact space. Let $\{A_i : i \in I\}$ be an open cover of X. Then $\{A_i : i \in I\}$ is a α^*-open cover of X as every open set is α^*-open set. Since X is α^*-compact, the α^*-open cover $\{A_i : i \in I\}$ of X has a finite subcover, say $\{A_i : i = 1,...,n\}$ for X. Hence X is compact. □

Definition 3.5. A function $f : X \longrightarrow Y$ is said to be α^*-continuous [9] if $f^{-1}(F)$ is α^*-closed in X for every closed set F of Y.

Definition 3.6. A function $f : X \longrightarrow Y$ is said to be α^*-irresolute [9] if $f^{-1}(F)$ is α^*-closed in X for every α^*closed-set F of Y.

Theorem 3.3. *Let $f : X \to Y$ be surjective, α^*-continuous function. If X is α^*-compact, then Y is compact.*

Proof. Let $\{A_i : i \in I\}$ be an open cover of Y. Since f is α^*-continuous function, then $\{f^{-1}(A_i) : i \in I\}$ is α^*-open cover of X has a finite subcover, say $\{f^{-1}(A_i) : i = 1,...,n\}$. Therefore, $X = \cup_{i=1}^{n} f^{-1}(A_i)$ which implies $f(X) = \cup_{i=1}^{n} f(A_i)$. Since f is surjective, $Y = \cup_{i=1}^{n} f(A_i)$. Thus, $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for Y. Hence Y is compact. □

Theorem 3.4. *If a map $f : X \to Y$ is α^*-irresolute and a subset B of X is α^*-compact relative to X, then the image $f(B)$ is α^*-compact relative to Y.*

Proof. Let $\{A_\alpha : \alpha \in I\}$ be any collection of α^*-open subsets of Y such that $f(B) \subset \cup\{A_\alpha : \alpha \in I\}$. Then, $B \subset \{f^{-1}(A_\alpha) : \alpha \in I\}$ holds. From the hypothesis, B is α^*-compact relative to X. Then, there exists a finite subset I_0 of I such that $B \subset \{f^{-1}(A_\alpha) : \alpha \in I_0\}$. Therefore, we have $f(B) \subset \cup\{A_\alpha : \alpha \in I_0\}$, which shows that $f(B)$ is α^*-compact relative to Y. □

4 α^*-connectedness

Definition 4.1. A topological space X is said to be α^*-connected if X can not be expressed as a disjoint union of two non-empty α^*-open sets. A subset of X is α^*-connected if it is α^*-connected as a subspace.

Example 4.1. *Let $X = \{a,b\}$ and let $\tau = \{X, \varphi, \{b\}\}$. Then, (X,τ) is α^*-connected.*

Remark 4.1. Every α^*-connected space is connected. But, the converse need not be true in general, which follows from the following example.

Example 4.2. *Let $X = \{a,b\}$ and let $\tau = \{X, \varphi\}$. Clearly, (X,τ) is connected. The α^*-open sets of X are $\{X, \varphi, \{a\}, \{b\}\}$. Therefore, (X,τ) is not a α^*-connected space, because $X = \{a\} \cup \{b\}$ where $\{a\}$ and $\{b\}$ are non-empty α^*-open sets.*

Theorem 4.3. *For a topological space X, the following are equivalent:*

(i) X is α^*-connected.

(ii) X and φ are the only subsets of X which are both α^*-open and α^*-closed.

(iii) Each α^*-continuous map of X into a discrete with at least two points is a constant map.

Proof. $(i) \Rightarrow (ii)$: Let F be any α^*-open and α^*-closed subset of X. Then, F^c is both α^*-open and α^*-closed. Since X is disjoint union of the α^*-open sets F and F^c implies from the hypothesis of (i) that either $F = \varphi$ or $F = X$.

$(ii) \Rightarrow (i)$: Suppose that $X = A \cup B$ where A and B are disjoint non-empty α^*-open subsets of X. Then, A is both α^*-open and α^*-closed. By assumption, $A = \varphi$ or X. Therefore, X is α^*-connected.

(ii) $\Rightarrow (iii)$: Let $f : X \to Y$ be a α^*-continuous map.Then, X is covered by α^*-open and α^*-closed covering $\{f^{-1}(y) : y \in Y\}$. By assumption, $f^{-1}(y) = \varphi$ or X for each $y \in Y$. If $f^{-1}(y) = \varphi$ for all $y \in Y$, then f fails to be a map. Then, there exists only one point $y \in Y$ such that $f^{-1}(y) = \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) $\Rightarrow (ii)$: Let F be both α^*-open and α^*-closed in X. Suppose $F \neq \varphi$. Let $f : X \to Y$ be a α^*-continuous map defined by $f(F) = \{x\}$ and $f(F^c) = \{y\}$ for some distinct points x and y in Y. By assumption f is constant, we have $F = X$. □

Theorem 4.4. *If $f : X \to Y$ is a α^*-continuous, onto and X is α^*-connected, then Y is connected.*

Proof. Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open set in Y. Since f is α^*-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty α^*-open sets in X. This contradicts the fact that X is α^*-connected. Hence Y is connected. □

Theorem 4.5. *If $f : X \to Y$ is a α^*-irresolute surjection and X is α^*-connected, then Y is α^*connected.*

Proof. Suppose that Y is not α^*-connected. Let $Y = A \cup B$ where A and B are disjoint non-empty α^*-open set in Y. Since f is α^*-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty α^*-open sets in X. This contradicts the fact that X is α^*-connected. Hence Y is connected. □

Theorem 4.6. *Suppose that X is a T_{α^*}-space. Then, X is connected if and only if it is α^*-connected.*

Proof. Suppose that X is connected. Then, X can not be expressed as disjoint union of two nonempty open subsets of X. Suppose X is not a α^*-connected space. Let A and B be any two α^*-open subsets of X such that $X = A \cup B$, where $A \cap B = \varphi$ and $A \subset X$, $B \subset X$. Since X is T_{α^*}-space and A, B are α^*-open, A, B are open subsets of X, which contradicts to the fact that X is connected. Therefore, X is α^*-connected. Conversely, assume that X is α^*-connected. Then, X can not be expressed as disjoint union of two non-empty α^*-open subsets of X. Since, X is a T_{α^*}-space, every α^*-open subset of X is open subset of X. Hence X can not be expressed as disjoint union of two non-empty open subsets of X. That is, X is connected. □

5 Conclusion

In this paper, we have introduced α^*-compactness and α^*-connectedness in the topological spaces by using α^*-closed sets and their properties were studied.

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