Fixed Point Theorem For Sequence of Self Mapping in Complete Metric Space

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Abstract:
In 1978 Kishorimohan Ghosh and S.K.Chatterjea [1] have investigated a fixed point theorem in metric space for two self mapping. In this paper a fixed point theorem for sequence of self mapping in the complete metric space has been proved

1 Introduction
In 1974M.Sen Gupta (Mrs.Das Gupta)[2] have proved that in a complete metric space (M,d) if there exists two operators $T_1$ and $T_2$ mapping M into itself and satisfying the relation

$$d(T_1X,T_2X) \leq \alpha d(X,T_1X) + \beta d(Y,T_2Y) + \gamma d(X,Y)$$

(1.1)

or

$$d(T_1X,T_2Y) \leq \alpha d(X,T_1X) + \beta d(Y,T_2Y) + \gamma d(X,Y)$$

(1.2)

For X,Y in M , Where $\alpha, \beta, \gamma$ are non-negative real number and $\alpha + \beta + \gamma < 1$, then $T_1$ and $T_2$ have a unique common fixed point

In 1971 the relation 1 was utilized by S.Reich [4] with the condition $T_1 = T_2$ to discuss some fixed point theorems and in 1972 the relation 1 yielding common fixed point of operators $T_1$ and $T_2$ was also discussed by S.Reich under the more general conditions imposed on $\alpha, \beta$ and $\gamma$

More generally the relation

$$d(T_1X,T_2Y) \leq \sum_{i=1}^{\infty} \alpha_i d(X,Y) + \sum_{i=1}^{\infty} \alpha_i d(X,T_1x) + \sum_{i=1}^{\infty} \alpha_i d(Y,T_2y) + \sum_{i=1}^{\infty} \alpha_i d(Y,T_1x)$$

(1.3)

where $\alpha_i > 0$ and $\sum_{i=1}^{\infty} \alpha_i < 1$ was considered by G. Hardy and T.Rogers [3] to discuss some fixed point theorems

In 1978 Kishorimohan Ghosh and S.K.Chatterjea [1] have investigated the following theorem

Theorem: Let(X,d) be metric space. $T_1$ and $T_2$ be two self mapping for which there exists non-negative real number $\alpha_i (i = 1, 2, \ldots, 5)$ and $\sum_{i=1}^{\infty} \alpha_i < 1$ such that

$$d(T_1X,T_2Y) \leq \alpha_1 d(X,Y) + \alpha_2 d(X,T_1x) + \alpha_3 d(Y,T_2y) + \alpha_4 d(X,T_2y) + \alpha_5 d(Y,T_1x)$$

(1.4)
for all \(X, Y \in X\) for any \(X_0 \in X\) the sequence \(X_1 = T_2 X_0\)

\[
X_{2n} = T_2 X_{2n-1}, X_{2n+1} = T_1 (X_{2n}) \tag{1.5}
\]

has a subsequence covering to \(u \in X\) then \(T_1\) and \(T_2\) have a unique common fixed point \(u\).

we can extend the above fixed point theorem for sequence of mapping in complete metric space.

Theorem:- Let \((X,d)\) be complete metric space \(T_i\) and \(T_j\) be two sequence of self mapping for which there exist non-negative real number \(\alpha (i=1,2,3,4,5)\) with

\[
0 \leq \alpha_i < 1 \text{ and } \sum_{i=1}^{n} \alpha_i < 1
\]

such that

\[
d(T_{i}, T_{j}) \leq \alpha_1 d(X,Y) + \alpha_2 d(X,T_i) + \alpha_3 d(Y,T_j) + \alpha_4 d(X,T_j) + \alpha_5 d(Y,T_i)
\]

For all \(X, Y \in X\) for any \(X_0 \in X\) the sequence

\[
X_1 = T_1 X_0, X_2 = T_2 X_1, \ldots, X_{2n} = T_2(X_{2n-1})
\]

\(X_{2n+1} = T_1(X_{2n})\) has a subsequence covering to \(u \in X\). Then \(T_i\) and \(T_j\) have a unique common fixed point \(u\).

Proof: - We will prove above theorem by considering following these steps i)First we will show that \(\{X_n\}\) is couch sequence.

ii)Existence of fixed point. iii)Uniqueness of fixed point.

Proof:- i) Let \(X_0\) be any point of \(X\) and consider the sequence

\[
X_1 = T_1 X_0, X_2 = T_2 X_1, \ldots, X_{2n} = T_2(X_{2n-1}), X_{2n+1} = T_1(X_{2n})
\]

we have for \(X, Y \in X\)

\[
d(T_i X, T_j Y) \leq \alpha_1 d(X, Y) + \alpha_2 d(X, T_i) + \alpha_3 d(Y, T_j) + \alpha_4 d(X, T_j) + \alpha_5 d(Y, T_i)
\]

By interchanging \(X\) with \(Y\) and \(T_i\) and \(T_j\), we get

\[
d(T_i Y, T_j X) \leq \alpha_1 d(Y, X) + \alpha_2 d(Y, T_i) + \alpha_3 d(X, T_j) + \alpha_4 d(Y, T_j) + \alpha_5 d(X, T_i)
\]

(1.7)

Now adding (A) and (B) we have

\[
d(T_i X, T_j Y) + d(T_i Y, T_j X) \leq \alpha_1 d(X, Y) + \alpha_1 d(Y, X) +
\]

\[
+ \alpha_2 d(X, T_i) + \alpha_2 d(Y, T_j) + \alpha_3 d(Y, T_i) + \alpha_3 d(X, T_j) +
\]

\[
+ \alpha_4 d(X, T_j) + \alpha_4 d(Y, T_j) +
\]

\[
+ \alpha_5 d(Y, T_i) + \alpha_5 d(X, T_i)
\]

Now by symmetric property of metric we have

\[
d(X, Y) = d(Y, X)
\]

\[
2d(T_i X, T_j Y) \leq 2\alpha_1 d(X, Y) + (\alpha_2 + \alpha_3) d(X, T_i) +
\]

\[
+ (\alpha_3 + \alpha_2) d(Y, T_j) + (\alpha_4 + \alpha_3) d(X, T_j) + (\alpha_5 + \alpha_4) d(Y, T_i)
\]

\[
2d(T_i Y, T_j X) \leq 2\alpha_1 d(Y, X) + (\alpha_2 + \alpha_3) d(X, T_j)
\]
\[ + d(Y,TjY) \{ \alpha_4 + \alpha_5 \} \{ d(X,TjY) + d(Y,TjX) \} \]

\[ \therefore d(TiX, TjY) \leq \alpha_1 d(X, Y) + \left( \frac{\alpha_2 + \alpha_3}{2} \right) \{ d(X, TiY) + d(Y, TiX) \} \]

put \( X = X_0 \) & \( Y = X_1 \)

\[ d(TiX_0, TjX_1) \leq \alpha_1 d(X_0, X_1) + \left( \frac{\alpha_2 + \alpha_3}{2} \right) \{ d(X_0, TiX_0) + d(X_1, TiX_0) \} \]

\[ \therefore T_0X_0 = X_3 \& T_jX_1 = X_2 \text{ we have} \]

\[ d(X_1, X_2) \leq \alpha_1 d(X_0, X_1) + \left( \frac{\alpha_2 + \alpha_3}{2} \right) \{ d(X_0, X_1) + d(X_1, X_0) \} \]

\[ \therefore d(X_1, X_2) = 0 \& d(X_0, X_3) \leq d(X_0, X_1) + d(X_1, X_2) \]

\[ \therefore \text{We have} \]

\[ d(X_1, X_2) \leq \alpha_1 d(X_0, X_1) + \left( \frac{\alpha_2 + \alpha_3}{2} \right) \{ d(X_0, X_1) + d(X_1, X_0) \} \]

\[ 2d(X_0X_2) - (X_2 + X_3)d(X_1X_2) - \left( \alpha_4 + \alpha_5 \right) d(X_1X_2) \leq 2\alpha_4 d(X_0X_1) + \left( \alpha_2 + \alpha_3 \right) d(X_0X_1) + \left( \alpha_4 + \alpha_5 \right) d(X_0X_1) \]

\[ \therefore d(X_1, X_2) \leq \frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{2 - \left( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \right)} \{ d(X_0, X_1) \} \]

\[ \therefore d(X_1, X_2) \leq r \{ d(X_0, X_1) \} \]

where

\[ r = \frac{2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5}{2 - \left( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \right)} \]

parallellly we can show that

\[ \therefore d(X_2X_3) \leq r d(X_2X_1) \]

\[ \leq r.r d(X_0X_1) \]

\[ \therefore d(X_2X_3) \leq r^2 d(X_0X_1) \]

By induction we prove that
∴ \( d(X_nX_{n+1}) \leq r^2 d(X_0X_1) \)

Hence

\[
d(X_nX_{n+p}) \leq d(X_nX_{n+1}) + d(X_{n+1}X_{n+2}) + \ldots + d(X_{n+p-1}X_{n+p})
\]

∴ \( d(X_n, X_{n+p}) \leq d \left( r^n, r^{n+1} + \ldots + r^{n+p-1} \right) d(X_0, X_1) \)

∴ \( d(X_n, X_{n+p}) \leq \frac{r^n (1 - r^{n+p-1})}{1 - r} d(X_0, X_1) \)

and owing to the assumption \( \alpha_i < 1 \)

\[ d(X_nX_{n+p}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]

So we have \( \{X_n\} \) is a cauchy

Subsequence \( \{X_{nk}\} \) of this sequence \( \{X_n\} \) converges to \( u \)

ii) Now we will prove that \( u \) is fixed point of \( T_i \) and \( T_j \) is we will prove that

\[ T_i u, T_j u = u \]

Now first consider

\[
d(T_iu, u) \leq d(T_iu, X_{2n}) + d(X_{2n}u) = d(T_iu, T_j X_{2n-1}) + d(X_{2n}u)
\]

∴ \( d(T_iu, u) \leq \alpha_i d(T_iu, X_{2n-1}) + \alpha_i d(u, T_iu) + \alpha_i d(X_{2n-1}, X_{2n}) + \alpha_i d(u, X_{2n}) + \alpha_i d(X_{2n-1}, T_j u) + d(X_{2n}u) \)

As \( n \rightarrow \infty \) we have

\[
d(T_iu, u) \leq (\alpha_i + \alpha_i) d(u, T_iu)
\]

\[ \lim_{n \rightarrow \infty} X_n = u \quad n \rightarrow \infty \]

If \( \lim_{n \rightarrow \infty} X_{2n-1} = u \) & \( d(v,u) = c \)

∴ \( d(T_iu, u) \leq (\alpha_i + \alpha_i) d(T_iu, u) \)

∴ \( d(T_iu, u) - (\alpha_i + \alpha_i) d(T_iu, u) \leq 0 \)

\( \{1 - (\alpha_i + \alpha_i)\} d(T_iu, u) \leq 0 \)

Which is possible if \( d(T_iu, u) = 0 \)

∴ \( 1 - (\alpha_i + \alpha_i) \neq 0 \)

parallelly we can prove that \( T_j u = u \)

\( T_i \) and \( T_j \) have common field point \( u \).

iii) Now considers the uniqueness of fixed point \( u \).
If possible let these be another fixed point \( v \) of \( T_i \) and \( T_j \)

\[ \therefore T_iv = v, T_jv = v \]

then \( d(u,v) = d(T_iu, T_jv) \)

\[ d(u,v) \leq \alpha_1 d(u,v) + \alpha_2 d(v, T_iu) + \alpha_3 d(v, T_jv) + \alpha_4 d(u, T_jv) + \alpha_5 d(v, T_iu) \]

\[ \therefore d(u,v) \leq \alpha_1 d(u,v) + \alpha_2 d(v, u) + \alpha_3 d(v, u) \]

\[ \alpha_4 d(u,v) + \alpha_5 d(u,v) \]

\[ \therefore d(u,v) - \alpha_1 d(u,v) - \alpha_4 d(u,v) - \alpha_5 d(u,v) \leq 0 \]

\[ \therefore d(u,u) = d(v,v) = 0 \& \ d(u,v) = d(v,u) \]

\[ [1 - (\alpha_1 + \alpha_4 + \alpha_5)]d(u,v) \leq 0 \]

Which is possible if \( d(u,v) = 0 \)

\[ \therefore [1 - (\alpha_1 + \alpha_4 + \alpha_5)] = 0 \]

\[ \therefore d(u,v) = 0 \& \ u = v \]

Hence \( T_i \) and \( T_j \) have unique common fixed point \( v \).

**References**


