SPECTRAL THEORY FOR SELF –ADJOINT OPERATORS IN Γ-HILBERT SPACE

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Abctract

Spectral theorem provides Spectral decomposition, Eigen Value decomposition of the vector spaces on which the operator acts. We discuss the Hilbert-Schimdt operator and function of an operator in Γ-Hilbert Space .We will show the generalization of operators explicitly in Γ-Hilbert Space while operator being self-adjoint and compact.

Key Words

Γ-Hilbert Space, Self-adjoint operator, Spectral theorem, Hilbert-Schimdt operator, Function of an operator.

1 INTRODUCTION AND PRELIMINARIES

Spectral theory for a self-adjoint operator is very important part of Γ-Hilbert Space.The definition of Γ-Hilbert Space was first introduced by Bhattacharya D.K. and T.E. Aman in their paper "Γ-Hilbert Space and linear quadretic control problem" in 2003 [1].

We shall denote Γ-Hilbert Space by H_Γ. Consider an operator $T: H_\Gamma \to H_\Gamma$. Spectral Theorem is a generalization of the familiar theorem from Linear algebra call that a self adjoint n x n matrix A can be diagnolized, there is a diagonal matrix D and unitary matrix U such that

$$A = UDU^{-1}.$$

Before Studying the Spectral theory for compact self-adjoint operator we have to reconstruct the Hilbert-Schmidt theorem.Also we present the Spectral theorem for Functional Calculas form. Here we consider both finite dimensional spaces and infinite dimensional spaces. At first we re call some important definitions :

Definition 1.1: Let E, Γ be two linear space over the field F. A mapping $\langle .,.,. \rangle: E \times \Gamma \times E \to \mathbb{R}$ is called a **Γ-**Inner product on E if

(i) $\langle .,.,. \rangle$ is linear in each variable.
(ii) $\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle \; \forall \; u, v \in E \; and \; \gamma \in \Gamma.$
(iii) $\langle u, \gamma, u \rangle > 0 \; \forall \; \gamma \neq 0 \; and \; u \neq 0.$
(iv) $\langle u, \gamma, v \rangle = 0$ if atleast one of u, v, γ is zero.

$[(E, \Gamma), \langle .,.,. \rangle]$ is called a Γ-inner product space over F.

A complete Γ-inner product space is called **Γ**-Hilbert Space.

Using the Γ-Inner product, we may define three types of norm in a **Γ**-Hilbert Space, namely

(i) γ- Norm
(ii) Γ_{inf}-Norm

(iii) Γ-Norm.

γ-norm 1.2 : Now if we write $\|u\|_\gamma{}^2 = \langle u, \gamma, u\rangle$, for $u \in H$ and $\gamma \in \Gamma$ then $\|u\|_\gamma{}^2$ satisfy all the Conditions of norm.

$\|u\|_{\Gamma_{inf}}$-Norm 1.3 : If we define $\|u\|_{\Gamma_{inf}} = \inf\{\|u\|_\gamma : \gamma \in \Gamma\}$.Clearly Γ_{inf}-Norm satisfy all the conditions of the norm for $u \in H$.

Γ-Norm 1.4 : If we if write $\|u\|_\Gamma = \{\|u\|_\gamma : \gamma \in \Gamma\}$ then this Norm is called the Γ-Norm of the Γ-Hilbert space.

γ-Orthogonal 1.5 : Let L be a non-empty subset of a Γ-Hilbert space H_Γ . Two elements x and y are said to be γ-Orthogonal if their inner product $\langle x, \gamma, y\rangle = 0$. In symbol , we write $x \perp_\gamma y$.

Self-adjoint operator 1.6 : Let A be a bounded operator on Γ-Hilbert space and we denote it by H_Γ.Then the operator A^*: $H_\Gamma \to H_\Gamma$ defined by

$$\langle Ax, \gamma, y\rangle = \langle x, \gamma, A^* y\rangle \quad \forall x, y \in H_\Gamma \text{ and } \gamma \in \Gamma$$

is called the adjoint operator of A.

If A=A^* then A is called self adjoint of H_Γ.

Compact Operator 1.7 : An operator A on a Γ-Hilbert space H_Γ is called a compact operator or continous operator if, for every bounded sequence (x_n) in H_Γ, the sequence (Ax_n) contains a convergent subsequence.

Orthonormal sequence 1.8 : A sequence (x_n) in H_Γ is said to be orthonormal sequence whose terms form an orthonormal set i.e it follows two conditions :

a) $\langle x, \gamma, y\rangle = 0 \ \forall x, y \in H_\Gamma$ and $\gamma \in \Gamma$.

b)) $\langle x, \gamma, x\rangle = 1 \ \forall x, y \in H_\Gamma$ and $\gamma \in \Gamma$.

Eigenvalue 1.9 : Let A be an operator on a complex vector space E. A complex number λ is called an eigenvalue of A if there is a nonzero vector $u \in E$ such that

$$Au = \lambda u .$$

Eigen Spectrum 1.10: The set of all Eigen values of A is called Eigen spectrum of A. It is denoted by $\sigma_{eig}(A)$ that is $\sigma_{eig}(A)$ of A consists of all k in **K** such that $A - kI$ is not injective.Thus $k \in \sigma_{eig}(A)$ if and only if there is some nonzero x in X such that $A(x) = kx$.

2 RESULTS AND DISCUSSION

At first we reconstruct the Hilbert-Schmidt theorem for Γ-Hilbert space H_Γ . Let H_Γ be a finite dimensional space. It is known from linear algebra that eigenvectors of a self-adjoint operator H_Γ form an orthogonal basis of H_Γ. The following theorems generalized this result to infinite dimensional spaces.

Theorem 2.1 : For every compact, self-adjoint operator A on an infinite dimensional Γ-Hilbert space H_Γ, there exists an orthonormal system of eigenvectors (u_n) corresponding to nonzero eigen value (λ_n) such that every element $x \in H_\Gamma$ has a unique representation in the form

$$x = \sum_{n=1}^{\infty} \alpha_n u_n + v ,$$

Where $\alpha_n \in \mathbb{C}$ and $v \in N(A)$.

Proof : We have known that there exist an eigen value λ_1 of A such that –

$$|\lambda_1| = \sup_{\|x\|\leq 1} |\langle Ax, \gamma, x\rangle| \text{ where } x \in H_\Gamma \text{ and } \gamma \in \Gamma .$$

Let u_1 be a normalized eigenvector correspondingf to λ_1 . We take-

$$Q_1 = \{x \in H_\Gamma : \ x \perp_\gamma \ u_1\} ,$$

Q_1 is the orthogonal complement of the set $\{u_1\}$. Thus, Q_1 is a closed linear subspace of H_Γ . If $x \in Q_1$, then we have-

$$\begin{aligned}\langle Ax, \gamma, u_1\rangle &= \langle x, \gamma, Au_1\rangle \\ &= \lambda_1\langle x, \gamma, u_1\rangle \\ &= 0 .\end{aligned}$$

Which means that $x \in Q_1$ implies $Ax \in Q_1$. So, A maps the Γ-Hilbert space Q_1 into itself. Again we apply the previous proceding , it gives an eigenvalue λ_2 such that

$$|\lambda_2| = \sup_{\|x\|\leq 1}\{|\langle Ax, \gamma, x\rangle| : x \in Q_1\}.$$

Let u_2 be the normalized eigenvector of λ_2 . Then clearly $u_1 \perp_\gamma u_2$. Next we set

$$Q_2 = \{x \in Q_1 : \ x \perp_\gamma \ u_2\} ,$$

and repeat the above argument . Having eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and the corresponding normalized eigenvectors $u_1, u_2, \dots, u_n$, we define

$$Q_n = \{x \in Q_{n-1} : \ x \perp_\gamma \ u_n\}$$

Also choose an eigenvalue λ_{n+1} such that

$$|\lambda_{n+1}| = \sup_{\|x\|\leq 1}\{ |\langle Ax, \gamma, x\rangle| : x \in Q_n\} . \quad \text{.....(i)}$$

For u_{n+1} we choose a normalized vector corresponding to λ_{n+1}.

Indeed , it can happen that there is a positive integer k such that $\langle Ax, \gamma, x\rangle = 0$ for every $x \in Q_k$ and $\gamma \in \Gamma$. Then every element x of H_Γ has a unique representation

$$x = \alpha_1 u_1 + \dots + \alpha_k u_k + v ,$$

Where $Av = 0$ and $Ax = \lambda_1\alpha_1 u_1 + \cdots + \lambda_k \alpha_k u_k$

Which proves the theorem in this case.

Now suppose that the described procedure yields an infinite sequence of eigenvalues (λ_n) and eigenvectors (u_n) . Let S be the closed space spanned by the vectors that is ,

$$S = \{ \textstyle\sum_{n=1}^{\infty} \alpha_n u_n : \sum_{n=1}^{\infty} |\alpha_n|^2 < \infty \} .$$

For every $x \in H_\Gamma$ has a unique decomposition $x = u + v$ or

$$x = \sum_{n=1}^{\infty} \alpha_n u_n + v,$$

Where $v \in s^{\perp\gamma}$. It remains to prove that $Av = 0$ for all $v \in s^{\perp\gamma}$.

Let $v \in s^{\perp\gamma}$, $v \neq 0$. Define $w = {}^{v}/_{\|v\|}$. Then

$$\langle Av, \gamma, v\rangle = \|v\|^2 \langle Aw, \gamma, w\rangle .$$

Since $w \in s^{\perp\gamma} \subset Q_n$ for $n \in \mathbb{N}$, by (i) we have-

$$\begin{aligned}\langle Av, \gamma, v\rangle &= \|v\|^2 \langle Aw, \gamma, w\rangle \\ &= \|v\|^2 \sup_{\|x\|\leq 1}\{ |\langle Ax, \gamma, x\rangle| : x \in Q_n\} \\ &= \|v\|^2 |\lambda_{n+1}| \to 0 .\end{aligned}$$

This implies $\langle Av, \gamma, v\rangle = 0$ for every $v \in s^{\perp\gamma}$ and $\gamma \in \Gamma$.

Therefore , $Av = 0$ for all $v \in s^{\perp\gamma}$.

Hilbert-Schimdt Operator 2.2 : A bounded operator A on H_Γ is called Hilbert-Schmidt Operator if $\sum_j \|A(v_j)\|^2_\gamma < \infty$ for some orthonormal basis $\{v_j\}$ for H_Γ and $\gamma \in \Gamma$.

Theorem 2.3 : Let $A \in BL(H_\Gamma)$ be a Hilbert-Schmidt operator. Then

(a) A is compact.

(b) A^* is a Hilbert-Schmidt operator on H_Γ.

Proof : Let $\{v_j\}$ be an orthonormal basis for H_Γ such that $\sum_j \|A(v_j)\|^2_\gamma < \infty$.

(a) For $x \in H_\Gamma$ and $\gamma \in \Gamma$, we consider its Fourier expansion

$$x = \sum_j \langle x, \gamma, v_j\rangle v_j .$$

Since A is continuous and linear , we have

$$A(x) = \sum_j \langle x, \gamma, v_j\rangle A(v_j) .$$

For $n = 1,2, \ldots,$ define

$$A_n(x) = \sum_{j=1}^{n} \sum_j \langle x, \gamma, v_j\rangle A(v_j) \ , x \in H_\Gamma \text{ and } \gamma \in \Gamma.$$

As $A_n(x) \in Span\{A(v_1), \ldots, A(v_n)\}$ for all $x \in H_\Gamma$, range of A_n is finite dimensional. It follows that A_n is compact. Now for all $x \in H_\Gamma$ and $\gamma \in \Gamma$, we have-

$$\begin{aligned}\|A(x) - A_n(x)\|_\gamma{}^2 &= \left\|\sum_{j>n} \langle x, \gamma, v_j\rangle A(v_j)\right\|^2 \\ &\leq \left(\sum_{j>n} |\langle x, \gamma, v_j\rangle| \|\gamma\| \, \|A(v_j)\|\right)^2 \\ &\leq \left(\sum_{j>n} |\langle x, \gamma, v_j\rangle|^2\right) \left(\sum_{j>n} \|A(v_j)\|^2\right) \\ &\leq \|x\|^2 \sum_{j>n} \|A(v_j)\|^2 \text{ [By Bessel's inequality]}\end{aligned}$$

Since $\sum_{j>n} \|A(v_j)\|_\gamma{}^2 \to 0$ as $n \to \infty$, we see that $\|A - A_n\|_\gamma \to 0$.

Hence A is compact .

(b) By Parseval's formula , we have

$$\sum_i \left\|A^*(v_j)\right\|_\gamma^{\ 2} = \sum_i \sum_j \left|\langle A^*(v_j), \gamma, v_j\rangle\right|^2 \text{ for all } \gamma \in \Gamma .$$
$$= \sum_j \sum_i \langle \left|v_i, \gamma, A(v_j)\right|^2 \rangle$$
$$= \sum_j \left\|A^*(v_j)\right\|_\gamma^{\ 2} < \infty .$$

Hence A^* IS A Hilbert-Schmidt operator on H_Γ .

Theorem 2.4 : (Spectral theorem for compact self-adjoint operators) Let A be be a compact, self-adjoint operator on an infinite-dimensional Γ-Hilbert space H_Γ . Then H_Γ has a complete orthonormal system (an orthonormal basis) $\{v_1, v_{2,\dots,} v_n\}$ consisting of eigenvectors of A. Moreover, for every $x \in H_\Gamma$,

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \gamma, v_n\rangle v_n ,$$

where λ_n is the eigenvalue corresponding to v_n and $\gamma \in \Gamma$.

Proof: To obtain a complete orthonormal system $\{v_1, v_{2,\dots,} v_n\}$, we need to complement the system $\{u_1, u_{2,\dots,} u_n\}$, defined in the proof of Theorem 2.1 , with an arbitrary orthonormal basis of $N(A)$. The eigenvalues corresponding to the vectors that form $N(A)$ are all equal to zero. Equality $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \gamma, v_n\rangle v_n$ follows from the continuity of A.

Corollary 2.5 : Let A be a compact self-adjoint operator on a Γ-Hilbert Space H_Γ. Then

$$A = \sum_{n=1}^{\infty} \lambda_n P_n$$

Where P_n is a projection on finite –dimensional of H_Γ.

Proof : Let $\{v_1, v_{2,\dots,}\}$ be a complete orthonormal system of eigenvectors of A corresponding to eigenvalues $\{\lambda_1, \lambda_{2,\dots,}\}$. Let P_n be the projection operator onto the one-dimensional space spanned by v_n. Since $P_n x = \langle x, \gamma, v_n\rangle v_n$, the equation $Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \gamma, v_n\rangle v_n$ can be written as $Ax = \sum_{n=1}^{\infty} \lambda_n P_n x$.

Here is another way of representing A in the form $A = \sum_{n=1}^{\infty} \lambda_n P_n$. Let $\{\lambda_1, \lambda_{2,\dots,}\}$ be all distinct nonzero eigenvalues of A and let P_n be the projection onto the eigenspace corresponding to λ_n. the by the theorem "The eigenspaces corresponding to nonzero eigenvalues of a compact self-adjoint operator are finite-dimensional", those eigenspaces are finite- dimensional . □

Above corollary is just another version of the Spectral theorem. This version is important because it has natural extensions to more general classes of operators. It is also useful as it leads to an elegant expression for powers and other functions of an operator.

Let $A, \lambda_n,$ and P_n be as mentioned in the above corollary. Then

$$A^2 = A(\sum_{n=1}^{\infty} \lambda_n P_n)$$
$$= \sum_{n=1}^{\infty} \lambda_n A P_n$$
$$= \sum_{n=1}^{\infty} \lambda_n^{\ 2} P_n ,$$

because $A P_n x = \lambda_n P_n x$ for every $x \in H_\Gamma$. Similarly , for any $k \in \mathbb{N}$, we get

$$A^k = \sum_{n=1}^{\infty} \lambda_n^{\ k} P_n ,$$

and hence for any polynomial $p(t) = \alpha_n t^n + ... + \alpha_1 t$, we have

$$p(A) = \sum_{n=1}^{\infty} p(\lambda_n) P_n \ .$$

The constant term in p must be zero, because otherwise the sequence $(p(\lambda_n))$ would not converge to zero.To deal polynomials with a nonzero constant term α_0, we add $\alpha_0 I$. In such case , $p(A)$ is not a compact operator.

Function of a operator 2.6 : Let f be a real –valued function $\mathbb{R}$ such that $f(\lambda) \to 0$ as $\lambda \to 0$

and $f(0) = 0$.

For a compact, self-adjoint operator A of H_Γ , $A = \sum_{n=1}^{\infty} \lambda_n P_n$, We define

$$f(A) = \sum_{n=1}^{\infty} f(\lambda_n) P_n \ .$$

Example 2.6.1 : Let $A = \sum_{n=1}^{\infty} \lambda_n P_n$ be a compact ,self-adjoint operator of a Γ-Hilbert Space H_Γ.
We can define sine of A by

$$SinA = \sum_{n=1}^{\infty} (Sin\lambda_n) P_n.$$

If $A = \sum_{n=1}^{\infty} \lambda_n P_n$ and $P_n x = \langle x, \gamma, v_n \rangle v_n$, then for any $x \in H_\Gamma$ and $\gamma \in \Gamma$, we have-

$$(f(A))x = \sum_{n=1}^{\infty} f(\lambda_n) \langle x, \gamma, v_n \rangle \, v_n,$$

Where the convergence of the series is justified, because

$$|f(\lambda_n) \langle x, \gamma, v_n \rangle|^2 \leq M |\langle x, \gamma, v_n \rangle|^2$$

for some constant M, and hence $(f(\lambda_n) \langle x, \gamma, \lambda_n \rangle \in l^2$.

Clearly, in this case, we cannot expect $f(A)$ to be a compact operator.

Theorem 2.6.2 : If eigenvectors $u_1, u_2, ...$ of a self-adjoint operator T on a Γ-Hilbert Space H_Γ form a complete γ-orthonormal system in H_Γ and all eigenvalues are positive (or non-negetive), then T is strictly positive (or positive).

Proof : Suppose $u_1, u_2, ...$ is a complete γ-orthonormal system of eigenvalues of T corresponding to eigenvalues $\lambda_1, \lambda_2, ...$. Then any non zero vector $u \in H_\Gamma$ can be represented as $= \sum_{n=1}^{\infty} \alpha_n u_n$, and we have

$$\begin{aligned}
\langle Tu, \gamma, u \rangle &= \langle Tu, \gamma, \textstyle\sum_{n=1}^{\infty} \alpha_n u_n \rangle \\
&= \textstyle\sum_{n=1}^{\infty} \overline{\alpha_n} \ \langle Tu, \gamma, u_n \rangle \\
&= \textstyle\sum_{n=1}^{\infty} \overline{\alpha_n} \ \langle u, \gamma, Tu_n \rangle \\
&= \textstyle\sum_{n=1}^{\infty} \overline{\alpha_n} \ \langle u, \gamma, \lambda_n u_n \rangle \\
&= \textstyle\sum_{n=1}^{\infty} \lambda_n \overline{\alpha_n} \ \langle u, \gamma, u_n \rangle \\
&= \textstyle\sum_{n=1}^{\infty} \lambda_n \overline{\alpha_n} \ \alpha_n \\
&= \textstyle\sum_{n=1}^{\infty} \lambda_n |\alpha_n|_\gamma^{\ 2} \geq 0
\end{aligned}$$

Where $\gamma \in \Gamma$ and all eigenvalues are non-negative. If all λ_n's are positive ,then the last inequality becomes strict.

Theorem 2.6.3 : For any two commuting , compact, self-adjoint operators A and B on a Γ -Hilbert Space H_Γ, there exists a complete orthonormal system in H_Γ of common eigenvectors of A and B.

3 CONCLUSION

we basically discussed the spectral representation of compact self adjoint operators. We tried to show that Compact Self-adjoint operators Satisfies the Spectral theorem and we show the extensions of the theorem by generalized the function of operator in Γ-Hilbert Space.

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