ON THE EXPONENTIAL DIOPHANTINE EQUATION $439^p + 457^q = r^2$

[1]Sanjay Kumar, [2]Sudhanshu Aggarwal
[1]*Associate Professor, Department of Mathematics, M. S. College, Saharanpur-247001, U.P., India*
Email: sanjay70gupta@yahoo.co.in
[2]*Assistant Professor, Department of Mathematics, National Post Graduate College, Barhalganj, Gorakhpur-273402, U.P., India*
Email:sudhanshu30187@gmail.com

Abstract: Diophantine equation is the subject undergoing intense study by the researchers in number theory as these equations help in solving various advance puzzle problems. In this article, authors discussed the existence of solution of exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers. Results show that the exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers, has no solution in whole number.

Keywords: Prime number; Diophantine equation; Solution, Integers; Number theory.

Mathematics Subject Classification: 11D61, 11D72, 11D45.

Introduction: Diophantine equations are those equations which are to be solved in integers. Diophantine equations are very important equations of theory of numbers and have many important applications in algebra, analytical geometry and trigonometry. Aggarwal et al. [1] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution. Aggarwal et al. [2] discussed the existence of solution of Diophantine equation $181^x + 199^y = z^2$. Bhatnagar and Aggarwal [3] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in whole number.

Gupta and Kumar [4] gave the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$. Kumar et al. [5] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [6] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [7] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution.

Mishra et al. [8] studied the existence of solution of Diophantine equation $211^{\alpha} + 229^{\beta} = \gamma^2$ and proved that the Diophantine equation $211^{\alpha} + 229^{\beta} = \gamma^2$ has no solution in whole number. Diophantine equations help us for finding the integer solution of famous Pythagoras theorem and Pell's equation [9-10]. The Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [11, 14]. He proved that these equations have a unique solution which is given by $\{x = 1, y = 0, z = 3\}$. Sroysang [12] proved that the exponential Diophantine equation $31^x + 32^y = z^2$ has no positive integer solution. Sroysang [13] discussed the Diophantine equation $3^x + 5^y = z^2$.

Goel et al. [15] discussed the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [16] proved that the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no solution in whole number. The exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ has studied by Kumar et al. [17]. Aggarwal and Sharma [18] studied the non-linear Diophantine equation$379^x + 397^y = z^2$ and proved that this equation has no solution in whole number.

The main aim of this article is to discuss the existence of solution of exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers.

Preliminaries:

Lemma: 1 *The exponential Diophantine equation* $439^p + 1 = r^2$, *where* p, r *are whole numbers, has no solution in whole number.*

Proof: Since 439 is an odd prime so 439^p is an odd number for all whole number p.

$\Rightarrow 439^p + 1 = r^2$ is an even number for all whole number p.

$\Rightarrow r$ is an even number.

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$ (1)

Now, $439 \equiv 1(mod3)$

$\Rightarrow 439^p \equiv 1(mod3)$, for all whole number p

$\Rightarrow 439^p + 1 \equiv 2(mod3)$, for all whole number p

$\Rightarrow r^2 \equiv 2(mod3)$ (2)

Equation (2) contradicts equation (1). Hence exponential Diophantine equation $439^p + 1 = r^2$, where p, r are whole numbers, has no solution in whole number.

Lemma: 2 *The exponential Diophantine equation* $457^q + 1 = r^2$, *where* q, r *are whole numbers, has no solution in whole number.*

Proof: Since 457 is an odd prime so 457^q is an odd number for all whole number q.

$\Rightarrow 457^q + 1 = r^2$ is an even number for all whole number q

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)$ or $r^2 \equiv 1(mod3)$ (3)

Now, $457 \equiv 1(mod3)$

$\Rightarrow 457^q \equiv 1(mod3)$, for all whole number q

$\Rightarrow 457^q + 1 \equiv 2(mod3)$, for all whole number q

$\Rightarrow r^2 \equiv 2(mod3)$ (4)

Equation (4) contradicts equation (3). Hence exponential Diophantine equation $457^q + 1 = r^2$, where q, r are whole numbers, has no solution in whole number.

Main Theorem: The exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers, has no solution in whole number.

Proof: There are four cases:

Case: 1 If $p = 0$ then the exponential Diophantine equation $439^p + 457^q = r^2$ becomes

$1 + 457^q = r^2$, which has no solution in whole number by lemma 2.

Case: 2 If $q = 0$ then the exponential Diophantine equation $439^p + 457^q = r^2$ becomes$439^p + 1 = r^2$, which has no solution in whole number by lemma 1.

Case: 3 If p, q are positive integers, then $439^p, 457^q$ are odd numbers.

$\Rightarrow 439^p + 457^q = r^2$ is an even number

$\Rightarrow r$ is an even number

$\Rightarrow r^2 \equiv 0(mod3)\ or\ r^2 \equiv 1(mod3)$ (5)

Now, $439 \equiv 1(mod3)$

$\Rightarrow 439^p \equiv 1(mod3)\ and\ 457 \equiv 1(mod3)$

$\Rightarrow 439^p \equiv 1(mod3)\ and\ 457^q \equiv 1(mod3)$

$\Rightarrow 439^p + 457^q \equiv 2(mod3)$

$\Rightarrow r^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5). Hence exponential Diophantine equation $439^p + 457^q = r^2$ has no solution in whole number.

Case: 4 If $p, q = 0$, then $439^p + 457^q = 1 + 1 = 2 = r^2$, which is impossible because r is a whole number. Hence exponential Diophantine equation $439^p + 457^q = r^2$ has no solution in whole number.

Conclusion: In this article, authors successfully discussed the solution of exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers and determined that the exponential Diophantine equation $439^p + 457^q = r^2$, where p, q, r are whole numbers, has no solution in whole number.

Conflict of Interests: Authors state that this paper has no conflict of interest.

References:

1. Aggarwal, S., Sharma, S.D. and Singhal, H. (2020) On the Diophantine equation $223^x + 241^y = z^2$, International Journal of Research and Innovation in Applied Science, 5 (8), 155-156.
2. Aggarwal, S., Sharma, S.D. and Vyas, A. (2020) On the existence of solution of Diophantine equation $181^x + 199^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science, 9 (8), 85-86.
3. Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $421^p + 439^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 128-129.
4. Gupta, D. and Kumar, S. (2020) On the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, International Journal of Interdisciplinary Global Studies, 14(4), 74-77.
5. Kumar, A., Chaudhary, L. and Aggarwal, S. (2020) On the exponential Diophantine equation $601^p + 619^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 29-30.

6. Kumar, S., Gupta, S. and Kishan, H. (2018) On the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$, Annals of Pure and Applied Mathematics, 18(1), 91-94.
7. Kumar, S., Gupta, D. and Kishan, H. (2018) On the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, Annals of Pure and Applied Mathematics, 18(2), 185-188.
8. Mishra, R., Aggarwal, S. And Kumar, A. (2020) On the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.
9. Mordell, L.J. (1969) Diophantine equations, Academic Press, London, New York.
10. Sierpinski, W. (1988) Elementary theory of numbers, 2nd edition, North-Holland, Amsterdam.
11. Sroysang, B. (2012) More on the Diophantine equation $8^x + 19^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 601-604.
12. Sroysang, B. (2012) On the Diophantine equation $31^x + 32^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 609-612.
13. Sroysang, B. (2012) On the Diophantine equation $3^x + 5^y = z^2$, International Journal of Pure and Applied Mathematics, 81(4), 605-608.
14. Sroysang, B. (2014) On the Diophantine equation $8^x + 13^y = z^2$, International Journal of Pure and Applied Mathematics, 90(1), 69-72.
15. Goel, P., Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 170-171.
16. Kumar, S., Bhatnagar, K., Kumar, A. and Aggarwal, S. (2020) On the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$, International Journal of Interdisciplinary Global Studies, 14(4), 183-184.
17. Kumar, S., Bhatnagar, K., Kumar, N. and Aggarwal, S. (2020) On the exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$, International Journal of Interdisciplinary Global Studies, 14(4), 181-182.
18. Aggarwal, S. and Sharma, N. (2020) On the non-linear Diophantine equation$379^x + 397^y = z^2$, Open Journal of Mathematical Sciences, 4(1), 397-399. DOI: 10.30538/oms2020.0129