ABSOLUTELY ALMOST SUMMABLE DIFFERENCE SEQUENCES OF ORDER r WITH RESPECT TO A MODULUS FUNCTION IN SEMINORMED SPACE

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Abstract. The purpose of this paper is to introduce and study a new sequence space that is Δ^r-absolutely almost summable with respect to a modulus function in seminormed complex linear space. Some topological results and certain inclusion relations on this space have been discussed. Furthermore, we construct the sequence space that is Δ^r-absolutely almost summable with respect to composite modulus function in seminormed complex linear space and give some inclusion relations on this space.

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1. Introduction

By w we shall denote the space of all scalar sequences. l_∞ and c, respectively, denote the Banach spaces of bounded and convergent sequences x = (x_k) with complex terms normed by $||x||_\infty$ = sup_k $|x_k|$. We write D for the shift operator; that is D((x_k)) = (x_{k+1}). It may be recalled that a Banach limit L (see Banach [1]) is a nonnegative linear functional on l_∞ such that L is invariant under the shift operator (that is, L(Dx) = L(x) for all x ∈ l_∞) and such that L(e) = 1, where

e = (1, 1, ...). Various types of limits, including Banach limit, are considered in Das [2]. Let B be the set of all Banach limits on l_∞. A sequence x ∈ l_∞ is said to be almost convergent to the value l (see Lorentz [7]) if L(x) = I for all L ∈ B. Let ĉ denote the space of all almost convergent sequences. For any sequence x, write

$$t_{mn} = t_{mn}(x) = (m+1)^{-1} \sum_{j=0}^{m} x_{j+n.}$$

Lorentz [7] proved that x ∈ ĉ if and only if t_{mn} (x) tends to a limit as m → ∞ uniformly in n.

We now extend the definition of t_{mn} (x) to m = -1 by taking $t_{-1,n}$ = $t_{-1,n}$(x) = 0.

We write, for m, n ≥ 0

$$\phi_{mn} = \phi_{mn}(\mathrm{x}) = t_{mn} - t_{m-1,n}.$$

A straightforward calculation shows that

$$\phi_{0n} = x_n;$$

$$\phi_{mn} = \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_{j+n} - x_{j+n-1}) \qquad (m \geq 1).$$

Note that for any sequences x, y and scalar λ , we have

ϕ_{mn} (x + y) = ϕ_{mn}(x) + ϕ_{mn}(y) and $\phi_{mn}(\lambda$x) = $\lambda\phi_{mn}$(x).

The sequence x is absolutely almost convergent (see Das et al. [3]) if $\sum_m |\phi_{mn}|$ converges uniformly in n. We denote the set of absolutely almost convergent sequences by $\hat{l}$.

The idea of modulus was structured in 1953 by Nakano [11]. Following Ruckle [13] and Maddox [9] we recall that a modulus f is a function from [0, ∞) to [0, ∞) such that

(i) f(x) = 0 if and only if x = 0,

(ii) f(x + y) ≤ f(x) + f(y) for all x ≥ 0, y ≥ 0,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

Because of (ii), |f(x) - f(y)| ≤ f(|x – y|) so that in view of (iv), f is continuous everywhere on [0, ∞). A modulus may be unbounded (for example, f(x) = x^p, 0 < p ≤ 1) or bounded (for example, f(x) = $\frac{x}{x+1}$).

It is easy to see that $f_1 + f_2$ is a modulus function when f_1 and f_2 are modulus functions, and that the function f^v (v is a positive integer), the composition of a modulus function f with itself v times, is also a modulus function.

Ruckle [13] used the idea of a modulus function f to construct a class of FK spaces

L(f) = $\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}$.

The space L(f) is closely related to the space l_1 which is an L(f) space with f(x) = x for all real x ≥ 0.

The notion of difference sequence spaces was introduced by Kizmaz [6]. It was generalized by Et and Colak [4] as follows:

Let m be a non-negative integer. Then

X(Δ^m) = {x = (x_k) : ($\Delta^m x_k$) ∈ X }

for X = l_∞, c, c_0; where Δ^0x = (x_k) and Δ^mx = ($\Delta^m x_k$) = ($\Delta^{m-1}x_k$ - $\Delta^{m-1}x_{k+1}$) for all k ∈ ℕ. The sequence spaces X(Δ^m) are BK spaces normed by $||x||_{\Delta} = \sum_{i=1}^{m} |x_i| + ||\Delta^m x||_\infty$,

X ∈ { l_∞, c, c_0}. Et and Nuray [5] defined a more general space Δ^m(X) = { x =(x_k) : ($\Delta^m x_k$) ∈ X } , where m ∈ ℕ and X is any sequence space.

Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if there exists a constant L such that q_2(x) ≤ Lq_1(x) for all $x \in$ X. If each is stronger than the other, q_1 and q_2 are said to be equivalent (Wilansky [14]).

Let X be a seminormed complex linear space with seminorm q, f be a modulus function, s ≥ 0 be a real number and p = (p_m) be a bounded sequence of strictly positive real numbers. The symbol w(X) denotes the space of all X-valued sequences.

We now introduce the following generalized difference absolutely almost summable X-valued sequence space with respect to a modulus function.

$$\hat{l}\,(\Delta^r, f, p, q, s) = \left\{x \in w(X): \sum_{m=1}^{\infty} m^{-s} \left[f\left(q\left(\Delta^r \phi_{mn}(x)\right)\right)\right]^{p_m} < \infty \; uniformly \; in \; n\right\}.$$

where $\Delta^r \phi_{mn}$(x) = $\Delta^{r-1}\phi_{mn}$ (x) - $\Delta^{r-1}\phi_{m+1,n}(x)$.

If we take X = ℂ, q(x) = |x|, f(x) = x and r = s = 0, then the sequence space defined above becomes $\hat{l}$(p) (see Das et al. [3]).

We denote $\hat{l}$ (Δ^r, f, p, q, s) by $\hat{l}$ (Δ^r, p, q, s) when f(x) = x and by $\hat{l}$ (Δ^r, f, p, q) when s = 0.

The following inequalities (see, e.g., [8, p. 190]) are needed throughout the paper.

Let p = (p_m) be a bounded sequence of strictly positive real numbers. If H = $sup_m p_m$, then for any complex x_m and y_m ,

$|x_m + y_m|^{p_m} \le$ C $(|x_m|^{p_m} + |(y_m)|^{p_m})$, (1)
where C = max(1,2^{H-1}). Also for any complex λ,

$|\lambda|^{p_m} \le$ max(1, $|\lambda|^H$). (2)

2. Main results

In this section we will prove the general results of this paper on the sequence space $\hat{l}\,(\Delta^r, f, p, q, s)$, those characterize the structure of this space.

Theorem 2.1. For any modulus f, $\hat{l}\,(\Delta^r, f, p, q, s)$ is a linear space over the complex field ℂ.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. $\hat{l}\,(\Delta^r, f, p, q, s)$ is a topological linear space, paranormed by

g_Δ(x) = $sup_n(\sum_{m=1}^{\infty} m^{-s} \, [f(q(\Delta^r \phi_{mn}(x)))]^{p_m})^{1/G}$

where G = max(1, $sup_m \, p_m$).

The proof follows by standard arguments and the fact that every paranormed space is a topological linear space [15, p. 37].

Remark 2.3. g_Δ need not be total, e.g., if x = (x_m) is defined by x_m = m then ϕ_{mn}(x) is constant for all m and hence g_Δ(x) is zero for r ≥ 1.

Lemma 2.4[12]. Let f be a modulus function and let 0 < δ < 1. Then for each x > δ we have f(x) ≤ 2f(1)δ^{-1}x.

Theorem 2.5. Let f, f_1, f_2 be modulus functions, then

(i) If s > 1, then $\hat{l}\,(\Delta^r, f, p, q, s) \subseteq \hat{l}\,(\Delta^r, fof_1, p, q, s)$,

(ii) Î($\hat{l}\,(\Delta^r, f_1, p, q, s) \cap \hat{l}\,(\Delta^r, f_2, p, q, s) \subseteq \hat{l}\,(\Delta^r, f_1 + f_2, p, q, s)$,

(iii) If s > 1 and lim $sup_{t\to\infty} \frac{f_1(t)}{f_2(t)} < \infty$, then $\hat{l}\,(\Delta^r, f_2, p, q, s) \subseteq \hat{l}\,(\Delta^r, f_1, p, q, s)$.

Proof. Let x ∈ $\hat{l}\,(\Delta^r, f_1, p, q, s)$. Let $\epsilon > 0$ and choose δ with 0 < δ < 1 such that f(t) < ϵ for 0 ≤ t ≤ δ. Write y_{mn} = f_1(q($\Delta^r \phi_{mn}(x)$)) and consider

$\sum_{m=1}^{\infty} m^{-s} [f(y_{mn})]^{p_m} = \sum_{y_{mn} \le \delta} m^{-s} [f(y_{mn})]^{p_m} + \sum_{y_{mn} > \delta} m^{-s} [f(y_{mn})]^{p_m}$

$< \max(1,\epsilon^H) \sum_{m=1}^{\infty} m^{-s} + \max(1, (2f(1)\,\delta^{-1})^H) \sum_{m=1}^{\infty} m^{-s} [y_{mn}]^{p_m}$

$< \infty$, uniformly in n,

by inequality (2) and Lemma 2.4 and hence $x \in \hat{l}(\Delta^r, f o f_1, p, q, s)$.

(ii) The proof follows trivially by using (1).

(iii) Let $x \in \hat{l}(\Delta^r, f_2, p, q, s)$ and $\lim sup_{t\to\infty} \frac{f_1(t)}{f_2(t)}$ =b <∞. Then for a given $\epsilon > 0$

there is a positive integer N such that for all t with t > N we have f_1(t) < (b+ϵ)f_2(t).

Let $\mathcal{Y}_{mn}$ = q($\Delta^r\ \phi_{mn}(x)$), then $\sum_{m=1}^{\infty} m^{-s}[f_1(\mathcal{Y}_{mn})]^{p_m} = \sum_1 + \sum_2$, where the

first summation is over $\mathcal{Y}_{mn} \le$ N and the second over $\mathcal{Y}_{mn} > N$. Then using (2)

$\sum_1 m^{-s}[f_1(\mathcal{Y}_{mn})]^{p_m} \le [N f_1(1)]^H \sum_{m=1}^{\infty} m^{-s}$

and

$\sum_2 m^{-s}[f_1(\mathcal{Y}_{mn})]^{p_m} \le \max(1, (b+\epsilon)^H) \sum_{m=1}^{\infty} m^{-s} [f_2(\mathcal{Y}_{mn})]^{p_m}$

and so $x \in \hat{l}(\Delta^r, f_1, p, q, s)$.

Proposition 2.6. For any modulus f and s >1, $\hat{l}(\Delta^r, p, q, s) \subseteq \hat{l}(\Delta^r, f, p, q, s)$.

The proof follows by taking $f_1(x) = x$ in Theorem 2.5(i).

Maddox [10, Prop. 1] proved that for any modulus f there exits $lim_{t\to\infty} \frac{f(t)}{t}$.

Using this result we give a sufficient condition for the inclusion $\hat{l}(\Delta^r, f, p, q, s) \subseteq \hat{l}(\Delta^r, p, q, s)$.

Theorem 2.7. For any modulus $f, if\ lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$ then $\hat{l}(\Delta^r, f, p, q, s) \subseteq \hat{l}(\Delta^r, p, q, s)$.

Proof. Following the proof of proposition 1 of Maddox [10], we have $\beta = lim_{t\to\infty} \frac{f(t)}{t} = \inf\{\frac{f(t)}{t} : t > 0\}$, so that $0 \le \beta \le f(1)$. Let $\beta > 0$. By definition of β we have $\beta t \le f(t)\ for\ all\ t \ge 0$. Since $\beta > 0$ we have $t \le \beta^{-1} f(t)$ for all $t \ge 0$. Now $x \in \hat{l}(\Delta^r, f, p, q, s)$ implies.

$\sum_{m=1}^{\infty} m^{-s} [q(\Delta^r \phi_{mn}(x))]^{p_m} \le \max(1, \beta^{-H}) \sum_{m=1}^{\infty} m^{-s} [f(q(\Delta^r \phi_{mn}(x)))]^{p_m}$

by (2), whence $x \in \hat{l}(\Delta^r, p, q, s)$ and the proof is complete.

Theorem 2.8. Let f be a modulus function, q, q_1, q_2 be seminorms and s, s_1, s_2 be non-negative real numbers . Then

i. $\hat{l}(\Delta^r, f, p, q_1, s) \cap \hat{l}(\Delta^r, f, p, q_2, s) \subseteq \hat{l}(\Delta^r, f, p, q_1 + q_2, s)$,
ii. $If\ q_1\ is\ stronger\ than\ q_2,\ then\ \hat{l}(\Delta^r, f, p, q_1, s) \subseteq \hat{l}(\Delta^r, f, p, q_2, s)$,
iii. $If\ q_1\ is\ equivalent\ to\ q_2,\ then\ \hat{l}(\Delta^r, f, p, q_1, s) = \hat{l}(\Delta^r, f, p, q_2, s)$,
iv. If $s_1 \le s_2$, then $\hat{l}(\Delta^r, f, p, q, s_1) \subseteq \hat{l}(\Delta^r, f, p, q, s_2)$.

Proof. The proof of (i) is straight forward using (1).

(ii) Let $x \in \widehat{l}\,(\Delta^r, f, p, q_1, s)$. Then

$$\sum_{m=1}^{\infty} m^{-s}[f(q_2(\Delta^r \quad \phi_{mn}(x)))]^{p_m} \leq \sum_{m=1}^{\infty} m^{-s}[f(L\, q_1(\Delta^r \quad \phi_{mn}(x)))]^{p_m}$$
$$\leq (1+[L])^H \sum_{m=1}^{\infty} m^{-s}[f(q_1(\Delta^r \quad \phi_{mn}(x)))]^{p_m}$$

by (2), whence $\in \widehat{l}\,(\Delta^r, f, p, q_2, s)$.

The proofs of (iii) and (iv) are trivial.

Theorem 2.9. Let r ≥ 1, then $\hat{l}(\Delta^{r-1}, f, p, q) \subseteq \hat{l}(\Delta^r, f, p, q)$.

Proof Let $x \in \hat{l}\,(\Delta^{r-1}, f, p, q)$ then

$$\sum_{m=1}^{\infty} [f(q(\Delta^r \phi_{mn}(x)))]^{p_m}$$

$$\leq C\, \{ \sum_{m=1}^{\infty} [f(q(\Delta^{r-1} \phi_{mn}(x)))]^{p_m} + \sum_{m=1}^{\infty} [f(q(\Delta^{r-1} \phi_{m+1,n}(x)))]^{p_m}\},$$

where C = max(1 , 2^{H-1}), Hence $x \in \hat{l}(\Delta^r, f, p, q)$.

In general $\hat{l}(\Delta^i f, p, q) \subseteq \hat{l}(\Delta^r, f, p, q)$ for all i = 1, 2, . . . , r – 1 and the inclusion is strict.

To show that the inclusion is strict, consider the following example.

Example 2.10. Let X = C, q$(x) = |\, x|, f(x) = x \text{ and } p_m = 1$ for all m. Let $x = (x_m)$ be defined by $x_m = m^3$, then $x \notin \hat{l}(\Delta^2, f, p, q)$ but $x \in \hat{l}(\Delta^3, f, p, q)$.

Theorem 2.11. If p = (p_m) and $t = (t_m)$ are bounded sequences of positive real numbers with $0 < p_m \leq t_m < \infty$ for each m, then for any modulus f,

i) $\hat{l}\,(\Delta^r, f, p, q) \subseteq \hat{l}\,(\Delta^r, f, t, q)$,
ii) $\hat{l}(\Delta^r, f, p, q) \subseteq \hat{l}\,(\Delta^r, f, p, q, s)$.

Proof. Let $x \in \hat{l}(\Delta^r, f, p, q)$. This implies that

$$f(q(\Delta^r \phi_{in}(x))) \leq 1$$

for sufficiently large values of i, say i ≥ m_0 for some fixed $m_0 \in N$. Since f is increasing, we have

$$\sum_{m\geq m_0}^{\infty} [f(q(\Delta^r \phi_{mn}(x)))]^{t_m} \leq \sum_{m\geq m_0}^{\infty} [f(q(\Delta^r \phi_{mn}(x)))]^{p_m} < \infty.$$

This shows that $x \in \hat{l}(\Delta^r, f, t, q)$.

The proof of (ii) is trivial.

3. Composite space $\hat{l}(\Delta^r, f^v, p, q, s)$ using composite modulus function f^v

Taking modulus function f^v instead of f in the space $\hat{l}(\Delta^r, f, p, q, s)$, we can define the composite space $\hat{l}(\Delta^r, f^v, p, q, s)$ as follows:

Definition 3.1. For a fixed natural v, we define

$\hat{l}(\Delta^r, f^v, p, q, s)$ = { $x \in w$(X) : $\sum_{m=1}^{\infty} m^{-s} [f^v(q(\Delta^r \phi_{mn}(x)))]^{p_m} < \infty$ uniformly in n}.

Theorem 3.2. For any modulus function f and v ∈ ℕ,

(i) $\hat{l}(\Delta^r, f^v, p, q, s) \subseteq \hat{l}(\Delta^r, p, q, s)$ if $\lim_{t\to\infty} \frac{f(t)}{t} = \beta > 0$,

(ii) $\hat{l}(\Delta^r, p, q, s) \subseteq \hat{l}(\Delta^r, f^v, p, q, s)$ if there exists a positive constant α such that f(t) ≤ αt for all $t \geq 0$.

Proof . Let ∈ $\hat{l}(\Delta^r, f^v, p, q, s)$. Following the proof of Proposition 1 of Maddox [10], we have $\beta = \lim_{t\to\infty} \frac{f(t)}{t}$ = inf { $\frac{f(t)}{t}$: t > 0 }, so that $0 \leq \beta \leq$ f(1). Let $\beta > 0$.

By definition of β we have $\beta t \leq$ f(t) for all t ≥ 0. Since f is increasing we have β^2t $\leq f^2$(t). So by induction, we have β^vt ≤ f^v(t). Now using inequality (2),

$$\sum_{m=1}^{\infty} m^{-s} \left[q\left(\Delta^r \phi_{mn}(x)\right)\right]^{p_m} \leq \sum_{m=1}^{\infty} m^{-s} \left[\beta^{-v} f^v \left(q\left(\Delta^r \phi_{mn}(x)\right)\right)\right]^{p_m}$$
$$\leq \max(1, \beta^{-vH}) \sum_{m=1}^{\infty} m^{-s} [f^v(q(\Delta^r \phi_{mn}(x)))]^{p_m}.$$

Hence $x \in \hat{l}(\Delta^r, p, q, s)$.

(ii) Let ∈ $\hat{l}(\Delta^r, p, q, s)$. Since f(t) ≤ αt for all t ≥ 0 and f is increasing, we have f^v(t) $\leq \alpha^v t$ for each $v \in$ ℕ. Again, using (2), we have

$$\sum_{m=1}^{\infty} m^{-s} \left[f^v \left(q\left(\Delta^r \phi_{mn}(x)\right)\right)\right]^{p_m} \leq \max(1, \alpha^{vH}) \sum_{m=1}^{\infty} m^{-s} \left[q\left(\Delta^r \phi_{mn}(x)\right)\right]^{p_m}.$$

Hence $x \in \hat{l}(\Delta^r, f^v, p, q, s)$ and this completes the proof.

Example 3.3. f_1(t)= $t + t^{1/2}$ and f_2(t)= log(1+t) for all t ≥ 0 satisfy the conditions given in Theorem 3.2(i), (ii) respectively.

Theorem 3.4. Let i, $v \in$ ℕ and $i < v$. If f is a modulus such that f(t) ≤ αt for all t ≥ 0, where α is a positive constant, then

$\hat{l}(\Delta^r, p, q, s) \subseteq \hat{l}(\Delta^r, f^i, p, q, s) \subseteq \hat{l}(\Delta^r, f^v, p, q, s)$.

Proof. Let $j = v - i$. Since $f(t) \leq \alpha t$, we have f^v(t) < $M^j f^i(t)$ < $M^v t$, where$M = 1 + [\alpha]. Let\ x \in \hat{l}(\Delta^r, p, q, s)$. By the above inequality and using (2), we get

$$\sum_{m=1}^{\infty} m^{-s} \left[f^v(q\left(\Delta^r \phi_{mn}(x)\right))\right]^{p_m} < M^{jH} \sum_{m=1}^{\infty} m^{-s} \left[f^i(q\left(\Delta^r \phi_{mn}(x)\right))\right]^{p_m}$$
$$< M^{vH} \sum_{m=1}^{\infty} m^{-s} \left[q\left(\Delta^r \phi_{mn}(x)\right)\right]^{p_m}.$$

Hence the required inclusion follows.

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