RECIPROCITY THEOREM OF RAMANUJAN AND ITS APPLICATIONS

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ABSTRACT

In his lost notebook, Ramanujan has stated a beautiful two variable reciprocity theorem. Its three and four variable generalizations were recently, given by Kang. In this paper, we give new and an elegant approach to establish all the three reciprocity theorems via their finite forms. Also we give some applications of the finite forms of reciprocity theorems.

***Keywords** : Reciprocity theorem, Theta function, Function identities etc.*

Introduction :

In this section, we obtain the generalizations of using Reciprocity theorem. Further, we deduce an interesting identity which contains the Jacobi's identity as a special case. There from we obtain a number of identities involving theta functions which are analogous to the theta function identities found in Ramanujan's second notebook [99].

Definition 1.1

We define F(a,b,c) by

$$F(a,b,c) := \sum_{n=0}^{\infty} \frac{a^{n(n+1)/2} b^{n(n-1)/2}}{(c/b;ab)_{n+1}} + \sum_{n=1}^{\infty} \frac{a^{n(n-1)/2} b^{n(n+1)/2}}{(-c;ab)_n}, \qquad (1.1)$$

Where $|ab| < 1, c \neq a^n b^{n+1}, -a^n b^n; \quad n = 0,1,2,...,.$

Then, we have

$$F(a,b,0)=f(a,b),$$

Where (a, b) is the Ramanujan's theta function.

Theorem 1.2.

We have

$$f(a,b,c)=\frac{(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty}{(-c;ab)_\infty(c/b;ab)_\infty}.$$

Proof. Changing $q\ to\ q^2 and\ z\ to\ z/q$ and then setting $qz=a\ and\ q/z=b$ in the resulting identity we obtain (1.3)

Theorem 1.3

We have

$$\begin{aligned}&F\left(q^{(k+m)/2},q^{(k-m)/2},cq^{k/2}\right)\times\\&\left[x+p\sum_{n=1}^{\infty}\left(\frac{q^{kn-(k-m)/2}}{1+q^{kn-(k-m)/2}}-\frac{q^{kn-(k+m)/2}}{1+q^{kn-(k+m)/2}}+\frac{cq^{kn-k+m/2}}{1-cq^{kn-k+m/2}}\right)\right]\\&=\sum_{n=0}^{\infty}\frac{(pn+x)q^{\left(kn^2+mn\right)/2}}{\left(cq^{m/2;}q^k\right)_{n+1}}+\sum_{n=1}^{\infty}\frac{(-pn+x)q^{(kn^2-mn)/2}}{\left(-cq^{k/2};q^k\right)_n}\\&+\sum_{n=0}^{\infty}\frac{q^{\left(kn^2+mn\right)/2}}{\left(cq^{m/2};q^k\right)_{n+1}}\left(\sum_{r=0}^{n}\frac{pcq^{rk+m/2}}{\left(1-cq^{rk+m/2}\right)}\right)\qquad(1.2)\end{aligned}$$

and

$$F\left(-q^{(k+m)/2},-q^{(k-m)/2},cq^{k/2}\right)\times$$
$$\left[x-p\sum_{n=1}^{\infty}\left(\frac{q^{kn-(k-m)/2}}{1-q^{kn-(k-m)/2}}-\frac{q^{kn-(k+m)/2}}{1-q^{kn-(k+m)/2}}+\frac{cq^{kn-k+m/2}}{1+cq^{kn-k+m/2}}\right)\right]$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n(pn+x)q^{\left(kn^2+mn\right)/2}}{\left(-cq^{m/2};q^k\right)_{n+1}}+\sum_{n=1}^{\infty}\frac{(-1)^n(-pn+x)q^{\left(kn^2-mn\right)/2}}{\left(-cq^{k/2};q^k\right)_n}$$

Proof.

$$-\sum_{n=0}^{\infty}\frac{(-1)^n q^{\left(kn^2+mn\right)/2}}{\left(-cq^{m/2};q^k\right)_{n+1}}\left(\sum_{r=0}^{\infty}\frac{pcq^{rk+m/2}}{\left(1+cq^{rk+m/2}\right)}\right). \quad (1.3)$$

Changing q to q^2, z to z/q c to cq

We obtain

$$\prod_{n=1}^{\infty}\frac{\left(1+zq^{2n-1}\right)\left(1+z^{-1}q^{2n-1}\right)\left(1-q^{2n}\right)}{\left(1+cq^{2n-1}\right)\left(1-czq^{2n-2}\right)}=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left(cz;q^2\right)_{n+1}}z^n+\sum_{n=1}^{\infty}\frac{q^{n^2}}{\left(cz;q^2\right)_n}z^{-n} \quad (1.4)$$

Replacing q by q^k and z by $z^p q^m$, (1.4) can be written as

$$\prod_{n=1}^{\infty}\frac{\left(1+z^p q^{2nk-k+m}\right)\left(1+z^{-p}q^{2nk-k-m}\right)\left(1-q^{2kn}\right)}{\left(1+cq^{2nk-k}\right)\left(1-cz^p q^{2nk-2k+m}\right)}$$
$$=\sum_{n=0}^{\infty}\frac{q^{kn^2+mn}z^{pn}}{\left(cz^p q^m;q^{2k}\right)_{n+1}}+\sum_{n=1}^{\infty}\frac{q^{kn^2-mn}z^{-pn}}{\left(-cq^k;q^{2k}\right)_n} \quad (1.5)$$

Change q to $q^{1/2}$ in (1.5), multiply the resulting identity throughout by z^x, to obtain

$$z^x\prod_{n=1}^{\infty}\frac{\left(1+z^p q^{nk-(k-m)/2}\right)\left(1+z^{-p}q^{nk-(k+m)/2}\right)\left(1-q^{kn}\right)}{\left(1+cq^{nk-k/2}\right)\left(1-cz^p q^{nk-k+m/2}\right)}$$
$$=\sum_{n=0}^{\infty}\frac{q^{\left(kn^2+mn\right)/2}z^{pn+x}}{\left(cz^p q^{m/2};q^k\right)_{n+1}}+\sum_{n=1}^{\infty}\frac{q^{\left(kn^2-mn\right)/2}z^{-pn+x}}{\left(-cq^{k/2};q^k\right)_n} \quad (1.6)$$

Differentiating (1.6) with respect to z and setting z = 1, we obtain (1.3) on using (1.2).

Change q to q^k, z to $-z^p q^m$ in (1.5); replace q by $q^{1/2}$ in the resulting identity and multiply throughout by z^x; differentiate the resulting identity with respect to and set z = 1, and use (1.2) to obtain (1.4).

Theorem 1.4.

We have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)q^{n(n+1)/2}}{(-c)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-c)_{n+1}}\left(\sum_{k=0}^{n}\frac{cq^k}{1+cq^k}\right)=\frac{(q)_\infty^3}{(-c)_\infty^2}. \qquad (1.7)$$

Proof. (2.1.1) can be written as

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}z^{-n-1}\left[\frac{1}{(-c)_{n+1}}+\frac{z^{2n+1}}{(cz)n+1}\right]=\frac{(1+z)(-qz)(-q/z)_\infty(q)_\infty}{z(-c)_\infty(cz)_\infty}. \qquad (1.8)$$

Multiplying (1.8) throughout by $\frac{z}{1+z}$ and then taking the limits as $z\to -1$, we obtain (1.9).

Corollary 1.5

$$f(-q)=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)q^{n(n+1)/2}}{(q)_{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(q)_{n+1}}\left(\sum_{k=1}^{n+1}\frac{q^k}{1-q^k}\right). \qquad (1.9)$$

Proof. Setting c = -*q*, we obtain (1.9).

Corollary 1.6

$$\varphi^2(-q)f(-q)=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)q^{n(n+1)/2}}{(-q)_{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q)_{n+1}}\left(\sum_{k=1}^{n+1}\frac{q^k}{1+q^k}\right). \qquad (1.10)$$

Corollary 1.7

$$\psi^2(-q)f(-q^2)=\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)q^{n(n+1)}}{(q;q^2)_{n+1}}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)}}{(q;q^2)_{n+1}}\left(\sum_{k=1}^{n+1}\frac{q^{2k+1}}{1-q^{2k+1}}\right). \qquad (1.11)$$

Proof. Changing q to q^2 and then setting c =-q, we obtain (1.11).

The Ramanujan's reciprocity theorem found in his lost notebook [101], can also be stated as :

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(cz)_{n+1}}z^{n}+\sum_{n=1}^{\infty}\frac{q^{n(n-1)/2}}{(-c)_{n}}z^{-n}=\frac{(-qz)_{\infty}(-1/z)_{\infty}(q)_{\infty}}{(-c)_{\infty}(cz)_{\infty}}, \qquad (1.12)$$

where $|q|<1, z\neq 0,\ c\neq -1,\ -q^{-n}$, where $n\in Z^{+}$. In fact setting a $=-cz/q$ *and* $b=c/q$ we obtain (1.12) of the present paper we give proof of (1.11).

The firs proof is based on the well-known Euler-Cauchy q-binomial theorem.

$$\sum_{n=0}^{\infty}\frac{(a)_{n}}{(q)_{n}}z^{n}=\frac{(az)_{\infty}}{(z)_{\infty}}.$$

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