Modal Analysis of Cantilever Beam Using Analytical and Finite Element Method

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Abstract : This paper represents theoretical modal analysis of a cantilever beam using Euler-Bernoulli beam theory and finite element analysis is performed which allows to obtain the modes of vibration and natural frequencies of a cantilever beam. While designing mechanical system, the modal analysis plays a crucial role. The paper presents the numerical approach method using ANSYS workbench to analyse all the natural frequency of the beam and analytical method using Euler-Bernoulli theory and MATLAB is used to represent the mode shapes. This analysis is done to compare the results obtained by the above mentioned tools. The results shows that there are minute errors found after comparison and that can be neglected. So the analytical results are found to be same as ANSYS workbench results.

***Keywords* – Modal analysis, Euler-Bernoulli theory, Natural frequency of beam, Ansys workbench.**

1. INTRODUCTION

Vibrations occur in many mechanical and structural systems whether it is desirable or undesirable. Their occurrence is uncontrollable as it depends on many factors. However, Undesirable vibrations must be controlled for proper and efficient operation of the systems. These vibrations needs to be controlled within the specified limits and for that it becomes essential to find out natural frequency of the system. The Euler–Bernoulli beam theory describes the relationship between the beam's deflection and the applied load and gives accurate results for most beams with solid cross-sections. This theory is valid for long, slender beams and does not account for shear deformations, which may have a significant effect on the lateral beam response of large-diameter beams [1], The Euler-Bernoulli beam theory is the most commonly used because it is simple and provides realistic engineering approximations for many problems [2]. Finite element analysis is an advanced technique for solving complex engineering problems such as analysis of cutting tools, engine parts simulating actual working conditions in the virtual environment to get the approximate results, where analytical solutions are not possible [3]. For accomplishing the finite element method there are multiple softwares like ANSYS, LS-DYNA, Hypermesh, Abaqus etc. In this paper we are going to analyse modal characteristics and harmonic response of cantilever beam using ANSYS workbench. The modal analysis is performed to understand the dynamic behaviour of the structure which results in natural frequency and mode shape of the structure [4]. The free vibration of cantilever beam can be solved using numerical approach and experimental approach. The numerical approach using the ANSYS workbench considers the free vibration of cantilever beam to find out mode shape and natural frequencies with high accuracy [5] [6]. The results obtained from both the approach are also compared and it is observed that very small error is there and can be neglected. MATLAB programming is helpful for plotting the mode shapes of the systems [7].

2. Mathematical Formulation

For deriving the equation of motion of a continuous system we can consider the free-body diagram of an infinitesimally small element of the system and apply Newton's second law of motion and find the natural frequencies and mode shapes of the system using harmonic solution [8]. The figure shown below is an elastic beam drawn in the deformed configurations. Transverse displacements, measured from the neutral axis at equilibrium, are designated as $y(x,t)$.

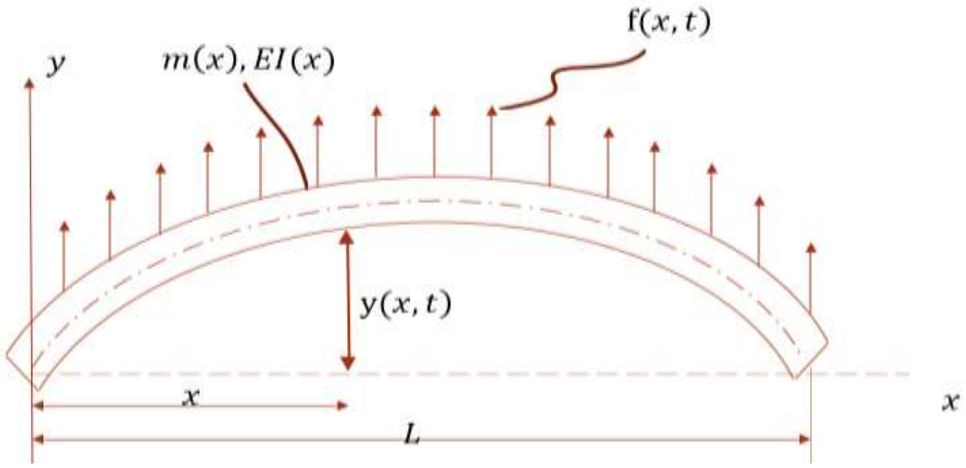

Fig 2-1 Beam in bending vibration

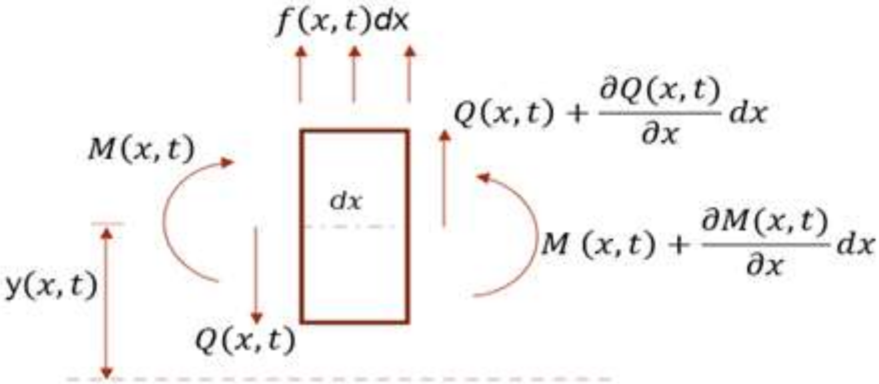

Fig 2-2 Free body diagram for beam element

Euler-Bernoulli beam theory specifies that rotatory inertia and shear deformation affects can be ignored.

Force equation of motion in vertical direction,

$$\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]-Q(x,t)+f(x,t)dx=m(x)dx\frac{\partial^2 y}{\partial t^2}, \quad 0<x<L \tag{1}$$

Under assumption that product of moment of inertia of element and angular acceleration is negligibly small, the moment equation of motion about an axis normal to x and y and passing through center of cross-section area is,

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx+f(x,t)dx\frac{dx}{2}=0,0<x<L \tag{2}$$

Ignoring second order terms in dx and cancelling appropriate terms for simplifying, the moment equation (2) reduces to,

$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0 \quad 0<x<L \tag{3}$$

Cancelling appropriate terms, dividing through by dx and using equation (3), the force equation (1) becomes,

$$-\frac{\partial^2 M(x,t)}{\partial x^2}+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \quad 0<x<L \tag{4}$$

Equation (4) relates bending moment M(x,t) and transverce force density F(x,t) to the bending displacement y(x,t).

To obtain an equation in terms of y(x,t) and f(x,t) alone, we recall from mechanics of materials that bending moment is related to bending displacement by,

$$M(x,t)=EI(x)\frac{\partial^2 y(x,t)}{\partial x^2} \tag{5}$$

So that using equation (3) the shearing force is related to bending displacement by,

$$Q(x,t)=-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] \tag{6}$$

Inserting equation (5) into equation (4), we obtain partial differential equation for bending vibration of beam in the form,

$$-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \quad 0<x<L \tag{7}$$

Above equation is fourth order

As the cantilever is having no transverse force acting, hence we can write equation (7) as,

$$-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=m(x)\frac{\partial^2 y(x,t)}{\partial t^2} \quad 0<x<L \tag{8}$$

Solution of above equation can be emulated using technique of separation of variables as

y(x,t) = Y(x)F(t)

Also m = ρA

Substituting values in equation (8)

$$EI\frac{\partial^4 \mathrm{Y(x)F(t)}}{\partial x^4}=-\rho A\frac{\partial^2 \mathrm{Y(x)F(t)}}{\partial t^2}$$

$$\frac{EI}{\rho A}\frac{1}{Y(x)}\frac{\partial^4 \mathrm{Y(x)}}{\partial x^4}=-\frac{1}{F(t)}\frac{\partial^2 \mathrm{F(t)}}{\partial t^2} \tag{9}$$

We can observe that left side of equation (9) depends only on X,while right side depends only on t. because x and t are independent variables, we conclude that both sides of equation (9) must be equal to a constant. Let this constant be ω^2, It follows that

$$-\frac{1}{F}\frac{\partial^2 F}{\partial t^2}=\omega^2$$

$$\frac{\partial^2 \mathrm{F(t)}}{\partial t^2}+\omega^2 F=0 \tag{10}$$

Which has solution in the form,

$$\mathrm{F(t)}=C_1 sin\omega t+C_2 cos\omega t \tag{11}$$

Also we have,

$$\frac{\partial^4 \mathrm{Y}}{\partial x^4}-\frac{\rho A}{EI}\omega^2 Y=0$$

Denoting $\beta^4=\frac{\rho A}{EI}\omega^2$ we get,

$$\frac{\partial^4 \mathrm{Y}}{\partial x^4}-\beta^4 Y=0 \tag{12}$$

Solution of equation (12) can be shown as

$$\mathrm{Y(x)}=A_1 cos\beta x+A_2 sinh\beta x+A_3 \cos\beta x+A_4 sin\beta x \tag{13}$$

For cantilever beam, the boundary conditions are,

At x=0 Y=0 & $\frac{dy}{dx}=0$

At x=L $\frac{d^2y}{dx^2}=0$ & $\frac{d^3y}{dx^3}=0$

Using boundary conditions on equation (13)

$A_1 + A_3 = 0$ (14)

$A_2 + A_4 = 0$ (15)

$A_1 cosh\beta x + A_2 sinh\beta x - A_3 \cos\beta x - A_4 sin\beta x = 0$ (16)

$A_1 sinh\beta x + A_2 cosh\beta x + A_3 \sin\beta x - A_4 cos\beta x = 0$ (17)

For non-trivial solution of the(A_1, A_2, A_3, A_4), we must have

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ cosh\beta x & sinh\beta x & -cos\beta x & -sin\beta x \\ sinh\beta x & cosh\beta x & sin\beta x & -cos\beta x \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0$$

$cos\beta l cosh\beta l + 1 = 0$ (18)

Which is the Characteristic equation for cantilever Euler-Bernoulli beam

$$cos\beta l = -\frac{1}{cosh\beta l}$$

$$cosz = -\frac{1}{coshz}$$

For higher modes (18) may be approximated as $cos\beta l = 0$.

The solution can be expressed as

$\beta_n l = \left(\frac{2n-1}{2}\pi + e_n\right)$ n=1, 2...

$\omega_n = \left(\frac{2n-1}{2}\pi + e_n\right)^2 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$ (19)

Where e_n are small (and rapidly diminishing) correction terms obtained as,

$e_1 = 0.3042$, $e_2 = -0.018$, $e_3 = 0.001$

Eliminating A_1 and A_2 from (16) using (14) and (15), we obtain

$A_1 = -\frac{sinh\beta_n l + sin\beta_n l}{cosh\beta_n l + cos\beta_n l} A_2 = \alpha_n A_2$ (20)

Therefore, taking A_2 = 1,

$A_1 = \alpha_n,\ \ A_1 = 1,\ \ A_1 = -\alpha_n,\ \ A_1 = -1$ (21)

Which yields the n[th] Eigen function as

$Y_n(x) = sinh\beta_n x - sin\beta_n x - \left[\frac{sinh\beta_n l + sin\beta_n l}{cosh\beta_n l + cos\beta_n l}\right](cosh\beta_n x - cos\beta_n x)$ (22)

The modes of vibration corresponding to the first three Eigen functions are shown as

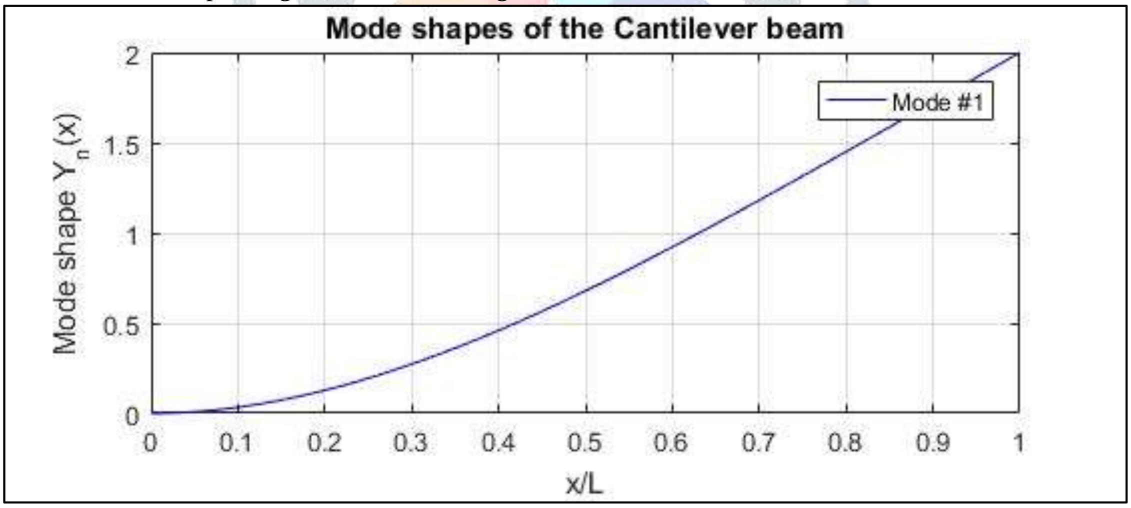

Fig 2-3 1st Mode shape

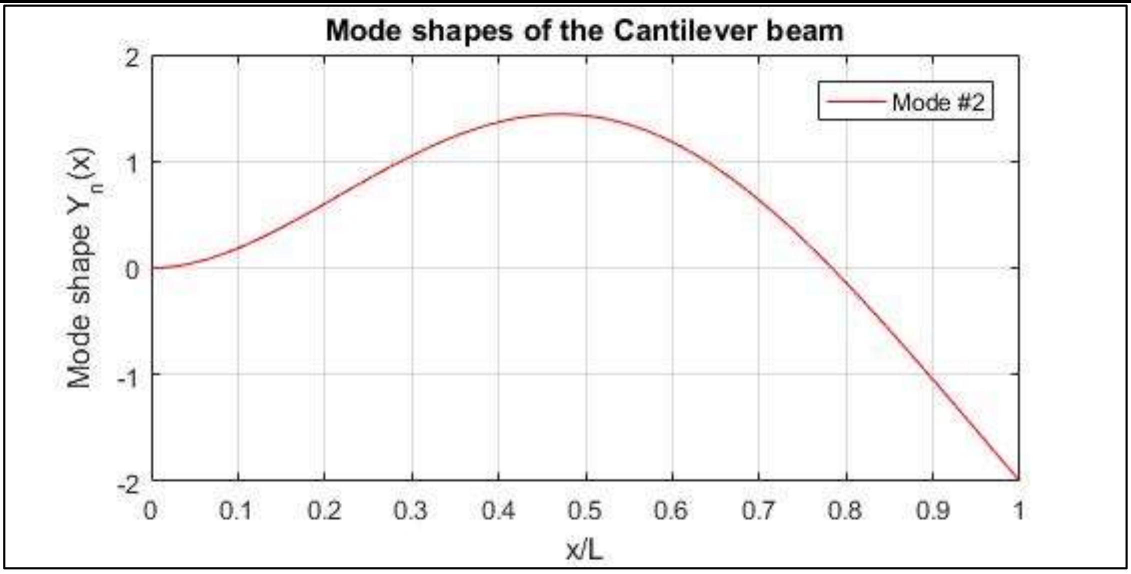

Fig 2-4 2nd Mode shape

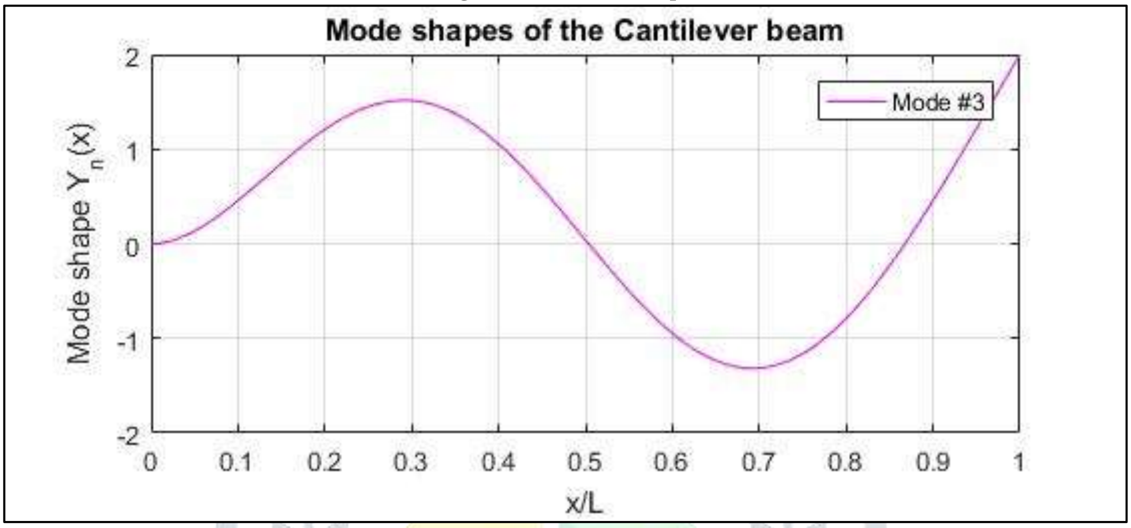

Fig 2-5 3rd Mode shape

For calculating the natural frequency of the cantilever beam, following parameters needs to be considered.

Table 1 Parameters

Dimensions of Cantilever Beam	160×25×1 mm
Material	Aluminium
Young's modulus	68×10^9
Poisson's ratio	0.29
Density	2700 kg/m^3

For calculating natural frequency of cantilever beam, using equation (19)

$$\omega_n = \left(\frac{2n-1}{2}\pi + e_n\right)^2 \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \quad (23)$$

$$I = \frac{bd^3}{12} = \frac{0.025*0.001^3}{12} = 2.083 \times 10^{-12}$$

For calculating first natural frequency, putting all the values in equation (23)

For n=1,

$$\omega_n = \left(\frac{2-1}{2}\pi + 0.3042\right)^2 \frac{1}{0.16^2}\sqrt{\frac{(68\times10^9)\times(2.083\times10^{-12})}{2700\times0.025\times0.001}}$$

$\omega_n = 198.933$ Rad/sec

And natural frequency f_n is given as,

$$f_n = \frac{\omega_n}{2\pi}$$

$\therefore f_n = 31.66\ Hz$

Similarly, next natural frequencies can be calculated by above formula and we get,

Table 2 Natural frequency

Sr. No.	f_n, (Hz)
1	31.66
2	198.456
3	555.684
4	1088.918
5	1800.060

3. Analysis in ANSYS CAE Software

3.1 Modal Analysis

Modal analysis is a technique to study the dynamic characteristics of a structure or system under vibrational excitation. Natural frequencies and mode shapes of a structure can be determined using modal analysis. Modal analysis allows the design to avoid resonating condition or to vibrate at a specified frequency and gives engineers an idea of how the design will respond to different types of dynamic loading conditions.

The objective of this study is to find out the natural frequencies and mode shapes of the cantilever beam. Modal analysis can be done in ANSYS software to find out natural frequency of the system and mode shapes.

3.1.1 Pre-processing

- **Material properties**

Table 3 Material property

Material	Aluminium
Young's modulus	68×10^{9}
Poisson's ratio	0.29
Bulk Modulus MPa	53968
Shear Modulus MPa	26357
Density	2700 kg/m^3
Dimensions of Cantilever Beam	160×25×1 mm

- **Meshing**

Meshing is an important step while doing finite element analysis, where complex geometries are divided into simple elements that can be used as discrete local approximations of the larger domain. The mesh has impact on accuracy, convergence and also the speed of solution.

Table 4 Meshing parameters

Element size	5 mm
Nodes	65
Elements	32

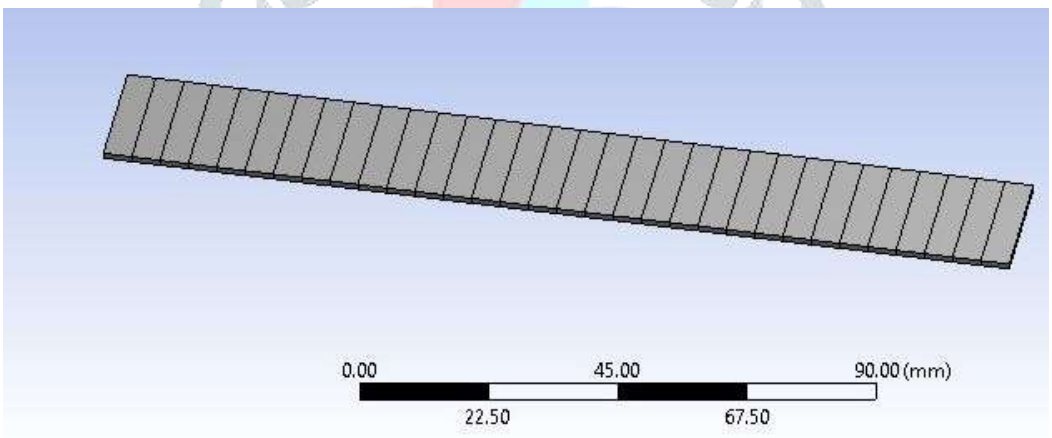

Fig 1-1 Meshing

- **Boundary conditions**

The cantilever beam has one fix end and a free end. So for the modal analysis, one end is fixed using fixed support option in loads.

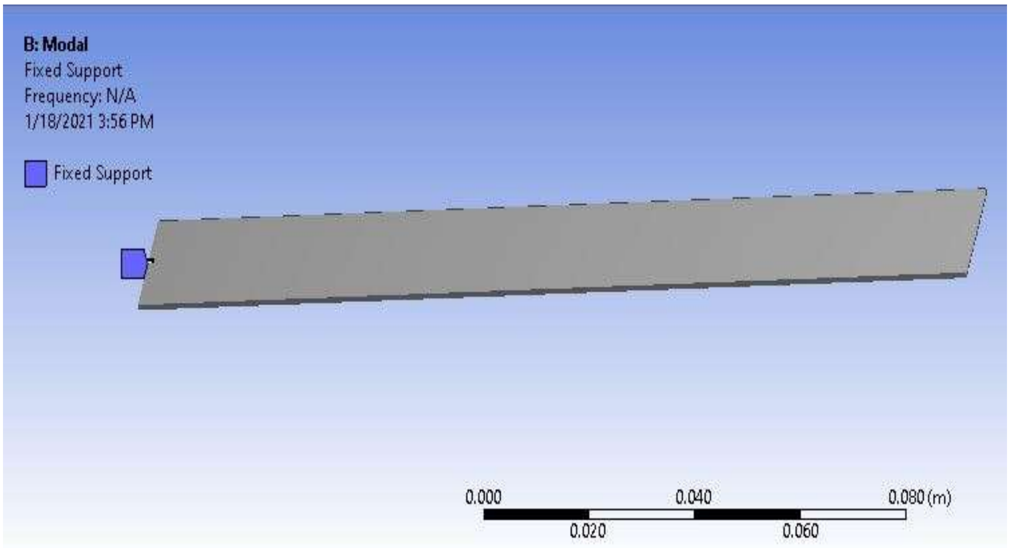

Fig 1-2 Applied boundary condition

4. Obtained Results in ANSYS

For getting result of modal analysis, Mechanical APDL solver is used. In solution, total deformation is selected and the result is obtained.

Object Name	*Modal (B5)*
State	Solved
Definition	
Physics Type	Structural
Analysis Type	Modal
Solver Target	Mechanical APDL
Options	
Environment Temperature	22. °C
Generate Input Only	No

Fig 4-1 Solver inputs

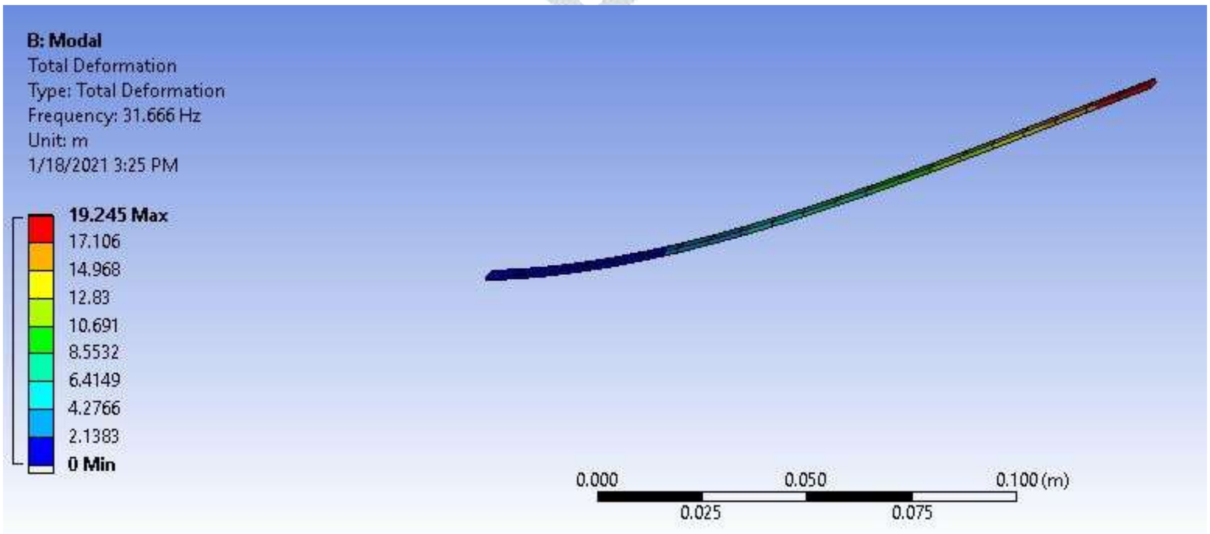

Fig 4-2 1st modal analysis result

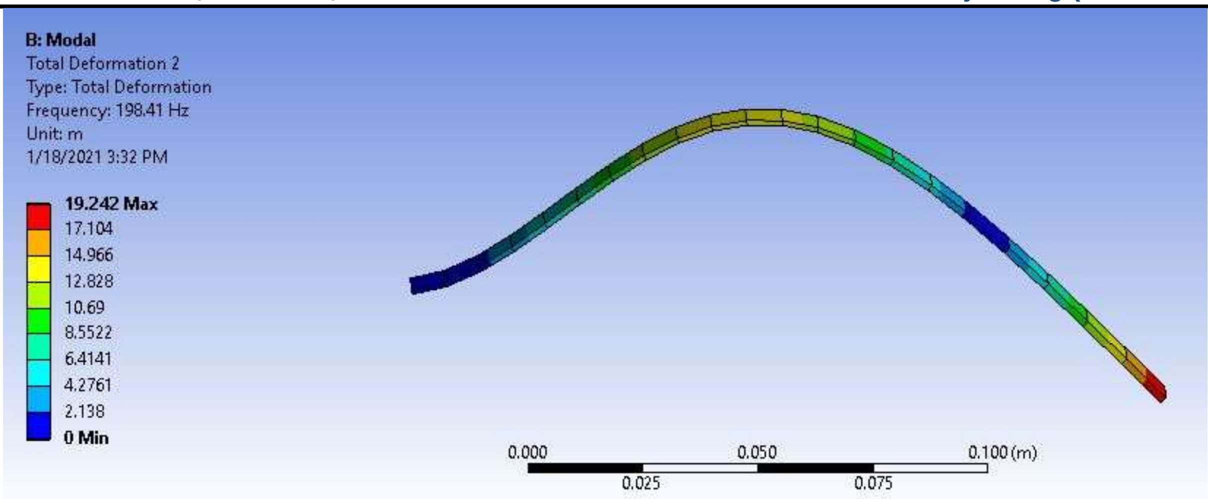

Fig 4-3 2nd modal analysis result

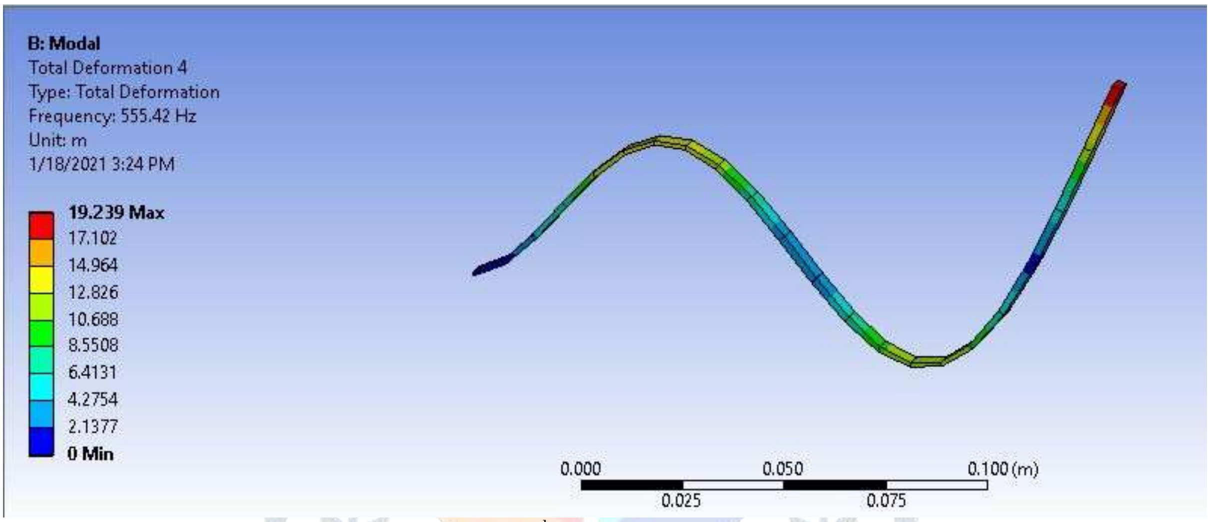

Fig 4-4 3rd modal analysis result

A. Harmonic response Analysis

A harmonic analysis is used to determine the response of the structure under a steady-state sinusoidal (harmonic) loading at a given frequency. Harmonic response analysis aims to calculating the response at the excitation frequency and Obtaining the frequency response curves to give an idea of the Peak response on the curve.

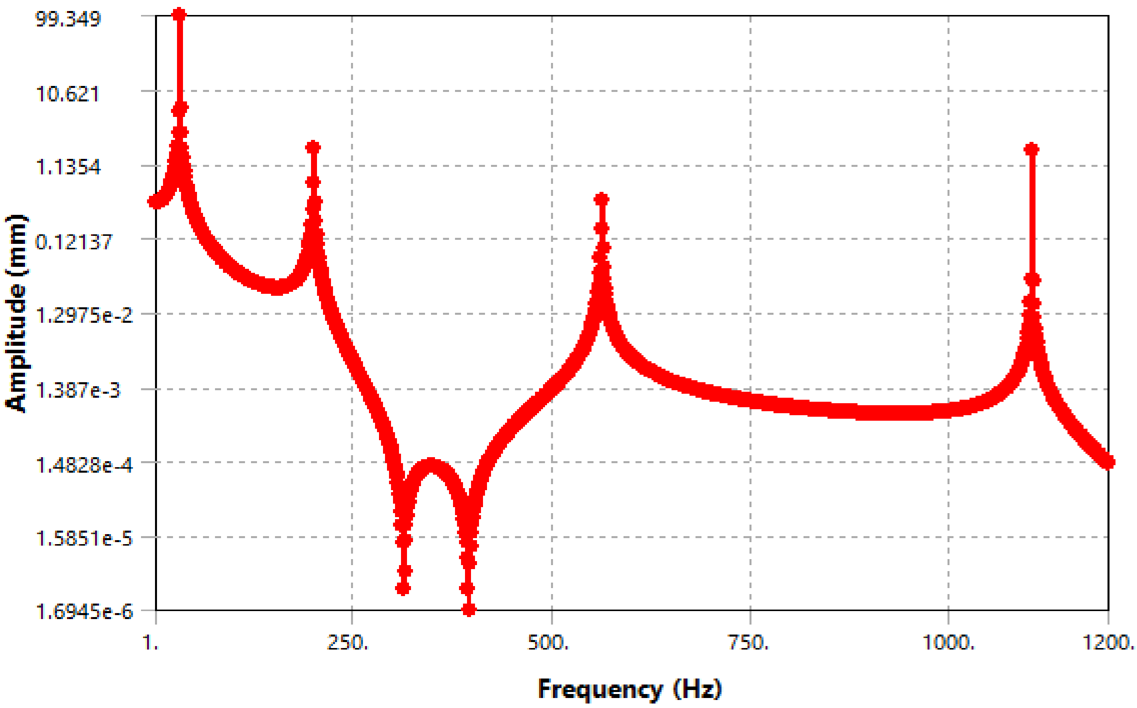

Fig 4-5 variation of amplitude with different exciting frequencies

Comparison between ANSYS workbench and Analytical solution –

The results obtained from Analytical method and Numerical method are compared in table given below

Table 5 Comparison between ANSYS workbench and Analytical solution

Mode	Natural frequency obtained analytically, Hz	Natural frequency obtained from ANSYS, Hz	% Error
1	31.66	31.666	0.018
2	198.456	198.41	0.023
3	555.684	555.42	0.047
4	1088.918	1087.9	0.093
5	1800.060	1797.5	0.14

5. Conclusion

The results obtained from analytical method and ANSYS workbench are found to be almost same. The maximum error found is 0.14% for the third mode while the minimum error is 0.018 % for first mode. Also error between analytical and ANSYS software is low at low frequencies, it increases with increase in frequency. The results show that the values obtained using analytical approach and numerical approach are not varying so much. But computation time is less in case of numerical approach. Thus, for complex engineering problems, FEA approach is found better than the numerical or analytical method. Also the plots of mode shapes using MATLAB and ANSYS workbench can be seen. For designing smart structure using active vibration control method we can use ANSYS for finding out natural frequencies and predetermining natural frequencies of the system helps us to modify design or apply some damping material to avoid resonance.

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