A note on Degree Partition Number of a Graph

N. Malathi[1], M. Bhuvaneshwari[2], Selvam Avadayappan[3]

[1]*Department of Mathematics, V.V.Vanniaperumal College for Women, Virudhunagar, India*

[2,3]*Research Department of Mathematics, VHN Senthikumara Nadar College, Virudhunagar, India*

Abstract: In pandemic period, there is a need to execute any system with partitioned labours in which every class in a partition shares more or less same potential. Let $\pi_k = \{V_1, V_2, \ldots, V_k\}$ be a partition of vertex set V of a graph G. Then π_k is called *similar degree partition* if the sum of degrees of vertices in each class $V_i, 1 \leq i \leq k$, differs from that of other by at most 1. The *degree partition number* of G, $\psi_D(G) = max\{k/\pi_k$ is a *similar degree partition* of $G\}$ or 1 if no such π_k exists. In this paper, we initiate the study on this parameter and prove some interesting facts.

Keywords: Partitioning, Degree Partition number, Degree partitioning.

AMS Subject Classification Code (2010): 05C (Primary)

1 INTRODUCTION

In this paper, we consider only finite, simple, undirected graphs. Basic notations and terminology that are not mentioned here can be referred from [3,4]. The degree of a vertex v is denoted by $d(v)$ and the degree set of a graph is denoted by $D(G)$. A graph G is said to be $r-regular$, if every vertex of G has degree r. A graph $G(V, E)$ is *connected* if every two vertices of G are connected by a path. The minimum and maximum degree of vertices in a graph are denoted by δ and Δ respectively. P_n and C_n denote path and cycle on n vertices respectively.

For any vertex $v \in V(G)$, the *open neighbourhood* $N(v)$ is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V(G) \,/\, uv \in E(G)\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. A vertex v is said to be *pendant* if $d(v) = 1$. A vertex v in G is said to be a *full vertex* if it is adjacent to every other vertex in G. The *join* of two graphs G and H denoted by $G \vee H$ is obtained by joining each vertex of G to every other vertices of H by means of edges. The *wheel graph* $W_n (n \geq 4)$ is nothing but $K_1 \vee C_{n-1}$.

In many places, there may arise a situation in which a particular set of people or components need to be divided into many groups so as to satisfy some necessary conditions. One of the ways of representing any system is through graph models. Our mathematicians develop many methods of partitioning to examine the nature and properties of a network.

The reduction of a graph to smaller graphs by partitioning its vertex set into mutually exclusive groups is called *graph partition*. A lot of research ideas are available in literature which are based on partitioning vertex set and edge set of a graph.

Vertex colouring and edge colouring [3,4] are well known examples for this kind of studies on Graph Theory.

E.Sampath Kumar and C.V.Venkatachalam initiated a study of new parameter *generalised chromatic number* $\chi_k(G)$ [5]. The *general chromatic number* $\chi_k(G)$ of a graph G is the minimum order of a partition P of V such that each set in P induces a subgraph H with $\chi(H) \leq k$. Also, some other partitions like *bilinear partitions*, *trilinear partitions* can be referred from [6].

The period in which this paper has been written is Corona Virus pandemic period which needs social distancing in every aspect. But for the economic and educational welfare of the country, every system must be dynamic. The system must be partitioned into smaller groups with more or less similar capacity to face the need of the hour. This forms the base for the study on *Degree Partition number* which is introduced in this paper.

Let $\pi_k = \{V_1, V_2, \dots, V_k\}, (k \geq 2)$ be a partition of the vertex set $V(G)$. π_k is called a *similar degree partition* (or simply *sim d-partition*) if $\left|\sum_{v \in V_i} d(v) - \sum_{v \in V_j} d(v)\right| \leq 1$. That is, the sum of degrees of vertices in any class of π_k differs from that of other by at most 1. When this difference equals zero for any two classes of a partition π_k, then it is called *perfect sim d-partition*. The *degree partition number* of a graph $\psi_D(G)$ is defined as max$\{k/\pi_k$ is a *sim d- partition of G*$\}$ or 1 if no such π_k exists.

For example, consider the following graph.

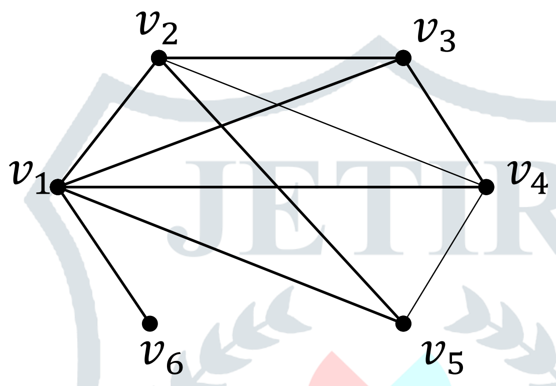

Figure 1

Let $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1\}, V_2 = \{v_2, v_3\}, V_3 = \{v_4, v_5, v_6\}$

$\pi_2 = \{V_1, V_2\}$ where $V_1 = \{v_1, v_2, v_6\}, V_2 = \{v_3, v_4, v_5\}$

$\pi_3' = \{V_1, V_2, V_3\}$ where $V_1 = \{v_1, v_6\}, V_2 = \{v_2, v_5\}, V_3 = \{v_3, v_4\}$

Here we can easily verify that π_3 is not a sim d-partition. π_2 *and* π_3' are sim d-partitions.

And $\psi_D(G) = 3$ as no other sim d-partition $\pi_k, k \geq 4$ exists for this graph.

In this paper, we initiate a study on this parameter and prove some interesting results. For further results on this parameter one can refer [1,2].

2 MAIN RESULTS

In this section, we first present some basic facts on the degree partition number of a graph.

Fact 1 For any complete graph K_n $(n \geq 3)$, $\psi_D(K_n) = n$.

Fact 2 In fact, for any regular graph G of order n, we have $\psi_D(G) = n$ and therefore

$\psi_D(C_n) = n$ where $n \geq 3$.

Fact 3 The converse of Fact 2 need not be true. As an example for this, $\psi(P_n) = n$, $(n \geq 3)$.

Fact 4 For a star, $\psi_D(K_{1,n}) = 2, (n \geq 3)$.

Fact 5 Regular graphs and stars $K_{1,n}, (n \geq 3)$ are *perfect sim d-graphs*.

Fact 6 For complete bipartite graph $K_{n,n+1}$, $\psi_D(K_{n,n+1}) = 2n + 1$ for all $n \geq 1$.

Fact 7 $\psi_D(W_6) = 1$. That is, for W_6 no *sim d-partition* exists.

Fact 8 If the range of D(G)=1, then $\psi_D(G) = n$. That is, if $|d(v_i) - d(v_j)| \leq 1$ for all $v_i, v_j \in V(G)$, then $\psi_D(G) = n$.

Proposition 9 Let G be any bipartite graph. Then $\psi_D(G) \geq 2$.

Proof If (V_1, V_2) is the bipartition of the given bipartite graph G, then $\pi_2 = \{V_1, V_2\}$ is a sim d-partition of G, as $\sum_{v_i \in V_1} d(v_i) = \sum_{w_i \in V_2} d(w_i)$. Therefore $\psi_D(G) \geq 2$. □

Theorem 10 For a complete bipartite graph $K_{m,n}, with\ |m-n| > 1, \psi_D(K_{m,n}) \geq 2k$ where $gcd(m,n) = k$.

Proof Let $K_{m,n}$ with $|m-n| > 1$ be the given graph. Since $gcd(m,n) = k$, we have $k|m\ and\ k|n$.

Let (V_1, V_2) be the bipartition of $K_{m,n}$ such that $|V_1| = m\ and\ |V_2| = n$.

Now $\pi_{2k} = \{U_1, U_2, \dots, U_k, W_1, W_2, \dots, W_k\}$ forms a sim d-partition of $K_{m,n}$ where U_i's are disjoint subsets of V_1 each having $\frac{m}{k}$ vertices & W_j's are disjoint subsets of V_2 each having $\frac{n}{k}$ vertices.

Here π_{2k} is a perfect sim d-partition as $\sum_{v_i \in U_i} d(u_i) = \sum_{w_i \in W_i} d(w_i) = \frac{mn}{k}$.

Thus, we conclude that $\psi_D(K_{m,n}) \geq 2k$. □

The following example stands as a proof for the existence of a complete bipartite graph with $\psi_D(K_{m,n}) > 2\gcd(m,n)$.

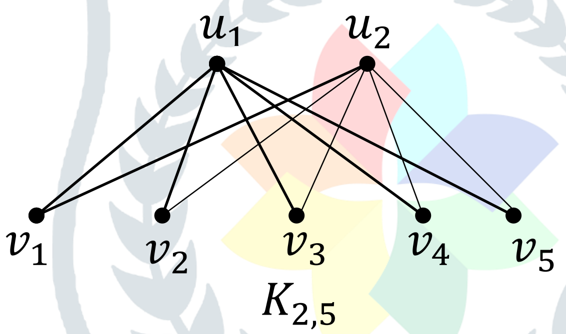

Figure 2

Here $K_{2,5}$ is a complete bipartite graph with bipartition (V_1, V_2) where $V_1 = \{u_1, u_2\}\ and\ V_2 = \{v_1, v_2, v_3, v_4, v_5\}$. Since $gcd(2,5) = 1$, there exists a perfect sim d-partition $\pi_2 = \{V_1, V_2\}$. But another partition $\pi_3 = \{U_1, U_2, U_3\}$ where $U_1 = \{u_1, v_1\}, U_2 = \{u_2, v_2\}$ and $U_3 = \{v_3, v_4, v_5\}$ stands as the maximum sim d-partition of $K_{2,5}$ forcing $\psi_D(K_{2,5}) = 3 > 2\gcd(2,5)$.

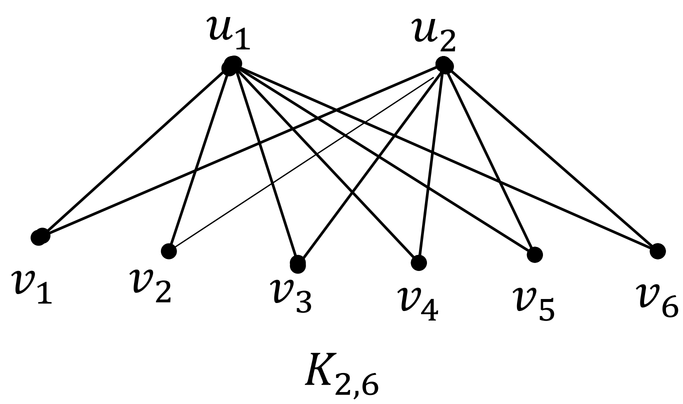

Figure 3

Another example $K_{2,6}$ is shown in Figure 3 which proves the inequality shown in the above theorem is sharp. Let $K_{2,6}$ have the bipartition (V_1, V_2) where $V_1 = \{u_1, u_2\}$ *and* $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

We know that $gcd(2,6) = 2$. We partition both V_1 *and* V_2 into 2 disjoint sets say $\{U_1, U_2\}$ *and* $\{W_1, W_2\}$ where $U_1 = \{u_1\}, U_2 = \{u_2\}, W_1 = \{v_1, v_2, v_3\}$ *and* $W_2 = \{v_4, v_5, v_6\}$.

Then $\pi_4 = \{U_1, U_2, W_1, W_2\}$ forms a perfect sim d-partition for $K_{2,6}$ and $\psi_D(K_{2,6}) = 4 = 2\gcd(2,6)$.

□

Theorem 11 For the graph $G \cong K_2 V K_{n-2}^C, (n \geq 3)$ the degree partition number

$$\psi_D(G) = \begin{Bmatrix} 4 & if\ n \equiv 0(mod\ 2) \\ 3 & if\ n \equiv 1(mod 2)\ and\ n \equiv 0(mod\ 3) \\ 1 & if\ n \equiv \pm 1(mod 6) \end{Bmatrix}$$

Proof Let v_1 and v_2 be the full vertices of $G \cong K_2 \vee K_{n-2}^C, (n \geq 3)$ and $u_1, u_2, \dots, u_{n-2}$ be the remaining vertices.

Case(i) n is even.

Consider the partition $\pi_4 = \{V_1, V_2, V_3, V_4\}$ where $V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \left\{u_1, u_2, \dots, u_{\frac{n-2}{2}}\right\}$ and $V_4 = \left\{u_{\frac{n-2}{2}+1}, u_{\frac{n-2}{2}+2}, \dots, u_{n-2}\right\}$.

It forms a sim d-partition with degree sum of each class being n-1, n-1, n-2, n-2 respectively.

Further we cannot find a sim d-partition π_k such that $k > 4$ for the considered graph as π_4 has optimum number of vertex classes with equivalent degree sum.

$$\therefore \psi_D(G) = 4\ if\ n \equiv 0(mod\ 2)$$

Case (ii) n is odd and $n \equiv 0(mod\ 3)$.

Since $n \equiv 0(mod\ 3)$, let $n = 3k$. Note that n is odd and hence k is also odd.

Consider the partition $\pi_3 = \{V_1, V_2, V_3\}$ where $V_1 = \left\{v_1, u_1, u_2, \dots, u_{\frac{k-1}{2}}\right\}$,

$V_2 = \left\{v_2, u_{\frac{k+1}{2}}, u_{\frac{k+3}{2}}, \dots, u_{k-1}\right\}$ and $V_3 = \{u_k, u_{k+1}, u_{k+2}, \dots, u_{n-2}\}$.

Now, we have

$$\sum_{v_i \in V_1} d(v_i) = n - 1 + \frac{2(k-1)}{2} = n + k - 2,$$

$$\sum_{v_i \in V_2} d(v_i) = n - 1 + 2\left((k-1) - \frac{(k-1)}{2}\right) = n - 1 + 2k - 2 - k + 1 = n + k - 2,$$

$$\sum_{v_i \in V_3} d(v_i) = 2(n - 2 - (k-1)) = 2n - 4 - 2k + 2 = 2n - 2k - 2 = n + 3k - 2k - 2 = n + k - 2.$$

Therefore, π_3 forms a sim d-partition of G. No other partition π_k exists (k>3) with equivalent degree sum classes.

$$\therefore \psi_D(G) = 3\ if\ n\ is\ odd\ and\ n \equiv 0(mod\ 3)$$

Case (iii) n is odd and not a multiple of 3.

Here, if we start partitioning full vertices to start partitioning full vertices to be singleton set, then such sim d-partition exists only when n is even which is explained in case (i).

Whereas if we partition full vertices, each with some vertices of degree 2, the required condition forces n to be a multiple of 3, again case (ii) repeats.

No partition is possible with v_1, v_2 both in same class as both are full vertices having maximum possible degree and the remaining vertices all together could make a sum of 2(n-2) only.

Hence in this case, $\psi_D(G) = 1$. □

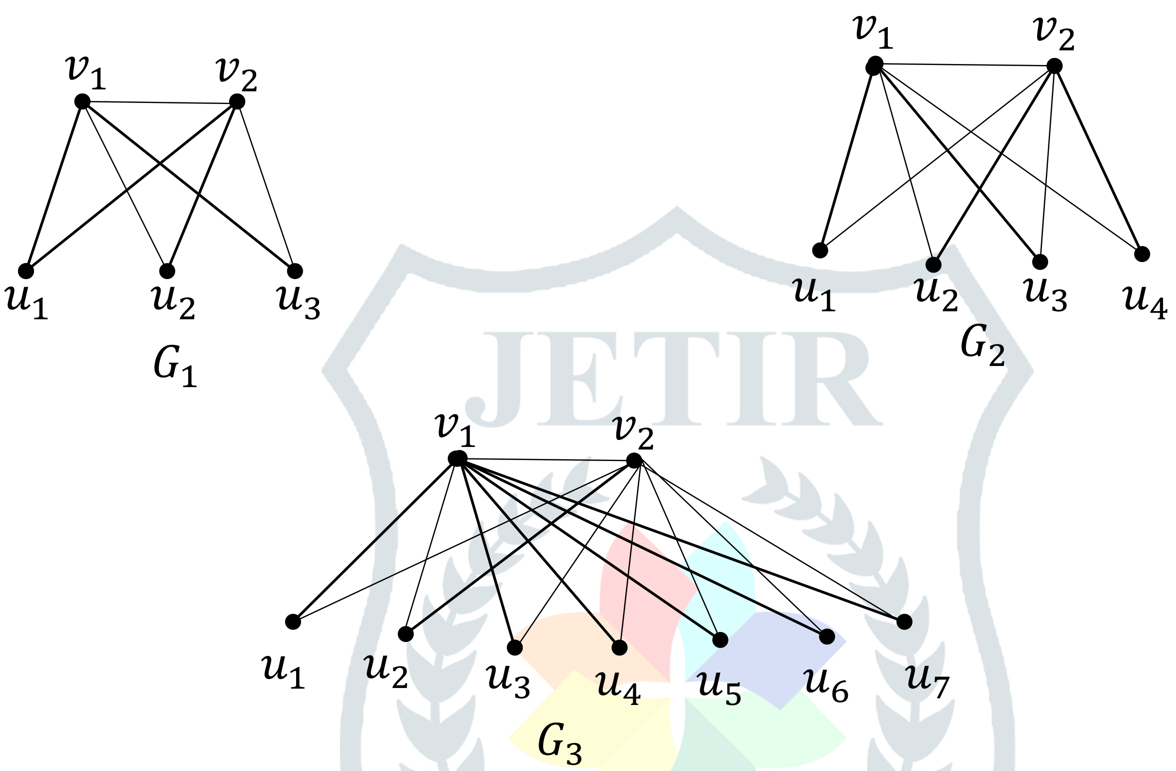

Figure 4

Consider the graphs given in Figure 4.

In G_1, n=5 which is odd and not a multiple of 3, so no sim d-partition exists and $\psi_D(G_1) = 1$.

In G_2, n=6 which is even, so $\psi_D(G_2) = 4$ with $\pi_4 = \{V_1, V_2, V_3, V_4\}$ where

$V_1 = \{v_1\}, V_2 = \{v_2\}, V_3 = \{u_1, u_2\}$ and $V_4 = \{u_3, u_4\}$ forms a sim d-partition of G_2.

In G_3, n=9 which is odd and multiple of 3, So $\psi_D(G_3) = 3$ with $\pi_3 = \{V_1, V_2, V_3\}$ where

$V_1 = \{v_1, u_1\}, V_2 = \{v_2, u_2\}$ and $V_3 = \{u_3, u_4, u_5, u_6, u_7\}$ forming a sim d-partition of G_3. □

Let I_n $(n \geq 4)$ denote the irregular most graph on n vertices with degree sequence

$n-1, n-2, \dots, \left\lfloor\frac{n}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor, \dots, 2,1$. That is, only two vertices have same degree in I_n and all the other vertices have distinct degrees.

For example, I_6 is shown for reference in Figure 5.

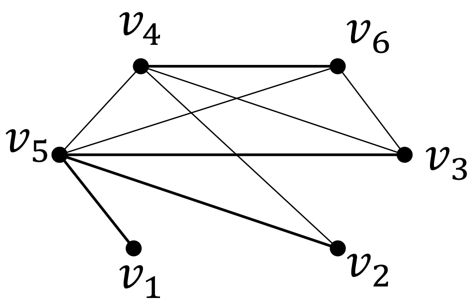

Figure 5

Theorem 12 For the irregular graph I_n, $\psi_D(I_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof Let $V(I_n) = \{v_1, v_2, \dots, v_n\}$ and $d(v_i) = i$ for $1 \le i \le n-1$ and $d(v_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

In the case when n is even, $\pi_{\frac{n}{2}+1} = \left\{V_1, V_2, \dots, V_{\frac{n}{2}+1}\right\}$ where $V_1 = \{v_1, v_{n-3}\}$,

$V_2 = \{v_2, v_{n-4}\}, \dots, V_{\frac{n}{2}-2} = \left\{v_{\frac{n}{2}-2}, v_{\frac{n}{2}}\right\}, V_{\frac{n}{2}-1} = \left\{v_{\frac{n}{2}-1}, v_n\right\}, V_{\frac{n}{2}} = \{v_{n-1}\}$ and $V_{\frac{n}{2}+1} = \{v_{n-2}\}$ forms a sim d-partition with all classes having degree sum $n-2$ except $V_{\frac{n}{2}}$ and $V_{\frac{n}{2}-1}$ have degree sum as $n-1$.

Hence $\psi_D(I_n) = \frac{n}{2} + 1$ when n is even.

As in the above case, when n is odd, $\pi_{\frac{n+1}{2}} = \left\{V_1, V_2, \dots, V_{\frac{n+1}{2}}\right\}$ where $V_1 = \{v_1, v_{n-2}\}$,

$V_2 = \{v_2, v_{n-3}\}, \dots, V_{\frac{n-3}{2}} = \left\{v_{\frac{n-3}{2}}, v_{\frac{n+1}{2}}\right\}, V_{\frac{n-1}{2}} = \left\{v_{\frac{n-1}{2}}, v_n\right\}$ *and* $V_{\frac{n+1}{2}} = \{v_{n-1}\}$ serves as a perfect sim d-partition with degree sum of all classes being n-1 and also forcing $\psi_D(I_n) = \frac{n+1}{2}$.

Therefore, we conclude that $\psi_D(I_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. □

For example, for I_6 shown in the Figure 5, $\pi_4 = \{V_1, V_2, V_3, V_4\}$ where $V_1 = \{v_1, v_3\}, V_2 = \{v_2, v_6\}, V_3 = \{v_5\}$ and $V_4 = \{v_4\}$ is a sim d-partition and $\psi_D(I_6) = 4 = \left\lfloor \frac{6}{2} \right\rfloor + 1$.

3 REFERENCES

[1] Selvam Avadayappan, M. Bhuvaneshwari and N. Malathi, *Degree Partition Number of Derived Graphs* (Communicated).

[2] Selvam Avadayappan, M. Bhuvaneshwari and N. Malathi, *More Results on Degree Partition* (Communicated).

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, The Macmillan press Ltd, Britain, 1976.

[4] F. Harary, Graph Theory, Addison-Wesly, Reading Mass, 1972.

[5] E. Sampathkumar and C.V. Venkatachalam, *Chromatic Partitions of a graph*, Discrete Mathematics 74 (1989) 227-239 North-Holland.

[6] E. Sampathkumar and L. Pushpa Latha, *Bilinear and Trilinear partitions of a graph*, Indian J. Pure Appl. Math., 25(8): 843-850, August 1994.