EULER CHARACTERISTIC AND SURFACES

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Abstract

The Euler characteristic is a topological notion appearing in many different topics throughout mathematics. The Euler formula shows that the expression of the Euler characteristic in terms of numbers of vertices, edges and regions (faces of different dimension in higher- dimensional cases) is a topological invariant. I had firstly come across the topic by reading the topological invariants. This formula is a powerful tool used in establishing many important mathematical results, from the classification of regular polyhedra to the nonplanarity criterion for graphs. I will describe some required tools in differential geometry. Gauss- Bonnet is also a deep result in differential geometry that illustrates a fundamental relationship between the curvature of a surface and its Euler characteristic. In this paper I introduce and examine properties of surfaces in effort to prove a discrete Gauss Bonnet analog.

Keyword: Euler Characteristic, Polyhydra, Gauss-Bonnet

Introduction

The Euler characteristic was originally defined for polyhedra and used to prove various theorems about them, including the classification of the Platonic solids. Euler’s characteristic is strictly easy to compute, and it really seems odd that it should be a topological invariant . After all, it is computed from cell decomposition and it is very easy to come up with a lot of different complexes on a single surface, all with varying numbers of faces, edges and vertices. Euler's formula : $V - E + F = 2$ is simple though it may look, this little formula encapsulates a fundamental property of those threedimensional solids we call polyhedra, which have fascinated mathematicians for over 4000 years. Actually I can go further and say that Euler's formula tells us something very deep about shape and space. The formula bears the name of the famous Swiss mathematician Leonhard Euler (1707 - 1783), who would have celebrated his 310th birthday this year. The Gauss Bonnet theorem bridges the gap between topology and differential geometry. Its importance lies in relating geometrical information of a surface to a purely topological characteristic, which has resulted in varied and powerful applications. Though this paper presents no original mathematics, it carefully works through the necessary tools for proving Gauss-Bonnet. Gauss first proved this theorem in 1827, for the case of a hyperbolic triangle. This theorem established a remarkable invariant relating curvature to the notion of angle within the surface. However, with the developments in topology in the 19th and 20thcenturies, this theorem has become an invaluable piece of modern mathematics.

Graphs And Maps

By an (undirected, finite) graph we will mean an ordered pair $G = (V;E)$, where the elements of the finite set V are called vertices, and elements of the finite set E are called edges, and each edge corresponds to a pair of vertices. Loops and multiple edges among two vertices are allowed. Simplicial complex is a graph without loops and multiple edges is a 1- dimensional simplicial complex. Let us also mention some special classes of graphs. By Kn we will denote the complete graph with n vertices, where each pair of different vertices is joined by an edge. By Km,n we will denote a complete bipartite graph on two groups of m and n vertices respectively, where each vertex from the first group is joined to each vertex of the second group, and no two vertices from the same group are joined. A map on some surface S (including the plane) is an injective continuous mapping $f : G \to S$ from some graph G to S. The images of the vertices are also called vertices, and the images of the edges are also called edges. So, edges of the map are some “curves” on surface S with no selfintersection, each two of which could intersect only at possible common vertex.

We could think of a map as of some "curved" image of a graph embedded in the surface. If there is a map of the graph G to surface S, we say that graph G could be embedded in S. If a graph could be embedded in the plane, than it could be embedded in the sphere as well, since the sphere without one point is homeomorphic to the plane (e.g. - stereographic projection). The vice-versa is true as well.

Namely, if there is an embedding of a graph in the sphere, there is a point on the sphere not belonging to the image of the embedding. (Otherwise this embedding would be a homeomorphism, which of course could not exist.) But, then a graph could be embedded in the sphere without a point, which is homeomorphic to the plane. We call such graphs planar.

The connected components of the complement of the map are called its regions, and each map determines a finite number of regions. If we consider a map in the plane, then some of its regions are unbounded and the other regions are bounded. On every closed surface (compact, without boundary), all regions have compact closures. We will be especially interested in the maps on closed surfaces whose regions are homeomorphic to an open disc. Note that this forces the graph G to be connected. In the case of the plane, there is one unbounded region which could not be homeomorphic to an open disc, and the graph G need not to be connected. In this case we impose the additional condition to the graph G to be connected. If we consider the image of a bipartite graph, then each region is bounded by at least four vertices and edges. Each map f : G → S determines a graph G* (in some sense) dual to the graph G, where the vertices of G* correspond to the regions of the map, and two vertices of G* are joined if and only if corresponding regions share a common edge on their boundaries. Note that this dual graph G* comes with a natural embedding in the surface S. Note also that this notion depends on a map of the graph G and not only on the graph G itself. It is obvious that a dual graph contains no loops or multiple edges by definition.

Under some assumptions, there would be a construction in the opposite direction. Starting from the graph G* embedded in S, there would be such graph G and a map f : G → S so that the dual graph is exactly the original graph G*. However, we will not use it here.

Euler Characteristic

Given a graph G = (V;E) with v vertices and e edges, its Euler characteristic is defined to be X(G) = v – e. For a graph G embedded in the plane, the following idea was used in to simply determine its Euler characteristic. In order to simplify the presentation, we present the idea in the case of the graph with no loops, although it works in general. Let us choose a line L in that plane, and project the drawing of G to the line L. We could choose a generic line L in the plane, so that no two vertices of G project to the same point in L, and so that every point x on L is the projection of finitely many, n(x), points from G. This function n(x) is constant in some intervals and let us donated by $a_0, a_1, \ldots, a_K$ the successive endpoints of these intervals in one direction.

These are the points of the line L in which function n(x) changes its value. We want hat every point which is projected to some of these points is a vertex of G. If some point which is not a vertex projects to some ai, we could introduce it as a vertex and split the edge containing it in two edges. In this process we increase the number of vertices and edges by 1 and so, preserve the Euler characteristic. If some vertex projects to a point in some open interval (a_i, a_{i+1}), it is an endpoint of two edges, and such a vertex could be removed from the set of vertices by joining these two edges in one edge (in a kind of reverse process). Notice that the Euler characteristic is again preserved, since we decrease the number of vertices and edges by 1. By applying these two processes we obtain an embedded graph whose vertices (and only vertices) are all projected to the points $a_0, a_1, \ldots, a_K$. So, the number of vertices of this graph is $\sum_{i=0}^{k} n(a_i)$. Also, the vertices of every edge are projected to two consecutive points in the order $a_0, a_1, \ldots, a_K$. Let us now choose some points $b_1 \in (a_0, a_1), \ldots, b_k \in (a_{K-1}, a_K)$ arbitrarily in these intervals. Each point projected to the point b_i belongs to an edge connecting two vertices which are projected to the points a_{i-1} and a_i. Different points projected to the same point bi correspond to different edges of G, and so $n(b_i)$ counts the number of edges whose vertices project to the points a_{i-1} and a_i. So, the number of edges of this graph is $\sum_{i=0}^{k} n(b_i)$, and we summarize all this in the following theorem.

Let us now consider a map f : G → S on the closed surface S, with v vertices, e edges, which determines r regions, all of which are homeomorphic to a disc. The Euler characteristic of this map is defined to be X(f) = v e + r.

In the case of the plane we denote by r the number of bounded regions (so we do not count one unbounded region), and also define the Euler characteristic in the same way as X(f) = v e + r.

In this case, we require that all regions except for the unbounded ones are homeomorphic to the disc, and that graph G is connected. The Euler- Poincare formula expresses this quantity in terms of Betti numbers of the surface S (their alternating sum). As a consequence, it turns out that the Euler characteristic is a topological invariant of the surface, and it does not depend on a map but only on the surface. We denote it by X(S).

Topological invariants

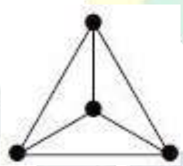
In order to understand if it should be true that the Euler characteristic "reads" the information of what kind of surface a certain polyhedron can be drawn on, we must understand what kinds of changes to a graph leave the Euler characteristic unchanged.

Lemma 3.1.

The Euler characteristic of a polyhedron does not change when we subdivide a face or an edge.

Proof : As an example, consider the following triangle that may be part of the underlying graph of a polyhedron :

Now, if we subdivide the face of triangle, this involves adding a single vertex, and then 3 edges to connect this vertex to the other 3 vertices of the triangular face :



In general, a face might be an n-gon. We still add a single vertex, so the number of vertices V changes by ΛV = 1.

This vertex must connect to all outer vertices on the n-gon, of which there are n. This means that E increases by n, so ΛE = n. These new edges divide the face into n compartments (sub-faces) where there used to just be 1. Thus F has increased by ΛF = n + 1. So in total the Euler characteristic changes by

$$\Lambda X_E = \Lambda V - \Lambda E + \Lambda F$$
$$= 1 - n + n - 1$$
$$= 0 .$$

So the process of subdividing a face does not change the global Euler characteristic. Similarly, if we "subdivide" an edge we introduce one new vertex in the middle of the edge, so ΛV = 1. We also have divided one edge into two, so ΛE = 1. Thus in total

$$\Lambda X_E = \Lambda V - \Lambda E + \Lambda F$$
$$= 1 - 1 + 0$$
$$= 0$$

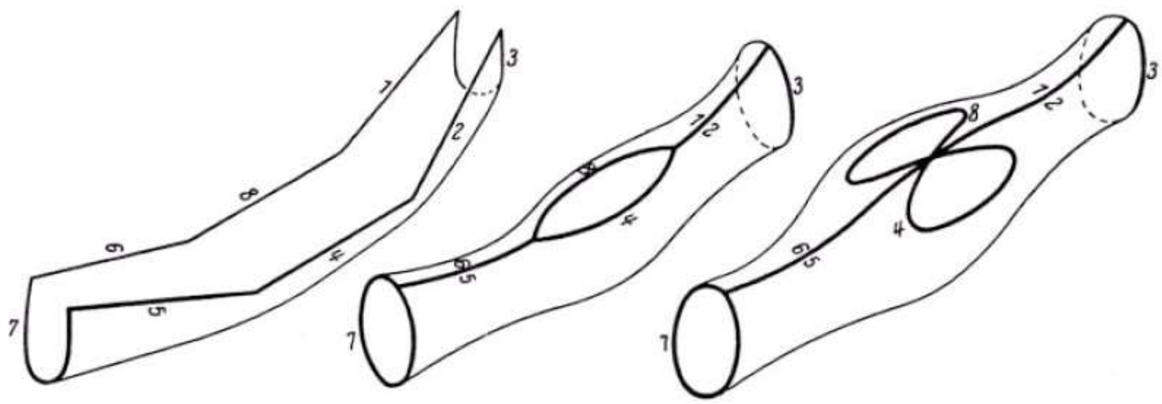
And again this process does not affect the global Euler characteristic.

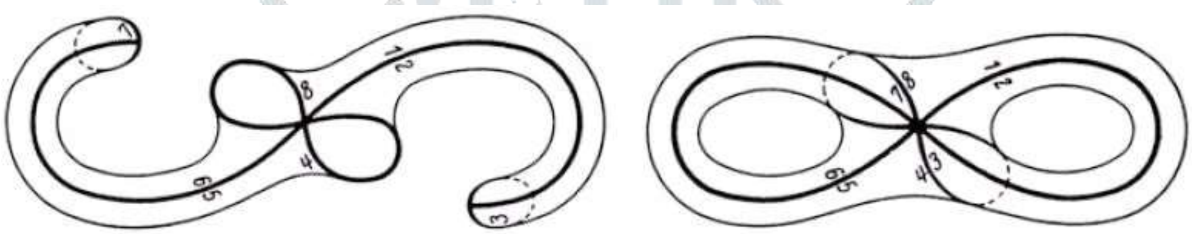
Theorem

Let Mg be a g-hole torus. Then we have $X_E(M_g) = 2 - 2g$. Remark. Note that in the cases we already know, this theorem agrees with what we have calculated. When g = 0, the surface is just the sphere, and so $X_E(M_0) = 2 - 2.0 = 2$.

And when g = 1, the surface is the torus so, $X_E(M_1) = 2 - 2.1 = 0$.

Diagrams describing polygons and torus with holes:





Gauss Bonnet

I now define the remaining elements in the Gauss-Bonnet formula and state its classical theorem.

Definition: The boundary of a surface M, is the set of points in M not belonging to the interior of M.

Definition: The boundary of a discrete surface is the set of edges that are contained in only one face.

Definition: Consider the unit tangent vector T of a curve C parametrized with respect to arc length at a point p. The geodesic curvature of C at p is the algebraic value of the covariant derivative of T at p. Intuitively, geodesic curvature measures how far a curve is from being a geodesic.

Definition: The exterior angle at the junction of two piecewise differentiable curves is the angular difference between the tangent vectors at the junction.

Definition: A subset S c R^n is a regular surface if for each p € S, there exists a neighborhood V in R^n and a map X : U → V ∩ S of an open set U c R^2 onto V ∩ S c R^n such that X is differentiable, X is a homeomorphism, and for each q € U the differential dXq : $R^2 \rightarrow R^3$ is one to one.

Definition: A simple region R of a surface S is a region such that R is homeomorphic to the disk.

Definition: Given a parametrization of a surface, X : U → S; we have the following quantities

These are the coefficients of the first fundamental form of a surface.

$$E = \langle x_u, x_u \rangle, F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle.$$

Definition: Given a parametrization of a surface, X : U → S ; we call that parametrization orthogonal if F = 0:

Definition: The geodesic curvature k_g of a curve is a measure of the amount of deviance of the curve from the shortest arc between two points on a surface.

Definition: The Gaussian curvature *k* of a surface is an intrinsic measure of the curvature of a surface at a point. It is calculated by considering the maximal and minimal curvatures on the surface at a point. Formally, these values are multiplied to give *k*.

Theorem (Gauss-Bonnet, Local}. *Set R C x [U) be a* maple region *o/ S with orthogonal param el* ri *zation, and* choose n : *I S such that a(I) —— dR. Assume* that n is *positinelg* oriented *and paramelrized precede bg* nrc-fenpth si. Ki *(8 ; ;*_ g be* the *ez:ternal angles oJ o at the ve rtices* {n(si))J 0, Men

$$\sum_{e=0}^{k}\int_{s_i}^{s_{i+1}} k_{g(s)}ds + \iint_{F} \kappa d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi.$$

Proof. We first let u == u(s) and r == r(s) be the expression of n in the parametrization x. We recall that

$$k_{g(s)} = \frac{1}{2\sqrt{EG}}(G_u\frac{dv}{ds} - E_v\frac{du}{ds}) + \frac{d\varphi}{ds},$$

where we denote the digerentiable function that measures the positive angle from x to n'(s) in /s, . •.+i) p(s i). We now integrate the above expression, adding up the vulues for each /s, , sigh) :

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_{g(s)} = \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds} - E_v\frac{du}{de}\right)ds + \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{dy}{de}ds,$$

now, using the Gauss-Green theorem in the us-plane on the right hand side of the above equation, we obtain the expression:

$$\iint_{\mathbf{x}^{-1}(R)} \frac{E_v}{2\sqrt{EG_v}} + \frac{G_u}{2\sqrt{EG_u}} \cdot dudv + \sum_{i=0}^{k}\int_{s}^{s_{i+1}} \frac{d\varphi_i}{de} ds.$$

also, recalling the Theorem of Turning Tangents, we know that,

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds}ds = \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \varphi_i(s_i+1) - \varphi_i(s_i) = \pm 2\pi - \sum_{i=0}^{k}\theta_i,$$

which we get because the theorem does not account for the discontinuities along the curve at the theta values. As we have assumed a positive orientation, we have,

$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_{g(s)}ds + \iint_{R} \kappa d\sigma + \sum_{i=0}^{k}\theta_i = 2\pi,$$

we note that we can obtain the opposite sign by assuming the opposite orientation, and we thus we have proven the local case of the Gauss-Bonnet Theorem.

Global Gauss-Bonnet Theorem

We have proven the local case of this theorem, and the global theorem tells us similar information. We prove this generalization by using the local theorem in each triangular region of our triangulation for the given surface. This theorem leads to a series very deep corollaries.

Theorem . *Let $R \subset S$ be a regular region of an oriented surface. Let ∂R be made up by closed, piecewise, simple, regular curves*

$$C_1 \dots C_n,$$

then

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi\chi(R).$$

Proof. Let J denote a triangulation of R such that each triangle T_j is contained in a neighborhood of orthogonal parametrizations compatible with the orientation of S. We note that such a triangulation exists by the proposition proven above. Now, we simply apply the local Gauss-Bonnet theorem to each T_j of the above triangulation, and we have:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{j,k=1}^{F,3} \theta_{jk} = 2\pi F,$$

where the indexing of each theta accounts for each external angle of the triangles in J. We note that F here is the number of faces in our triangulation. We denote the interior angles of the triangles by $\varphi_{jk} = \pi - \theta_{jk}$. We calculate, in general, that,

$$\sum_{i,k=1} \theta_{jk} = 3\pi F - \sum_{i,k=1} \varphi_{jk}.$$

Now, we introduce notation to assist in counting the vertices and edges of our triangulation, so, the number of external/internal edge and vertices are V_e, E_e and V_i, E_i, respectively. Since the C_i are closed, however, we know $V_e = E_e$, and thus, inductively, $3F = 2E_i + E_e$.This implies,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E_i + \pi E_e - \sum_{j,k=1} \varphi_{jk}.$$

We note that the vertices must belong to either some T_j or a C_i, so $V_e = V_{et} + V_{ec}$, and then, since the sum of the angles around each internal vertex is 2π,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E_i + \pi E_e - 2\pi V_i - \pi V_{et} - \sum_i (\pi - \theta_i).$$

By adding πE_e and subtracting πE_e to the right hand side of the above equation, we have,

$$\sum_{j,k=1} \theta_{jk} = 2\pi E - 2\pi V + \sum_i \theta_i.$$

Now, we collect the terms to find:

$$\sum_{i=0}^{k} \int_{C_i} k_{g(s)} ds + \iint_R \kappa d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi(F - E + V).$$

But, by the definition of a triangulation $F - E + V = \chi(R)$, thus,

$$= 2\pi\chi(R).$$

Applications of the Euler Characteristic

Brussel Sprouts.

In the course of playing Brussels Sprouts, you create a geometric object known as a planer graph. The game of Brussels Sprouts goes as follows:

- Draw some number n of four-pointed crosses. In the game below, n = 2.
- On each round, players alternate joining crosses by drawing a line from one point of one cross, to

another point of another cross, without intersecting any the other lines already drawn. Then adding a dash in the middle of the line, so as to create another cross in the middle (a cross that is already occupied by in two positions)

- The game ends when it is no longer possible to add any lines without there being an inter-section somewhere.
- Here is an example of a game starting with 2 crosses. As you will see, there are 8 rounds and player 2 wins (Figure-1). In fact, a game with 2 crosses will always be a win for the second player, and you can prove this using the Euler characteristic.

Platonic Solids :

The Platonic solids are polyhedra built out of regular polygons (like squares, pentagons, triangles) such that the number of edges meeting each vertex is constant (the same for each vertex).

Fixed Point Theory :

Let S be a surface. Then in an appropriate sense that I cannot make less vague here, XE equals the sum of the fixed points of S (with appropriate coefficients) under an infinitesimal deformation of S. Consider the case that S is just a sphere. Then one such "infinitesimal deformation" of S is just to draw a line through the sphere, passing through the north and south poles and rotate the sphere about this axis. This map has 2 fixed points, the north and south poles where the line forming the axis of the rotation met the sphere. We can visualize this idea by thinking of a "hairy" sphere. Then by combing the hairs in some particular way gives us a map of the sphere, send each point to the corresponding point at the end of the ahir. A fixed point would correspond to a hair pointing straight up, i.e. a cowlick of hair. The fact that XE of a sphere is 2 means that any "combing" of the sphere always has at least one cowlick! This is also known as the hairy-ball theorem.

References

1. A Guide to the Classification Theorem for Compact Surfaces: By Jean Gallier, Dianna Xu.
2. Renzo's Math 490 : Introduction to Topology.
3. An Introduction to Topology The Classification theorem for Surfaces - E. C. Zeeman.
4. Topology of surfaces - L.Chiristine Kinsey
5. Elementary Differential Geometry - Andrew Pressley