JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Analytical Solution of Transport of Pollutants in Two-Dimensional Advection Dispersion Equation with Adsorption

[1]Niranjan C M and [2]S R Sudheendra
[1]Assistant Professor and [2]Professor
[1]Department of Mathematics,
[1]Acharya Institute of Technology,
Bangalore, INDIA
niranjancm@acharya.ac.in

Abstract

Ground water plays a vital role in the field of agriculture, industry and for drinking purpose. Around 53% of the rural as well as urban population in the developing countries like India depends on groundwater for drinking. Because of the growing demand of water for various purposes resulting in wasteful use and contamination is taking place all over the world. In order to face this challenges it is much required to have proper plan for managing the wastage of water and chemical flow. Contaminants containing different chemicals will pass through different hydrogeological zones as they migrate through the soil to the water table. The water table is the upper surface of the groundwater system. The pore space between soil particles above the water table are occupied by both air and water. Flow in this unsaturated zone is vertically downward, as liquid contaminants or solutions of contaminants and precipitation move under the force of gravity. The solution is obtained for the given mathematical model in a finite length initially solute free domain. The input condition is considered continuous of uniform and of increasing nature both. The solution has been obtained using Laplace transform, moving coordinates and Duhamel's theorem is used to get the solution in terms of complementary error function.

Key words:

Advection, dispersion, adsorption, Integral transforms, Fick's law, Moving coordinates, Duhamel's theorem

1. Introduction

Many researchers are working the field of agriculture, hydrology and in the field of engineering by choosing an important topic as the transport of chemicals in porous media using the advection and dispersion equations. Still the investigation is active in all dimensions and this transport properties and mass balancing equations have been studied in this present study. The dispersion in the direction of flow (longitudinal dispersion) is noticeably different from the dispersion perpendicular to the direction of flow (transverse dispersion).

The solutions of two-dimensional deterministic advection-dispersion equations have been investigated in numerous publications before and are still actively studied. Previous works closely related to the work carried out by Yeh (1981), Domenico and Robbins (1984), Domenico (1987), Batu (1989, 1993) provided a two-dimensional analytical solute transport model in a bounded aquifer by using the same source dimensional analytic solute transport model in a bounded aquifer by using the same source dimension as the aquifer thickness along the z-axis and include the contaminant source as a boundary condition. An analytical approach was developed for non-equilibrium transport of reactive solutes in the unsaturated zone during an infiltration–redistribution cycle (Severino and Indelman 2004, Sudheendra and Niranjan 2014). Analytical solutions were presented for solute transport in rivers including the effects of transient storage and first order decay (Smedt 2006, Sudheendra 2012).

Here the solutions have one or the other restrictions. Therefore, our main objective is to locate an analytical solution for two dimensional transport problem which accounts for both longitudinal and transverse dispersion. The advective flux can be easily obtained by solving the contaminated groundwater flow problem and the dispersive flux is because of differences in concentration. The solution is obtained by using advection and dispersion. The Advection-dispersion equation is a useful tool for research to prevent and control the groundwater pollution. The total solute flux and advective flux are the two terms got by achieving the solutions of contamination flow problem. In the present work, we have considered the longitudinal dispersion coefficientD_L, and the transverse dispersion coefficientD_T. The Values of D_T are more complicated to obtain than values for D_L , because of the concentration distribution are required to be measured in perpendicular to the flow direction.

2. Mathematical Formulation and Model

Mathematical Models are being applied extensively in groundwater studies. Groundwater modelling can be classified into two models namely, the groundwater flow and solute transport models. The solute transport models are applied in connection with groundwater quality problems. The solute transport models are often extended with chemical sub-models for description of the fate of non-conservative polluting species while in some cases may be sufficient only to study the aquifer. It is often necessary to include some of the overlying layers in the hydro-geological description. This is of importance of both with respect to estimation of groundwater recharge and the assessment of the sources of groundwater pollution. Hence, a comprehensive groundwater modelling package has to include models for the unsaturated zone as well as integrated ground water/surface water models.

The Perception is assumed that the transport of solute is obtained with advection-dispersion equation. Consider the two-dimensional dispersion with one dimensional steady state flow is of the form

$$\frac{\partial C}{\partial t} = D_L \frac{\partial^2 C}{\partial z^2} - v\frac{\partial c}{\partial z} + D_T \frac{\partial^2 C}{\partial x^2} - \left[\frac{1-n}{n}\right] K_d C \qquad (1)$$

$$\frac{\partial C}{\partial t} = Dz \frac{\partial^2 C}{\partial z^2} - v\frac{\partial c}{\partial z} + Dx \frac{\partial^2 C}{\partial x^2} + Dy \frac{\partial^2 C}{\partial y^2} - \left[\frac{1-n}{n}\right] K_d C$$

The water flow is assumed strictly vertical and consequently the same restriction applies to the convective transport of the dissolved solutes and model allows for both longitudinal and transverse dispersion. The model can in its present version simulate the dispersion process from an area and line sources. The convective transport is solved by displacing the concentration pro le according to the velocity field. The changes in concentration concerned by hydrodynamic dispersion and chemical reactions are subsequently superimposed by the convicted concentration profile.

If the aquifer is isotropic, then dispersion coefficient may be characterized by a longitudinal and a transverse coefficient (Fig.1). For this problem conditions are as follows:

$$C(x,z,0) = f(x), 0 < z < \infty, \quad -\infty < x < \infty \qquad (2a)$$

$$\left.\frac{\partial C}{\partial z}\right|_{z\to\infty} = 0, \quad -\infty < x < \infty, t > 0 \qquad (2b)$$

$$\left.\begin{aligned} C(0,x,t) &= g(x) = C_L, x < 0, \quad t > 0 \\ C(0,x,t) &= g(x) = \frac{C_L + C_R}{2}, x = 0, \quad t > 0 \\ C(0,x,t) &= g(x) = C_R, x > 0, \quad t > 0 \end{aligned}\right\} \qquad (2c)$$

$$\left.\frac{\partial C}{\partial x}\right|_{x\to\pm\infty} = 0, \quad 0 < z < \infty, t > 0 \qquad (2d)$$

Further the solutions continued by Harleman and Rumer (1963), ignoring the longitudinal dispersion $\left(D_L\left(\frac{\partial^2 C}{\partial z^2}\right) << D_T\left(\frac{\partial^2 C}{\partial x^2}\right)\right)$. The simultaneous determination of D_L and D_T was simple to obtain the solution. The required solutions for infinite conditions for the finite model are presented using separation of variables by Brunch and street (1967).

The boundary conditions as equation (2a) is subjected to find the solution of equation (1), which can be obtained with integral transform as, Fourier transform for x variable and Laplace transform for variable t.

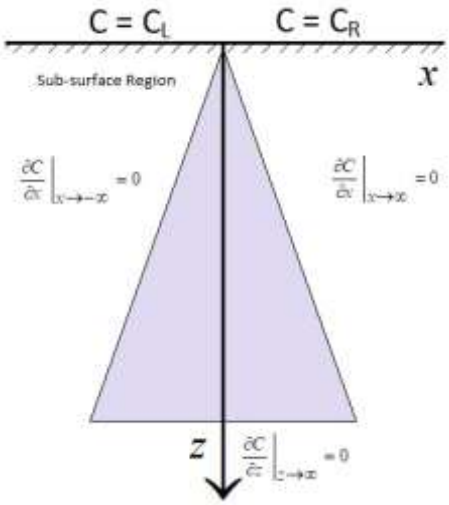

Fig.1 : Schematic of one-dimensional advection and two-dimensional dispersion in a half plane for an isotropic medium

The solution for the problem is solved first by using the well-known Laplace transforms applications for C and $\frac{\partial C}{\partial t}$ with respect to t the utilization of the initial condition (2a), for the transform equation (1) is transformed in the form

$$L[C(x,z,t)] = \int_0^\infty \exp(-pt)\, C(x,z,t)dt = C'(x,z,p) \tag{3}$$

$$L\left[\frac{\partial C}{\partial t}\right] = pC'(x,z,p) - C(x,z,0) \tag{4}$$

Applying the Laplace transforms to equations (1) and (2), it transforms in the form

$$pC'(x,z,p) - C(x,z,0) = D_L \frac{\partial^2 C}{\partial z^2} - v\frac{\partial c}{\partial z} + D_T \frac{\partial^2 C}{\partial x^2} - \left[\frac{1-n}{n}\right] K_d C$$

$$pC'(x,z,p) - f = D_L \frac{\partial^2 C'}{\partial z^2} + D_T \frac{\partial^2 C'}{\partial x^2} - u_1 \frac{\partial C'}{\partial x} - v_1 \frac{\partial C'}{\partial z} - \left[\frac{1-n}{n}\right] K_d C' \tag{5}$$

and boundary conditions reduces to

$$\left.\frac{\partial C'}{\partial z}\right|_{z\to\infty} = 0 \tag{6a}$$

$$C'(0,x,p) = \frac{g}{p} \tag{6b}$$

$$\left.\frac{\partial C'}{\partial x}\right|_{x\to\pm\infty} = 0 \tag{6c}$$

For the infinite x domain the Fourier transform is applied and for C' and $\frac{\partial^2 C'}{\partial t^2}$ the Fourier transform is given by

$$F[C'(z,x,p)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\alpha x) C'(z,x,p)dx = C'(z,\alpha,p) \tag{7}$$

And

$$F\left[\frac{\partial^2 C'}{\partial x^2}\right] = -\alpha^2 C' \tag{8}$$

The Fourier transforms of equation (5) and (6) therefore becomes

$$D_L \frac{d^2\overline{C}'}{dz^2} - D_T\alpha^2\overline{C}' - u_1\frac{d\overline{C}'}{dx} - v_1\frac{d\overline{C}'}{dz} - \left[\frac{1-n}{n}\right] K_d\overline{C}' - p\overline{C}'(x,z,p) + f = 0$$

$$D_L \frac{d^2\overline{C}'}{dz^2} - v_1 \frac{d\overline{C}'}{dz} - u_1 \frac{d\overline{C}'}{dx} - (D_T\alpha^2 + p + \left[\frac{1-n}{n}\right]K_d)\overline{C}' + \bar{f} = 0 \tag{9}$$

And the reduced boundary conditions are

$$\left.\begin{aligned} &\left.\frac{d\overline{C}'}{dz}\right|_{z\to\infty} = 0 \\ &\overline{C}'(0,x,p) = \frac{\bar{g}}{p} \\ &\left.\frac{d\overline{C}'}{dx}\right|_{x\to\pm\infty} = 0 \end{aligned}\right\} \tag{10}$$

The Fourier transforms of f and g are $\bar{f}$ and $\bar{g}$

The transport equation (9) which is an ODE, subjected to the boundary condition (10) is of the form

$$\overline{C}'(z,\alpha,p) = A\exp(R_1 z) + B\exp(-R_2 z) + \frac{\bar{f}}{D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p} \tag{11}$$

where

$$R_1 = \frac{v_1}{2D_L} + \frac{\sqrt{v_1^2 + 4D_L(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p)}}{2D_L} \tag{12a}$$

$$R_2 = \frac{v_1}{2D_L} - \frac{\sqrt{v_1^2 + 4D_L(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p)}}{2D_L} \tag{12b}$$

Applying the inner boundary condition of equation (10), we arrive at

$$B = \frac{\bar{g}}{p} - \frac{\bar{f}}{D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p} \tag{13}$$

Substituting (13) in (11) we get

$$\overline{C}'(z,\alpha,p) = \left[\frac{\bar{g}}{p} - \frac{\bar{f}}{D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p}\right]\exp(Rz) + \frac{\bar{f}}{D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p} \tag{14}$$

Inverse Laplace transform is applied for the domain z, x, t in order to obtain the solution. This transport can also be solved using various numerical technique but we prefer to solve it using analytical method. The RHS of equation (14) has divided into sum of three terms, among the first term reduces to

$$\overline{C}(z,\alpha,p) = L^{-1}\left[\frac{\bar{g}}{p}\exp(Rz)\right]$$

$$\overline{C}(z,\alpha,p) = \bar{g}\exp\left[\frac{v_1 z}{2D_L}\right]L^{-1}\left\{\frac{1}{p}\exp\left(-\frac{z}{\sqrt{D_L}}\left(\frac{v_1^2}{4D_L} + D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p\right)^{\frac{1}{2}}\right)\right\} \tag{15}$$

by the application of convolution theorem it is more preferable to solve the inverse Laplace transform than inverse Fourier transform.

$$f * g = \int_0^t f(\tau)g(t-\tau)d\tau$$

Here the convolution $f * g$ is a function of t. The main limitation of this convolution is finding the Laplace transform of any two functions without evaluating the integral.

Consider

$$L^{-1}[h'(p)k'(p)] = h * k$$

$$L^{-1}[h'(p)k'(p)] = \int_0^t h(\tau)k(t-\tau)d\tau = \int_0^t h(t-\tau)gk(\tau)d\tau \tag{16}$$

Here τ is a Replica variable. The difference in two functions in the expression (15) is of the form

$$\left.\begin{aligned} h'(p) &= \frac{1}{p} \\ k'(p) &= \exp\left[-\frac{z}{\sqrt{D_L}}\left(\frac{v_1^{\,2}}{4D_L} + D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d + p\right)^{\frac{1}{2}}\right] \end{aligned}\right\} \tag{17}$$

The values of $h(t)$ and $k(t)$ are evaluated using shift theorem and by the table of Laplace transforms [Oberhettinger and Baddi (1973)], given by

$$\left.\begin{aligned} h(t) &= 1 \\ k(t) &= \frac{z}{t\sqrt{4\pi D_L}}\exp\left[-\left(\frac{v_1^{\,2}}{4D_L} + D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)t - \frac{z^2}{4D_L t}\right] \end{aligned}\right\} \tag{18}$$

Substitution of (18) into equation (16) and subsequently into equation (15) reduces to

$$\overline{C}'(z,\alpha,p) = \frac{\overline{g}z}{\sqrt{4\pi D_L}}\int_0^t \tau^{\frac{3}{2}}\exp\left[\frac{-(z-v_1\tau)^2}{4D_L\tau}\right]\exp\left[(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)\tau\right]d\tau \tag{19}$$

By using the table of Laplace transforms, the expression (19) can be transformed in the form of complimentary error function. For the integral the error function of probability is defined as

$$erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\eta^2}d\eta$$

Similarly the complementary error function of $f(z)$ is given by

$$erfc(z) = 1 - erf(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\eta^2 d\eta}$$

Now applying the complimentary error function to the expression (19), becomes

$$\begin{aligned} \overline{C}_1(z,\alpha,t) = \frac{\overline{g}}{2}\exp\left(\frac{v_1 z}{2D_L}\right)\Bigg\{&\exp\left[\frac{z}{\sqrt{D_L}}\left(\frac{v_1^{\,2}}{4D_L} + D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)^{\frac{1}{2}}\right] * \\ &erfc\left[\frac{z}{\sqrt{4D_L T}} + \sqrt{\frac{v_1^{\,2}t}{4D_L} + D_T\alpha^2 t + \left[\frac{1-n}{n}\right]K_d t}\right] + \\ &\exp\left[\frac{-z}{\sqrt{D_L}}\left(\frac{v_1^{\,2}}{4D_L} + D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)^{\frac{1}{2}}\right] * \\ &erfc\left[\frac{z}{\sqrt{4D_L T}} - \sqrt{\frac{v_1^{\,2}t}{4D_L} + D_T\alpha^2 t + \left[\frac{1-n}{n}\right]K_d t}\right] \end{aligned} \tag{20}$$

Inverse Fourier transform of the expression (20) appears more difficult than for expression (14). The Fourier transforms variable displays in the argument of both exponential and error functions. Applying the inverse Fourier transform to second term $\overline{C}_2(z,\alpha,p)$ of RHS of the expression (14), we have

$$\bar{C}_2(z,\alpha,t) = L^{-1}\left[-\frac{\bar{f}}{D_T\alpha^2 + p + \left[\frac{1-n}{n}\right]K_d}\exp(Rz)\right] \tag{21}$$

$$\bar{C}_2(z,\alpha,t) = -\frac{\bar{f}}{2}\exp\left[-(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)t\right]\left\{erfc\left[\frac{z-v_1t}{\sqrt{4D_Lt}}\right] + \exp\left[\frac{v_1z}{D_L}\right].erfc\left[\frac{z+v_1t}{\sqrt{4D_Lt}}\right]\right\} \tag{22}$$

Now applying inverse Laplace transform of the third term of the RHS of expression (6.2.14) can be written in the form [Oberhettinger and Baddi (1973)]

$$\bar{C}_3(z,\alpha,t) = L^{-1}\left[\frac{\bar{f}}{D_T\alpha^2 + p + \left[\frac{1-n}{n}\right]K_d}\right] = \bar{f}\exp\left[-(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)t\right] \tag{23}$$

Finally the required expression for $\bar{C}(z,\alpha,t)$ can be written in the form

$$\bar{C}(z,\alpha,t) = \bar{C}_1(z,\alpha,t) + \bar{C}_2(z,\alpha,t) + \bar{C}_3(z,\alpha,t) \tag{24}$$

The final step of the solution is the other form of the application of the Fourier inversion to all terms of the expression (24). The inverse Fourier transform of $\bar{C}(z,\alpha,t)$ is given by

$$C(z,x,t) = F^{-1}\left[\bar{C}(z,x,t)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-i\alpha x]\bar{C}(z,\alpha,t)\,d\alpha \tag{25}$$

It is left with the Fourier inversion of the first term of the expression (24) that can be expressed as

$$F^{-1}[C_1(z,x,t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\alpha x)\frac{\bar{g}z}{\sqrt{4\pi D_L}} *$$

$$\int_0^t \tau^{-\frac{3}{2}}\exp\left[\frac{(z-v_1\tau)^2}{4D_L\tau}\right]\exp\left[-(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)\tau\right]d\tau d\alpha$$

$$F^{-1}[C_1(z,x,t)] = \frac{z}{\sqrt{4\pi D_L}}\int_0^t \tau^{-\frac{3}{2}}\exp\left[-\frac{(z-v_1\tau)^2}{4D_L\tau}\right] *$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-i\alpha x)\left\{\bar{g}\exp\left[-(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)\tau\right]\right\}d\alpha\, d\tau \tag{26}$$

Applying convolution we have,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{h}(\alpha)\bar{k}(\alpha)\exp(-i\alpha x)d\alpha = h*k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}h(x-\gamma)k(\gamma)\,d\gamma \tag{27}$$

Where γ is the dummy variable. After verifying the last term for $C_1(z,x,t)$ in equation (26), using the convolution integral for the other two functions are

$$\left.\begin{aligned}\bar{h}(\alpha) &= \bar{g}\\ \bar{k}(\alpha) &= \exp\left[-\left(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)\tau\right]\end{aligned}\right\} \tag{28}$$

It is obvious that by the Fourier transform of the single step function we must have $h(x)$ should be equal to $g(x)$, by not determining $\bar{g}$ and to find the inverse transform of $\bar{k}(\alpha)$, we have

$$F^{-1}\left[\exp\left[-\left(D_T\alpha^2+\left[\frac{1-n}{n}\right]K_d\right)\tau\right]\right]=\frac{1}{\sqrt{2(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}\exp\left[-\frac{x^2}{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}\right] \tag{29}$$

The resulting $h(x)$ and $k(x)$ are

$$\left.\begin{aligned} h(x)=g(x)=C_L,\ x<0,\\ h(x)=g(x)=\frac{C_L+C_R}{2},\ x=0\\ h(x)=g(x)=C_R,\ x>0,\end{aligned}\right\} \tag{30}$$

And

$$k(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}k(\alpha)\exp(-i\alpha x)d\alpha=\frac{1}{\sqrt{2(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}\exp\left[-\frac{x^2}{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}\right] \tag{31}$$

Now from equation (27) it can be followed that

$$h*k=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{g}{\sqrt{2(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}\exp\left[-\frac{(x-\gamma)^2}{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}\right]d\gamma \tag{32}$$

The above expression can be evaluated by

$$\rho=\frac{\gamma-x}{\sqrt{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}$$

The conditions of equation (30) and the complimentary error function [Abramowiz and Stegan (1970 and Crank (1975))] converts to

$$h*k=\frac{C_L}{2}erfc\left[\frac{x}{\sqrt{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}\right]+\frac{C_R}{2}erfc\left[\frac{x}{\sqrt{4(D_T+\left[\frac{1-n}{n}\right]K_d)\tau}}\right] \tag{33}$$

The result is then substituted in (30) to get the required expression for $C_1(z,x,t)$, is

$$C_1(z,x,t)=\int_0^t h*k=\frac{z}{\sqrt{4\pi D_L}}\int_0^t\tau^{-\frac{3}{2}}\exp\left[-\frac{(z-v_1\tau)^2}{4D_L\tau}\right]*$$

$$\left\{\frac{C_L}{2}erfc\left[\frac{x}{\sqrt{4(D_L+\left[\frac{1-n}{n}\right]K_d)\tau}}\right]+\frac{C_R}{2}erfc\left[\frac{x}{\sqrt{4(D_L+\left[\frac{1-n}{n}\right]K_d)\tau}}\right]\right\}d\tau \tag{34}$$

The inverse of the second term of the equation (26), $C_2(z,x,t)$ is given as follows

$$C_2(z,x,t)=-\frac{1}{2}\left\{erfc\left[\frac{z-\gamma t}{\sqrt{4D_Lt}}\right]+\exp\left(\frac{\gamma z}{D_L}\right)erfc\left[\frac{z+\gamma t}{\sqrt{4D_Lt}}\right]\right\}*$$

$$F^{-1}\left[\bar{f}\exp\left[-(D_T+\left[\frac{1-n}{n}\right]K_d)t\right]\right] \tag{35}$$

Thus,

$$\left.\begin{aligned}\bar{h}(\alpha) &= \bar{f} \\ \bar{k}(\alpha) &= \exp\left[-\left(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)\tau\right]\end{aligned}\right\} \tag{36}$$

Let us assume that the initial concentration C_i is constant and equation (29) is utilized to find $k(x)$,

$$\left.\begin{aligned}h(x) &= C_i \\ k(x) &= \frac{1}{\sqrt{2\left(D_T + \left[\frac{1-n}{n}\right]K_d\right)t}}\exp\left[-\frac{x^2}{4\left(D_T + \left[\frac{1-n}{n}\right]K_d\right)t}\right]\end{aligned}\right\} \tag{37}$$

The inverse transformation of equation is carried out by using the properties or error function

$$F^{-1}\left[\bar{f}\exp\left[-\left(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d\right)t\right]\right] = h * k =$$

$$= \frac{C_i}{\sqrt{4\pi(D_T + \left[\frac{1-n}{n}\right]K_d)t}}\int_{-\infty}^{\infty}\exp\left[-\frac{(x-\gamma)^2}{\sqrt{4\pi(D_T + \left[\frac{1-n}{n}\right]K_d)t}}\right]d\gamma = C_i \tag{38}$$

$$C_2(z,x,t) = -\frac{C_i}{2}\left\{erfc\left[\frac{z-\gamma t}{\sqrt{4D_L t}}\right] + \exp\left(\frac{\gamma z}{D_L}\right)erfc\left[\frac{z+\gamma t}{\sqrt{4D_L t}}\right]\right\} \tag{39}$$

Similarly $C_3(z,x,t)$ is already evaluated in equation (38) and we get

$$C_3(z,x,t) = F^{-1}\left[\bar{f}\exp\left[-(D_T\alpha^2 + \left[\frac{1-n}{n}\right]K_d)t\right]\right] = C_i$$

$$C_3(z,x,t) = \frac{C_i}{\sqrt{4\pi(D_T + \left[\frac{1-n}{n}\right]K_d)t}}\int_{-\infty}^{\infty}\exp\left[-\frac{x-\gamma}{\sqrt{4\pi(D_T + \left[\frac{1-n}{n}\right]K_d)t}}\right]d\gamma \tag{40}$$

Substitution of $C_1(z,x,t), C_2(z,x,t)$ and $C_3(z,x,t)$ in equation (24) reduces to

$$C(z,x,t) = \frac{z}{\sqrt{4\pi D_L}}\int_0^t \tau^{-\frac{3}{2}}\left\{\frac{C_L}{2}erfc\left[\frac{x}{\sqrt{4(D_T + \left[\frac{1-n}{n}\right]K_d)\tau}}\right] + \frac{C_R}{2}erfc\left[\frac{-x}{\sqrt{4(D_T + \left[\frac{1-n}{n}\right]K_d)\tau}}\right]\right\}$$

$$\exp\left[-\frac{(z-\gamma\tau)^2}{4D_L\tau}\right]d\tau - \frac{C_i}{2}\left\{erfc\left[\frac{z-\gamma t}{\sqrt{4D_L t}}\right] + \exp\left(\frac{\gamma z}{D_L}\right)erfc\left[\frac{z+\gamma t}{\sqrt{4D_L t}}\right]\right\} + C \tag{41}$$

Various numerical methods can be applied to evaluate equation (41), among the suitable is with Gauss-Chebyshev quadrature Leij and Dane (1989). The flow of solutes using the rough layer can be solved by splitting the variables D_T; D_T and v with the factor of retardation. The solution can be different for various initial and inlet conditions, like f and g, respectively.

3. Results and Conclusion

The Advection-dispersion equation is a useful tool for research to prevent and control the groundwater pollution. The total solute flux and advective flux are the two terms are obtained by solving the contaminant flow problem. The dispersive flux occurs when the differences in concentrations. The dispersion in the flow direction (longitudinal dispersion) is different from the dispersion in the perpendicular to the flow direction (transverse dispersion). The derived solution is an effective and useful for further application to verify the newly developed numerical transport model for predicting the two dimensional time-dependent transport of contaminants. The application results reveal that the solute transport process at the test site obeys the linearly time dependent dispersion model and that the linearly time-dependent assumption is valid in this real world example. The proposed solution can be applied to field problems where the hydrological properties of the medium and prevailing boundary and initial conditions are the same as, or can be approximated by, the ones considered in this study. For the different sections and for various $'n'$ the resulting break through curves (BTC) have been shown.

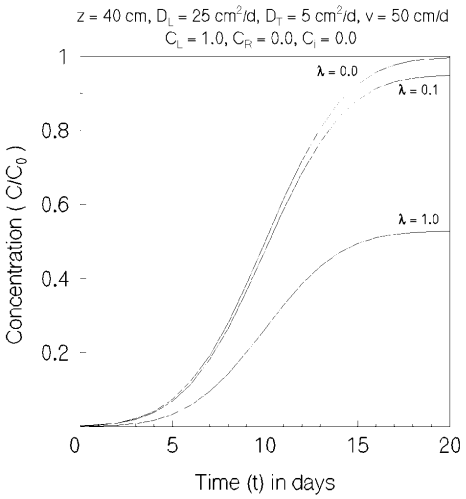

Fig 1: BTC for C/C_0vs depth for n = 1.0 at t = 1.00d. (z = 50cm, D_L = 25cm^2/d, D_T = 5 cm^2/d, v = 50cm/d, C_L = 1, C_R = 0, C_i = 0

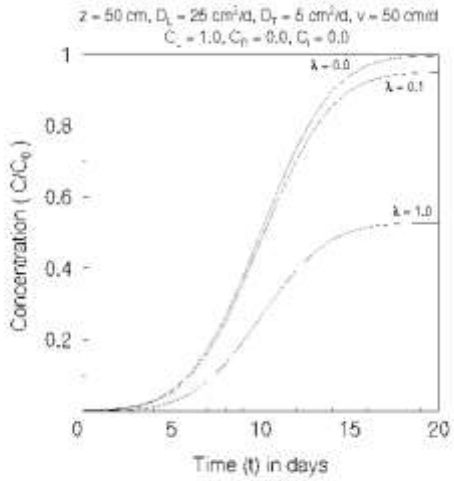

Fig 2: for n = 1.0 at t = 1.00d.(z = 50cm, D_L = 25cm^2/d, D_T = 5 cm^2/d, v = 50cm/d, C_L = 1, C_R = 0, C_i = 0)

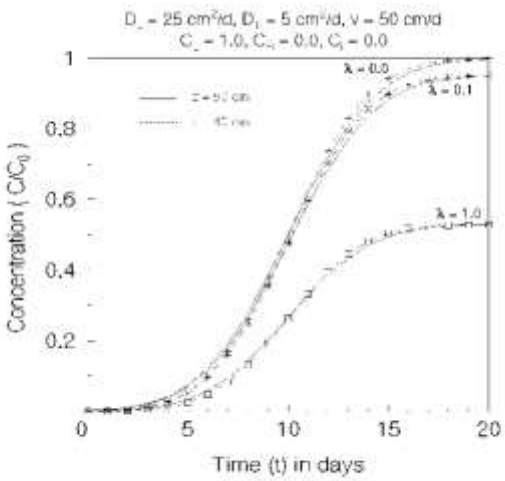

Fig 3: for n = 1.0 and 0.0 at t = 1.00d.(z = 50cm, D_L = 25cm^2/d, D_T = 5 cm^2/d, v = 50cm/d, C_L = 1, C_R = 0, C_i = 0)

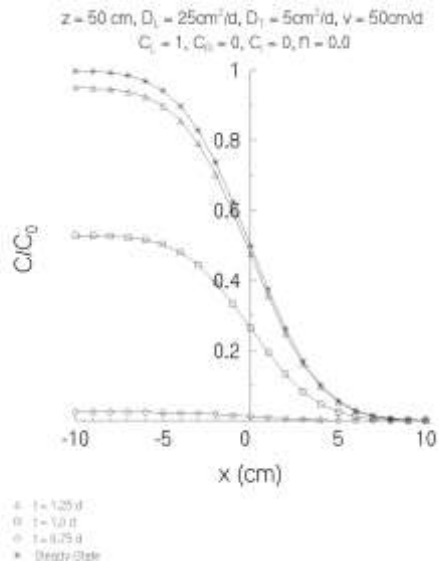

Fig 4: for n = 0.0, at t = 0.75d, 1.0d 1.25d, and steady-state (t = 2.0).

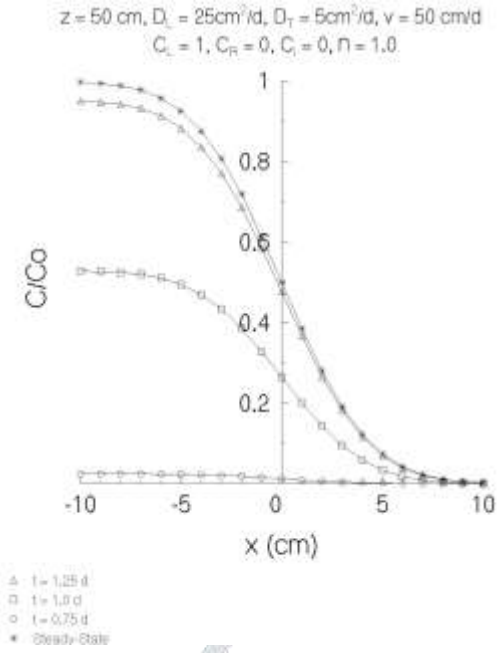

Fig 5: for n = 1.0, at t = 0.75d, 1.0d 1.25d, and steady-state (t = 2.0).

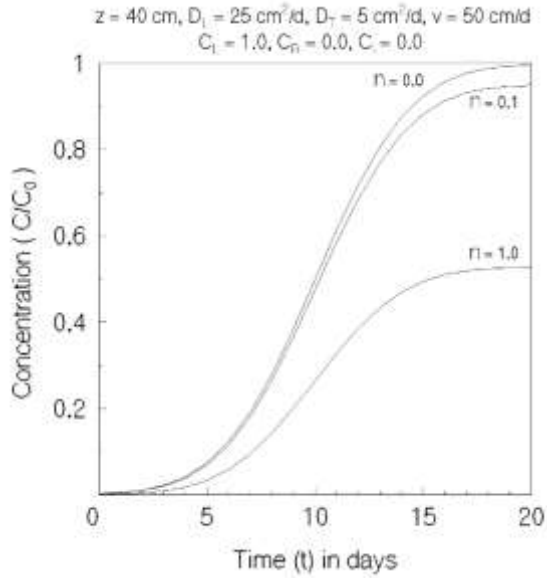

Fig 6: for n = 0, 0.1, 1.0 at t = 1.00d.

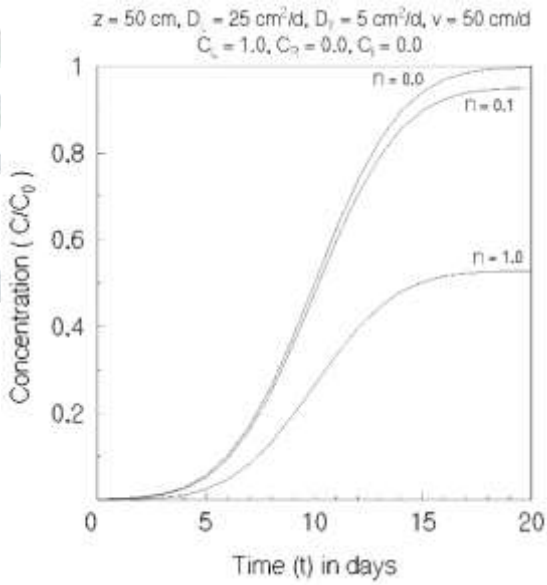

Fig 7: for n = 0, 0.1, 1.0at t = 1.00d.

4. References :

[1] A. Verruijt (1990): Modelling Groundwater Flow and Pollution. D Reidel Publishing Co., Tokyo.

[2] Al-Niami A N S and Rushton K R 1977 Analysis of flow against dispersion in porous media; J. Hydrol. 33 87–97.

[3] Aral M M and Liao B 1996 Analytical solutions for two-dimensional transport equation with time-dependent dispersion coefficients; J. Hydrol. Engg. 1(1) 20–32.

[3] A. Kumar, D. K. Jaiswal and N. Kumar, "Analytical Solutions to One-Dimensional Advection-Diffusion Equation with Variable Coefficients in Finite Domain," Journal of Earth System Science, Vol. 118, No. 5, 2009, pp. 539-549

[4] Bear, J., and A. Verruijt (1990): Modelling Groundwater Flow and Pollution. D Reidel Publishing Co., Tokyo.

[5] Branes, C.J (1989): Solute and water movement in unsaturated soils. Water Resour. Res., 25(1): 38-42.

[6] Bruce, J.C.; and Street, R.L. (1966). Studies of free surface flow and two-dimensional dispersion in porous media. Report No. 63, Civil Eng. Dept. Stanford University.

[7] Celia, M.A., E.T. Bouloutas and R.L. Zarba (1990): A general mass conservative numerical solution of the unsaturated flow equation, Water Resour. Res. 26 (7): 1483-1496.

[8] De Smedt, F. and Wierenga, P.J., 1979b. Mass transfer in porous media with immobile water. J. Hydrol., 41: 59967.

[9] Domenico, P.A. and G.A. Robbins 1984: A Dispersion Scale Effect in Model calibrations & Field Tracer Expts. J. of Hydrology 70: 123-132.

[10] Eungyer Park & Hongbin Zhan (2001) : Analytical solutions of contaminant transport from finite one-, two- and three-dimensional sources in a finite-thickness aquifer. J. of Contaminant Hydrology. 53 : 41-61

[11] Gelhar L W, Welty C and Rehfeldt K R 1992 A critical review of data on field-scale dispersion in aquifers; Water Resour. Res. 28(7) 1955–1974.

[12] Jaiswal, D.K., Kumar, A., Kumar, N., Yadav, R.R., 2009. Analytical solutions for temporally and spatially dependent solute dispersion of pulse type input concentration in one-dimensional semi-infinite media. Journal of Hydro environmental Research 2, 254–263.

[13] Kim Kue-Young, Kim T, Kim Y and Woo Nam-Chil 2007 A semi-analytical solution for ground water responses to stream–stage variations and tidal fluctuations in a coastal aquifers; Hydrol. Process. 21 665–674.

[14] Liping Pang & Bruce Hunt (2001): Solutions and verification of a scale-dependent dispersion model. J. Contam. Hydrol., in press.

[15] Pang L, Gottz M and Close M 2003 Application of the method of temporal moment to interpret solute transport with sorption and degradation; J. Contam. Hydrol. 60 123–134.

[16] Severino, G., Tartakovsky, D., Srinivasan, G., & Viswanathan, H. (2012). Lagrangian models of reactive transport in heterogeneous porous media with uncertain properties. In , Vol. 468. Proceedings of the royal society of london A: Mathematical, physical and engineering sciences (pp. 1154e1174).

[18] Smedt, F.D. (2007). Analytical solution and analysis of solute transport in rivers affected by diffusive transfer in the hyporheic zone. Journal of Hydrology, 339(1-2), 29-38.

[19] Sudheendra S.R., 2010 A solution of the differential equation of longitudinal dispersion with variable coefficients in a finite domain, Int. J. of Applied Mathematics & Physics, Vol.2, No. 2, 193-204.

[20] Sudheendra S.R., 2011. A solution of the differential equation of dependent dispersion along uniform and non-uniform flow with variable coefficients in a finite domain, Int. J. of Mathematical Analysis, Vol.3, No. 2, 89-105.

[21] Sudheendra S.R. 2012. An analytical solution of one-dimensional advection-diffusion equation in a porous media in presence of radioactive decay, Global Journal of Pure and Applied Mathematics, Vol.8, No. 2, 113-124.

[22] Sudheendra S.R., Raji J, & Niranjan CM, 2014. Mathematical Solutions of transport of pollutants through unsaturated porous media with adsorption in a finite domain, Int. J. of Combined Research & Development, Vol. 2, No. 2, 32-40.

[23] Yates S R 1992 An analytical solution for one-dimensional transport in porous media with an exponential dispersion function; Water Resour. Res. 28 2149–2154.