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A STUDY ON FIRST ORDER AUTOREGRESSIVE PROCESS AR (1) WITH CHANGING AUTOREGRESSIVE COEFFICIENT AND A CHANGE POINT MODEL FROM BAYESIAN PERSPECTIVE

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ABSTRACT:

In this research paper, we shall apply the concept of change point inference problem. For that let us consider first order autoregressive process with changing autoregressive coefficient at some point of time, say ‘***m***’. This is called change point inference problem. Here, we have used **RWM-H (Random Walk Metropolis - Hasting) Algorithm and Gibbs Sampling Technique** for the Bayes Estimation of ***‘m’*** and autoregressive coefficient. Further, we have studied the effects of prior information on the Bayes Estimates obtained.

KEY WORDS:

First Order Auto Regressive AR (1) Process, AR (1) Model, Autoregressive Coefficient, RWM-H Algorithm, Gibbs Sampling Technique, Bayes Estimates, Change Point

1. INTRODUCTION:

Many researchers have studied the Bayes Estimators of ***m, β1 and β2*** under Linex Loss Function and General Entropy Loss Function which are Asymmetric in nature. It was found that those estimators were of changing auto regressive process with normal error. **Zacks (1983) and Tsurumi (1987)** are the noteworthy and useful references on structural changes. Later on further research was carried out where the experts studied the Bayesian Analysis of the Autoregressive Model $X_t = \beta_1 X_{t-1} + \varepsilon_t$ *(where t = 1, 2,…,m)* and $X_t = \beta_2 X_{t-1} + \varepsilon_t$ *(where t = m+1,…,n)* and also $0 < \beta_1, \beta_2 < 1$. It was found at the end of the research work that ε_t was an independent random variable with an exponential distribution with mean θ_1 and it gets reflected in the sequence after ε_m is changed in mean θ_2.

2. PROPOSED FIRST ORDER AUTOREGRESSIVE AR (1) MODEL:

Let us assume the first order autoregressive model AR (1) as under:

$$X_i = \begin{cases} \beta_1 X_{i-1} + \epsilon_i, & i = 1, 2, \ldots, m. \\ \beta_2 X_{i-1} + \epsilon_i, & i = m+1, \ldots, n. \end{cases} \qquad (1)$$

where, β_1 *and* β_2 are unknown autocorrelation coefficients, x_i is the i^{th} observation of the dependent variable, the error terms ϵ_i are the independent random variables following the normal distribution with $N(0, \sigma_1^2)$ *for i = 1,2,…,m and* $N(0, \sigma_2^2)$ *for i = m+1,…,n* and σ_1^2 *and* σ_2^2 both are known. Here, we note that ***'m'*** is the unknown change point and x_0 is the initial quantity.

3. BAYES ESTIMATION PROCEDURE:

We clearly know that the procedure of Bayes Estimation is totally based on a posterior density, say, $g(\beta_1, \beta_2, m \mid Z)$, which is proportional to the product of the likelihood function $L(\beta_1, \beta_2, m \mid Z)$, with a joint prior density, say, $g(\beta_1, \beta_2, m)$ representing uncertainty on the values of parameters.

Hence, the likelihood function of β_1, β_2 and m, given the sample information

$Z_t = (x_{t-1}, x_t)$ where ***t = 1, 2… , m, m+1…, n*** will be:

$$L(\beta_1, \beta_2, m|Z) = K_1 \,.\, exp\left(-\frac{1}{2}\beta_1{}^2\left(\frac{S_{m_1}}{\sigma_1{}^2}\right) + \beta_1\left(\frac{S_{m_2}}{\sigma_1{}^2}\right) - \frac{A_{1_m}}{2\sigma_1{}^2}\right) .\, exp\left(-\frac{1}{2}\beta_2{}^2\left(\frac{S_{n_1} - S_{m_1}}{\sigma_2{}^2}\right) + \beta_2\left(\frac{S_{n_2} - S_{m_2}}{\sigma_2{}^2}\right) - \frac{A_{2_m}}{2\sigma_2{}^2}\right) \sigma_1^{-m}\, \sigma_2^{-(n-m)} \qquad (2)$$

where we have:

$$S_{k_1} = \sum_{i=1}^{k} x_{i-1}{}^2 \qquad S_{k_2} = \sum_{i=1}^{k} x_i\, x_{i-1}$$

$$\boldsymbol{A_{1_m} = \sum_{i=1}^{m} x_i^2 \qquad A_{2_m} = \sum_{i=m+1}^{n} x_i^2}$$

$$\boldsymbol{k_1 = (2\pi)^{-\frac{n}{2}}}$$

(3)

4. POSTERIOR DENSITY OF CHANGE POINT USING INFORMATIVE PRIORS (NORMAL DISTRIBUTION) ON β_1, β_2:

Here, we have derived the posterior density of change point ***m, β_1 and β_2*** of the model explained in ***equation (1)*** under informative priors.

Further, we have considered the ***AR (1) model*** as shown in ***equation (1)*** with unknown $\boldsymbol{\sigma^{-2}}$. Also, we suppose uniform prior of change point same as ***Broemeling (1987)*** and we also suppose that ***m, β_1 and β_2*** are independent.

Thus we can write $g(m) = \frac{1}{n-1}$

Now, the normal prior density on ***β_1 and β_2*** will be:

$$g(\beta_1) = \frac{1}{\sqrt{2\pi}a_1} e^{-\frac{1}{2}\left(\frac{\beta_1}{a_1}\right)^2}$$

$$g(\beta_2) = \frac{1}{\sqrt{2\pi}a_2} e^{-\frac{1}{2}\left(\frac{\beta_2}{a_2}\right)^2}$$

Hence, joint prior p.d.f. of ***β_1, β_2 and m*** will be the joint prior density say $\boldsymbol{g(\beta_1, \beta_2, m)}$ which is as under:

$$g(\beta_1, \beta_2, m) = \frac{1}{2\pi a_1 a_2\,(n-1)}\, e^{-\frac{1}{2}\left(\frac{\beta_1}{a_1}\right)^2} e^{-\frac{1}{2}\left(\frac{\beta_2}{a_2}\right)^2} \qquad \textbf{(4)}$$

Now, using the likelihood function shown in ***equation (2)*** with the joint prior density in ***equation (4)***, the joint posterior density of $\boldsymbol{\beta_1, \beta_2, m}$ ***say*** $\boldsymbol{g(\beta_1, \beta_2, m|Z)}$ will be:

$$g(\beta_1, \beta_2, m|Z) = \frac{K_1}{h_1(z)}\left[L(\beta_1, \beta_2, m|Z)\,.\,g(\beta_1, \beta_2, m)\right]$$

$$= \frac{K_2}{h_1(z)}\left[e^{\left[-\frac{1}{2}\beta_1{}^2A_1 + \beta_1B_1\right]} e^{\left[-\frac{1}{2}\beta_2{}^2A_2 + \beta_2\ B_2\right]} e^{\left[-\left(\frac{A_{1m}}{2\sigma_1{}^2} + \frac{A_{2m}}{2\sigma_2{}^2}\right)\right]}\right] \sigma_1^{-m}\,\sigma_2^{-(n-m)}$$

(5)

where we have:

$$\boldsymbol{K_2 = \frac{K_1}{2\pi\, a_1 a_2\,(n-1)}}$$

$$\boldsymbol{A_1} = \frac{S_{m1}}{\sigma_1^{\,2}} + \frac{1}{a_1^{\,2}} \qquad \boldsymbol{B_1} = \frac{S_{m2}}{\sigma_1^{\,2}}$$

$$\boldsymbol{A_2} = \frac{S_{n1} - S_{m1}}{\sigma_2^{\,2}} + \frac{1}{a_2^{\,2}} \qquad \boldsymbol{B_2} = \frac{S_{n2} - S_{m2}}{\sigma_2^{\,2}} \qquad \textbf{(6)}$$

Here, we note that $h_1(Z)$ is the marginal density of z which is as under:

$$h_1(Z) = \sum_{m=1}^{n-1} \int_{\beta_1} \int_{\beta_2} L(\beta_1, \beta_2, m \mid \underline{X}) \, . \mathrm{g}(\beta_1, \beta_2, m) \, d\beta_1 d\beta_2$$

$$= \sum_{m=1}^{n-1} e^{\left[-\left(\frac{A_{1m}}{2\sigma_1^{\,2}} + \frac{A_{2m}}{2\sigma_2^{\,2}}\right)\right]} \sigma_1^{-m} \, \sigma_2^{-(n-m)} \int_{-\infty}^{\infty} e^{\left[-\frac{1}{2}\beta_1^{\,2} A_1 + \beta_1 B_1\right]} \, d\beta_1 \int_{-\infty}^{\infty} e^{\left[-\frac{1}{2}\beta_2^{\,2} A_2 + \beta_2 \; B_2\right]} \, d\beta_2$$

$$= k_3 \sum_{m=1}^{n-1} T_1(m) \qquad \textbf{(7)}$$

where we have:

$$T_1(m) = \; k_m G_{1m} G_{2m} \qquad \textbf{(8)}$$

$$G_{1m} = \int_{-\infty}^{\infty} \exp\left[-\; \frac{1}{2} \; \beta_1^{\,2} A_1 \; + \; \beta_1 B_1\right] d\beta_1 = \frac{e^{\frac{B_1^{\,2}}{2A_1}} \sqrt{2\pi}}{\sqrt{A_1}} \qquad \textbf{(9)}$$

$$G_{2m} = \int_{-\infty}^{\infty} \exp\left[- \; \frac{1}{2} \; \beta_2^{\,2} A_2 + \; \beta_2 \;\; B_2\right] d\beta_2 = \frac{e^{\frac{B_2^{\,2}}{2A_2}} \sqrt{2\pi}}{\sqrt{A_2}} \qquad \textbf{(10)}$$

$$k_m = e^{\left[-\left(\frac{A_{1m}}{2\sigma_1^{\,2}} + \frac{A_{2m}}{2\sigma_2^{\,2}}\right)\right]} \sigma_1^{-m} \, \sigma_2^{-(n-m)} \qquad \textbf{(11)}$$

Now, the marginal posterior density of the change point ***m,*** $\boldsymbol{\beta_1}$ ***and*** $\boldsymbol{\beta_2}$ will be:

$$g_1(m|x) = \frac{T_1(m)}{\sum_{m=1}^{n-1} T_1(m)} \qquad \textbf{(12)}$$

$$g_1(\beta_1|X) = \frac{k_3}{h_1(X)} \left[\sum_{m=1}^{n-1} k_m e^{\left[-\frac{1}{2}\beta_1^{\,2} A_1 + \beta_1 B_1\right]}\right] G_{1m} \qquad \textbf{(13)}$$

$$g_1(\beta_2|X) = \frac{k_3}{h_1(X)} \left[\sum_{m=1}^{n-1} k_m \; e^{\left[-\frac{1}{2}\beta_2^{\,2} A_2 + \beta_2 \;\; B_2\right]}\right] G_{2m} \qquad \textbf{(14)}$$

Here, $\boldsymbol{G_{1m}}$, $\boldsymbol{G_{2m}}$ ***and*** $\boldsymbol{k_m}$ are same as defined and shown in ***equations (9), (10) and (11)*** respectively.

Now, the Bayes estimator of any function of parameter α, say $g(\alpha)$ under the squared loss function is,

$$E_{\alpha|z}(g(\alpha|Z)) = \int_0^\infty \alpha(g(\alpha|Z))\, d\alpha \qquad (*)$$

Here, $g\ (\alpha \mid Z)$ is marginal posterior density of α. It is very complicated to compute the ***equation (*)*** analytically in this case. Therefore, we shall apply ***MCMC methods*** to find the Bayes Estimates of β_1, β_2 ***and*** m.

5. ALGORITHM USING GIBBS SAMPLING TECHNIQUE:

We can easily identify the full conditional distribution $g\ (\alpha_i|Z, \alpha_j)$ ***where*** $j \neq i$ up to proportionality by regarding $g\ (\alpha|Z)$ as a function of $\alpha_i\ (i = 1, \ldots, k)$ only, corresponding to all other α_j, ***where*** $j \neq i$, to be fixed given a posterior distribution $g\ (\alpha|Z)$ for unknown parameters $\alpha = (\alpha_1, \ldots, \alpha_k)$ defined, at least up to proportionality, by multiplying the likelihood function with the corresponding prior distribution.

For implementing the Gibbs Sampling Technique, we have to re-write ***equation (13)*** as the full conditional of β_1 by fixing all other parameters i.e. β_2 ***and m.*** Hence full conditional density of β_1 given β_2 ***and m*** is as follows:

$$g(\beta_1 \mid \beta_2, m, Z) \propto N\left(\frac{B_1}{A_1}, \left(\frac{1}{\sqrt{A_1}}\right)^2\right) \qquad (15)$$

where A_1 and B_1 are the same as shown in ***equation (6)***.

Now we shall re-write ***equation (14)*** as full conditional density of β_2 by fixing all other parameters β_1 ***and m***. Hence, we get the full conditional density of β_2 given β_1, σ^2 ***and m*** is as follows:

$$g(\beta_2 \mid \beta_1, m, Z) \propto N\left(\frac{B_2}{A_2}, \left(\frac{1}{\sqrt{A_2}}\right)^2\right) \qquad (16)$$

where A_2 and B_2 are the same as shown in ***equation (6)***.

Now, in order to estimate the parameters β_1 ***and*** β_2, we shall apply the Gibbs Sampling Technique to generate sample from the full conditional density of β_1 **and** β_2 which are given respectively in the ***equations (15) and (16)***. We shall use the Gibbs Sampling Algorithm which is as under:

Initialize $\beta_1 = \beta_{10}, \beta_2 = \beta_{20}$ ***and*** $m = m_0$ and then follow the steps given below.

Step-1: Generate $\beta_1 \sim N\left(\frac{A_1}{B_1}, \left(\frac{1}{\sqrt{B_1}}\right)^2\right)$, using Gibbs Sampling Technique.

Step-2: Generate $\beta_2 \sim N\left(\frac{A_2}{B_2}, \left(\frac{1}{\sqrt{B_2}}\right)^2\right)$, using Gibbs Sampling Technique.

Step-3: Repeat the above steps.

6. APPLYING MCMC TECHNIQUES:

Here, we notice that the posterior distribution of the change point shown in ***equation (12)*** has no closed form. Hence, we propose to use MCMC techniques to generate the samples from the posterior distribution. To implement the MCMC Techniques, we re-write ***equation (12)*** as target function of $\boldsymbol{m}$, by fixing all other parameters i.e. $\boldsymbol{\beta_1}$ **and** $\boldsymbol{\beta_2}$. Hence target function of $\boldsymbol{m}$ given $\boldsymbol{\beta_1}$ **and** $\boldsymbol{\beta_2}$ will be:

$$g(m\,|\beta_1,\beta_2\,,Z) \propto\; k_m\, e^{\left[-\frac{1}{2}\beta_1{}^2A_1+\beta_1B_1\right]}\, e^{\left[-\frac{1}{2}\beta_2{}^2A_2+\beta_2B_2\right]} \qquad \textbf{(17)}$$

where $\boldsymbol{A_1, B_1, A_2, B_2}$ ***and*** $\boldsymbol{k_m}$ are the same as shown in the ***equations (6) and (11)*** respectively.

7. APPLICATION TO GENERATED DATA USING NUMERICAL EXAMPLE:

Let us assume an AR (1) model as under:

$$\boldsymbol{X_1} = \begin{cases} \boldsymbol{0.1\,X_{i-1} + \epsilon_i\,,\;\; i = 1,2,\dots,10} \\ \boldsymbol{0.3\,X_{i-1} + \epsilon_i\;,\;\; i = 11,12,\dots,20} \end{cases} \qquad \textbf{(18)}$$

Here, in the above equation, the error terms $\boldsymbol{\epsilon_i}$ are independent random variables following Normal Distribution $\boldsymbol{N}(\boldsymbol{0},\boldsymbol{1})$ for ***i = 1,2,…,10*** and $\boldsymbol{N}(\boldsymbol{0},\boldsymbol{4})$ for ***i = 11,12,13,…,20***. Also we note that here $\boldsymbol{\sigma_1^2}$ and $\boldsymbol{\sigma_2^2}$ are known. Further, we note that $\boldsymbol{m}$ is the unknown change point and $\boldsymbol{x_0} = \boldsymbol{0.1}$ is the initial quantity. Here, we have generated 20 random observations from the proposed **AR (1) model** given in **equation (18)**. Out of total twenty random observations, the first ten observations are from normal distribution with $\boldsymbol{\sigma_1{}^2 = 1}$ and next ten observations are from normal distribution with $\boldsymbol{\sigma_2{}^2 = 4}$. Also, we note that $\boldsymbol{\beta_1}$ **and** $\boldsymbol{\beta_2}$ themselves are random observations from the normal distribution with prior means $\boldsymbol{\mu_1}$ **= 0.1,** $\boldsymbol{\mu_2}$ **= 0.3** and variances $\boldsymbol{a_1}$ ***= 0.1*** and $\boldsymbol{a_2}$ ***= 0.1***. These observations are given in the following **TABLE 1**.

TABLE 1

GENERATED OBSERVATIONS FROM PROPOSED AR (1) MODEL

i	1	2	3	4	5	6	7	8	9	10
X_i	0.167	-0.204	0.399	-0.259	-0.784	-1.058	0.819	0.404	1.215	1.537
ϵ_i	0.157	-0.221	0.420	-0.299	-0.758	-0.979	0.925	0.322	1.175	1.416
i	11	12	13	14	15	16	17	18	19	20
X_i	-3.833	-16.173	9.441	11.857	20.645	1.458	13.249	-9.335	19.812	30.657
ϵ_i	-4.294	-15.023	14.293	9.025	17.088	-4.734	12.812	-13.310	22.613	24.713

Here, the target function is bounded. In order to generate a random sample using the RWM-H algorithm, the selected proposal is **uniform (2, 19)** same as prior, which is **symmetric around 10** with small steps. The initial distribution is chosen as **uniform (1, 19)**. Further, we truncate the initial distribution and then we get integer value of the **Bayes Estimate of change point (*m*)** as 10, when selected proposal is **uniform (1, 19)** and initial distribution is **uniform (3, 14)**. Here, the results are shown in **TABLE 2** for the data given in **TABLE 1** when given value of $\beta_1 = 0.1$, $\beta_2 = 0.3$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 16$.

TABLE 2

BAYES ESTIMATES OF CHANGE POINT (m) USING RWM-H ALGORITHM UNDER SQUARED ERROR LOSS FUNCTION

Bounded	*Selected Proposal*	*Initial Distribution*	*Bayes Estimate of change point* (m)	*Integer value of Bayes Estimate of change point* (m)
BD (2,19)	U (1,19)	U (1,19)	8.4	8
BD (2,19)	U (2,19)	U (2,19)	8.6	9
BD (3,19)	U (1,19)	U (1,19)	10.3	10
BD (3,19)	U (1,19)	U (3,14)	10.2	10

Further, we also compute the Bayes Estimates of **'*m*'** using **RWM-H algorithm** for different priors under consideration for the data given in **TABLE 1**. The results are shown in the following **TABLE 3**.

TABLE 3

BAYES ESTIMATES OF CHANGE POINT (m) USING RWM-H ALGORITHM UNDER SQUARED ERROR LOSS FUNCTION FOR DIFFERENT PRIORS UNDER CONSIDERATION

Serial Number	a_1^2	a_2^2	*Bayes Estimate of change point (m) (Posterior Mean)*
1	0.0100	0.01	10
2	0.0400	0.04	10
3	**0.0490**	**0.04**	**10**
4	0.0550	0.09	10
5	0.0600	0.25	10
6	0.0625	0.49	10
7	0.0900	0.64	10
8	0.4900	0.81	10
9	0.8100	1.00	10
10	1.0000	4.00	10

Now we compute the Bayes Estimates of β_1 (when given value of $\beta_2 = 0.3, m = 10, \sigma_1^2 = 1$ *and* $\sigma_2^2 = 16$) and β_2 (when given value of $\beta_1 = 0.1, m = 10, \sigma_1^2 = 1$ *and* $\sigma_2^2 = 16$) using **Gibbs Sampling and MCMC algorithm** for different priors under consideration for the data given in **TABLE 1**. The results are shown in the following **TABLE 4**.

TABLE 4

BAYES ESTIMATES OF β_1 AND β_2 USING GIBBS SAMPLING MCMC ALGORITHM UNDER SQUARED ERROR LOSS FUNCTION FOR DIFFERENT PRIORS UNDER CONSIDERATION

Serial Number	a_1^2	a_2^2	*Bayes Estimates of*		*S.D. of Bayes Estimates of*	
			β_1	β_2	β_1	β_2
1	0.0100	0.01	0.025	0.255	0.048	0.008
2	0.0400	0.04	0.090	0.305	0.048	0.008
3	**0.0490**	**0.04**	**0.107**	**0.305**	**0.048**	**0.008**
4	0.0550	0.09	0.118	0.344	0.048	0.008
5	0.0600	0.25	0.126	0.367	0.048	0.008
6	0.0625	0.49	0.130	0.374	0.048	0.008
7	0.0900	0.64	0.172	0.376	0.048	0.008
8	0.4900	0.81	0.415	0.377	0.048	0.008
9	0.8100	1.00	0.475	0.378	0.048	0.008
10	1.0000	4.00	0.496	0.381	0.048	0.008

FIGURE 1 shows the graph of the full conditional of β_1 when a sample of size ***10,000*** is generated. Here, Gibbs Sampling with MCMC algorithm has been run for $\boldsymbol{\beta_2 = 0.3, m = 10, \sigma_1^2 = 1}$ ***and*** $\boldsymbol{\sigma_2^2 = 16}$.

Figure 1: Full Conditional of β_1

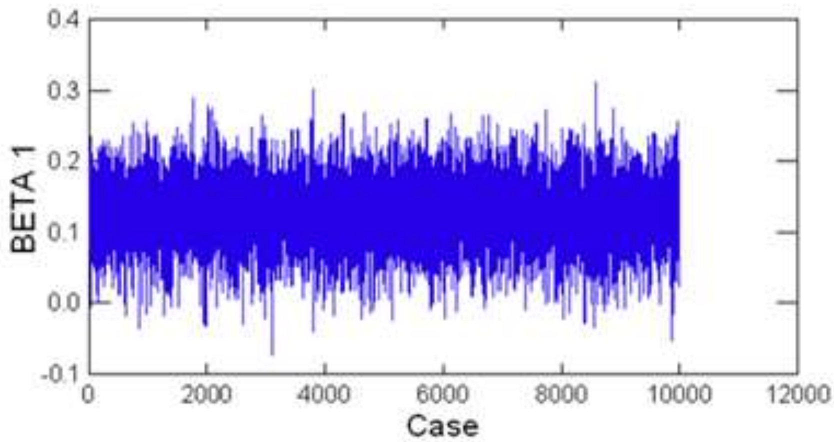
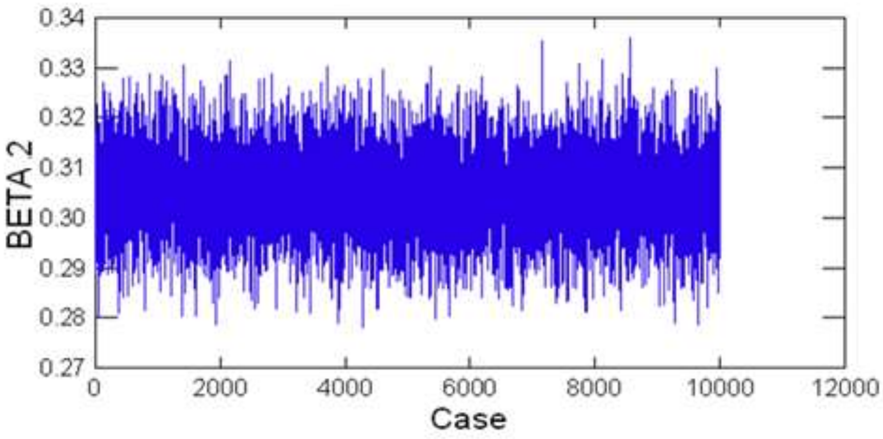


FIGURE 2 shows the graph of the full conditional of $\boldsymbol{\beta_2}$ when a sample of size ***10,000*** is generated. Here, Gibbs Sampling with MCMC algorithm has been run for $\boldsymbol{\beta_1 = 0.1, m = 10, \sigma_1^2 = 1}$ **and** $\boldsymbol{\sigma_2^2 = 16}$.

Figure 2: Full Conditional of $\boldsymbol{\beta_2}$



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