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Linear Algebra Approach to Daubechies Wavelet Approximation

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Abstract: The Wavelet Transform is a relatively discovery in signal and image processing that allows us to simultaneously analyse temporal and frequency data. The enormous research on the topic necessitates a mathematical study. Linear algebra and Wavelet have a solid relationship. As a result, the investigation takes a linear algebra method. In this study, we examine Daubechies wavelet transforms approximation and details spaces using a representation based on linear maps and their accompanying properties.

***IndexTerms* - Wavelet transform, Approximation space, linear map, Null space.**

1. Introduction

The wavelet transform (WT) is a fast-growing tool for time-frequency analysis [1], [2]. If we need temporal information, the Fourier transform (FT) is an excellent tool to apply, as it is beneficial in analyzing stationary signals [3]. However, for non-stationary signals, the FT transform only gives the signal's overall frequency content, and temporal information is lost. Both time and frequency information are required in many signal and image processing applications. The advent of the wavelet transform, which decomposes data into multiple frequency bands, is a significant invention in this discipline. The inner product of the signal with the wavelet and scaling functions yields WT.

An essential quality of WT is Multi resolution analysis [4], which is satisfied by the Daubechies family of wavelets. In our study, we have considered the Daubechies family for the mathematical construction of the WT.

2. Preliminaries

A wavelet is a tiny wave that decays rapidly and integrates to zero. The wavelet transformations express a signal f(t) using a single wavelet called the mother wavelet $\varphi^d(t)$ and its dilations and translations, resulting in a time-frequency representation. There are various wavelets, the most well-known of which is the Daubechies family [5]. This family has finite vanishing moments and compact support [6]. Daubechies family of wavelets follows Multi-Resolution Analysis (MRA) [7]. As a result of MRA, we have two functions $\varphi^d(t)$ and $\Psi^d\,(t)$ whose dyadic translations form the approximation and detail spaces. The functions denote the dyadic translates $\varphi^d{}_{j,k}(t)$ and $\Psi^d{}_{j,k}(t)$ are given below [8], [9],

$$\varphi^d{}_{j,k}(t) = 2^{-\frac{j}{2}}\varphi^d(2^{-j}t-k), j,k \in \mathbb{Z} \qquad (1)$$

$$\Psi^d{}_{j,k}(t) = 2^{-\frac{j}{2}}\Psi^d(2^{-j}t-k), j,k \in \mathbb{Z} \qquad (2)$$

Next are some mathematical theories we have used in this paper.

Definition 2.1. Null space [10]: Let T be a linear map from a vector space V into a vector space W. Null space of T denoted by N(T) is the set of elements in V that are mapped to the zero vector in W that is N(T) = {v ∈ V: T (v) = 0} which is always a subspace of V. If V is of finite dimensional dim(N(T)) is called Nullity of T.

Definition 2.2. Rank of T [11]: Rank of T is the dimension of the Range of T.

3. Daubechies Wavelet Transform Properties [12]

In the coming sections, we will discuss the mathematical concepts in the Daubechies wavelet transform.

3.1 Approximation Space in Daubechies Wavelet Transform

Let V_0^d be a 2^n dimensional vector space spanned by integer translations of Daubechies wavelet $\varphi^d(t)$ of the length of support N. Then

$$V_0^d = span\left\{\varphi^d{}_{0,k}(t): k = 0,1,2,\ldots,2^n-1\right\}, n \in \mathbb{N}$$

$$= \left\{ \sum_{k=0}^{2^n-1} x(k)\ \varphi^d{}_{0,k}(t) : x(k)\ \epsilon\ \mathbb{R}, 0\ \le k\ \le\ 2^n-1 \right\}$$

Similarly, define the space V_{-1}^d spanned by $\varphi^d{}_{-1,k}(t),\ 0\ \le k\ \le\ 2^{n-1}-1$

$$V_{-1}^d = \left\{ \sum_{k=0}^{2^{n-1}-1} y(k)\ \varphi^d{}_{-1,k}(t) : y(k)\ \epsilon\ \mathbb{R}, 0\ \le k\ \le\ 2^{n-1}-1 \right\}$$

Therefore,

$$\dim V_{-1}^d = \frac{dim V_0^d}{2}$$

In general, we can define V_{-m}^d as the space spanned by $\varphi^d{}_{-m,k}(t)$,$0\ \le k\ \le\ 2^{n-m}-1$ where $0\ \le m\ \le\ n$. For each V_{-m}^d form a vector space of dimension 2^{n-m} over $\mathbb{R}$ under the operations,

i) $f(t) + g(t) = \sum_{k=0}^{2^{n-m-1}-1} x(k)\ \varphi^d{}_{m,k}(t) + \sum_{k=0}^{2^{n-m-1}-1} y(k)\ \varphi^d{}_{m,k}(t)$

$$= \sum_{k=0}^{2^{n-m-1}-1} [x(k) + y(k)]\ \varphi^d{}_{m,k}(t)\ , \forall\ f(t), g(t) \in\ V_{-m}^d$$

ii) $\alpha f(t) = \sum_{k=0}^{2^{n-m-1}-1} \alpha x(k)\ \varphi^d{}_{m,k}(t)$

$$= \alpha \sum_{k=0}^{2^{n-m-1}-1} x(k)\ \varphi^d{}_{m,k}(t)$$

Similarly, we can define the detail space,

$W_{-m}^d = span\{ \Psi^d{}_{-m,k}(t) : k = 0, 1, 2, \ldots, 2^{n-m}-1 \}$ where $0\ \le m\ \le\ n$.

3.2 Approximation and Detail Map

Consider the signal $x(t) \in\ L^2([0, 2^n])$. Let f(t) be its approximation on the space V_0^d which is given by $f(t) \sum_{k=0}^{2^n-1} x(k)\ \varphi^d{}_{0,k}(t)$. Define a map $A_d^1 : V_0^d \to V_{-1}^d$ as,

$$A_d^1[f(t)] = A_d^1 \left(\sum_{k=0}^{2^n-1} x(k)\ \varphi^d{}_{0,k}(t) \right)$$

$$= \sum_{k=0}^{2^{n-1}-1} a^1(k)\ \varphi^d{}_{-1,k}(t)$$

Where $a^1(k) = \sum_{m=2k}^{2k+N-1} x(m)\, h^d(m-2k)$ and $\{h^d(n) : n = 0, 1, \ldots, N-1\}$ is the low pass filter coefficients [13]. This map is well defined linear map. Also, this map A_d^1 is called the first Approximation map. Similarly, define the second approximation map $A_d^2 = V_{-1}^d \to V_{-2}^d$ as,

$$A_d^2[g(t)] = A_d^2 \left(\sum_{k=0}^{2^{n-1}-1} a^1(k)\ \varphi^d{}_{-1,k}(t) \right)$$

$$= \sum_{k=0}^{2^{n-2}-1} a^2(k)\ \varphi^d{}_{-2,k}(t)$$

where $a^2(k) = \sum_{m=2k}^{2k+N-1} a^1(k)\, h^d(m-2k)$, $0\ \le k\ \le\ 2^{n-2}-1$. In general, m^th approximation map $A_d^m = V_{-(m-1)}^d \to V_{-m}^d$ where $1\ \le m\ \le n$ is defined as

$$A_d^m[h(t)] = A_d^m \left(\sum_{k=0}^{2^{n-(m-1)}-1} a^{m-1}(k)\ \varphi^d{}_{-(m-1),k}(t) \right) = \sum_{k=0}^{2^{n-m}} a^m(k)\ \varphi^d{}_{-m,k}(t)$$

where $a^m(k) = \sum_{m=2k}^{2k+N-1} a^{m-1}(k)\, h^d(m-2k)$, $0\ \le k\ \le\ 2^{n-m}-1$.

In the same manner, we can define the detail map $D_d^1 : V_0^d \to W_{-1}^d$ as $D_d^1[f(t)] = D_d^1 \left(\sum_{k=0}^{2^n-1} d^1(k)\ \Psi^d{}_{0,k}(t) \right)$, where $d^1(k) = \sum_{m=2k}^{2k+N-1} x(m)\, g^d(m-2k)$, $0\ \le k\ \le\ 2^n-1$. And $\{g^d(n) : n = 0, 1, \ldots, N-1\}$ are the high pass filter coefficients. In general, the m^th detail map $D_d^m = V_{-(m-1)}^d \to W_{-m}^d$ where $1\ \le m\ \le n$ is defined as

$$D_d^m[h(t)] = D_d^m\left(\sum_{k=0}^{2^{n-(m-1)}-1} a^{m-1}(k)\ \varphi^d{}_{-(m-1),k}(t)\right) = \sum_{k=0}^{2^{n-m}} d^m(k)\ \Psi^d{}_{-m,k}(t)$$

where $d^m(k) = \sum_{m=2k}^{2k+N-1} a^{m-1}(k)\, g^d(m-2k), 0 \le k \le 2^{n-m}-1$.

3.3 Matrix representation of Approximation and Detail map

Let $X = [x(0)\ \ x(1)\ \dots x(2^n-1)]^T$, the mth approximation and detail coefficients are given by $[a^m] = [\,a^{m-1}(0)\ \ a^{m-1}(1)\ \dots\ a^{m-1}(2^{n-m}-1)]^T$ and $[d^m] = [\,d^{m-1}(0)\ \ d^{m-1}(1)\ \dots\ d^{m-1}(2^{n-m}-1)]^T$ respectively. Now consider the first approximation map A_d^1

For any $f(t) \in V_0^d, A_d^1[f(t)] = [a^1]^T = \begin{bmatrix} \varphi^d{}_{-1,0} \\ \varphi^d{}_{-1,1} \\ \vdots \\ \varphi^d{}_{-1,2^{n-1}-1} \end{bmatrix}$ where $f(t) = \sum_{k=0}^{2^n-1} x(k)\ \varphi^d{}_{0,k}(t) = X^T \begin{bmatrix} \varphi^d{}_{0,0} \\ \varphi^d{}_{0,1} \\ \vdots \\ \varphi^d{}_{0,2^n-1} \end{bmatrix}$ and

$$[a^1] = \underbrace{\begin{bmatrix} h^d(0) & h^d(1) & h^d(2) & \cdots & \cdots & h^d(N-1) & 0 & \cdots & 0 \\ 0 & 0 & h^d(0) & h^d(1) & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & h^d(0) & h^d(1) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^d(0) & h^d(1) \end{bmatrix}}_{L_d} X^T$$

Matrix L_d is called the low pass filter matrix. This matrix representation can be generalized to any arbitrary approximation map A_d^m.

Similarly, for detail map D_d^1 the matrix representation is, $D_d^1[f(t)] = [d^1]^T = \begin{bmatrix} \Psi^d{}_{-1,0} \\ \Psi^d{}_{-1,1} \\ \vdots \\ \Psi^d{}_{-1,2^{n-1}-1} \end{bmatrix}$ where

$$[d^1] = \underbrace{\begin{bmatrix} g^d(0) & g^d(1) & g^d(2) & \cdots & \cdots & g^d(N-1) & 0 & \cdots & 0 \\ 0 & 0 & g^d(0) & g^d(1) & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & g^d(0) & g^d(1) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g^d(0) & g(1) \end{bmatrix}}_{H_d} X^T$$

The matrix H_d is called the High pass filter matrix.

4 Nullity of Approximation map

This section is about the properties of linear maps corresponding to the Daubechies wavelet transform.

Theorem 4.1. Nullity of the approximation map A_d^1 is 2^{n-1}.
Proof. Consider the Null space of the approximation map

$$N(A_d^1) = \{f(t) \in V_0^{;d} : A_d^1(f(t) = 0\}$$
$$=\left\{f(t) = \textstyle\sum_{k=0}^{2^n-1} x(k)\ \varphi^d{}_{0,k}(t) \in V_0^{;d} : \sum_{k=0}^{2^{n-1}-1} a^1(k)\ \varphi^d{}_{-1,k}(t) = 0\right\}$$
$$\Rightarrow a^1(k) = 0,0\ \le k \le 2^{n-1}-1$$
$$\Rightarrow a^1(k) = \textstyle\sum_{m=2k}^{2k+N-1} x(m)\, h^d(m-2k) = 0,,0 \le k \le 2^{n-1}-1$$

We get a homogenous system of equations that has 2^{n-1} equations with 2^n unknowns. In matrix notation is $L_d X = 0$. That is

$$\underbrace{\begin{bmatrix} h^d(0) & h^d(1) & h^d(2) & \cdots & \cdots & h^d(N-1) & 0 & \cdots & 0 \\ 0 & 0 & h^d(0) & h^d(1) & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & h^d(0) & h^d(1) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^d(0) & h^d(1) \end{bmatrix}} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(2^n-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix L_d is 2^{n-1}. Therefore Nullity of $A_d^1 = 2^{n-1}$.

Corollary 4.2. Similarly, we can prove that Nullity of $D_d^1 = 2^{n-1}$.

Corollary 4.3. Rank of A_d^1= Rank of D_d^1=2^{n-1}.
Proof. Using Rank Nullity theorem [14].

Corollary 4.4. For $1 \leq m \leq n$, Nullity of A_d^m=Rank of A_d^m=Nullity of D_d^m= Rank of $D_d^m = 2^{n-m-1}$.

Remark 4.1. This result implies that the approximation map A_d^m and detail map D_d^m is an onto map.

6 Conclusion

The mathematical aspects of Daubechies wavelet transform are discussed in this study. Daubechies wavelet is a linear transformation on finite dimensional space, as we previously demonstrated. Approximation and detail space are represented by linear maps, which can be used to forecast approximation and detail spaces at various levels of decomposition. Our work is limited to one dimension, but it can also be expanded to two dimensions.

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