JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

A Fractional Homotopy Perturbation Transform Method for Solving Riccati Differential Equations

Subodh Pratap Singh[1], Amardeep Singh[*1]

[1, *]Department of Mathematics, Glocal University, Saharanpur @sp740956@gmail.com amrdeep.srars@gmail.com

Abstract: The paper includes the solution of Riccati differential equation by the Frcational Homotopy Perturbation Transform Method (FHPTM). Traditional Variational Iteration Method (VIM) is used to solve the Riccati equation. It is observed that good approximations by VIM are obtained for the solution of Riccati equation. However, the limitation is that solution of Riccati equations are obtained only in the neighborhood of the initial position. Thus, the method FHPTM has an advantage that convergence region of iteration solutions is increased. Numerical results suggest the application of the FHPTM techniques effectively.

Keywords: Fractional Homotopy Perturbation Transform Method; Fractional Partial Differential Equations; Riccati Differential Equations; FHPTM; Riccati Equation; Nonlinear Partial Differential Equation.

1. Introduction

A Fractional Homotopy Perturbation Transform Method (FHPTM) is presented for solving the Riccati differential equation as follows:

$$u'(x) = p(x) + q(x)u(x) + r(x)u^2(x),\ 0 \le x \le X,$$
$$u(0) = \alpha, \tag{1}$$

where p(x), q(x), r(x) are continuous, which plays a key role in many desciplines of applied mathematics as well as sciences, [1]. For example, it is well known that static Schrödinger equation is closely related to a Riccati differential equation. Also, solitary wave solution of a nonlinear partial differential equation (NPDE) is expressed as a polynomial in two basic elementary functions, which satisfy a projective Riccati equation, [2]. These problems are also useful in analyzing optimal control strategies. Optimal control analysis is a very useful tool in the application of mathematical modeling of infectious disease dynamics. Several researchers are interested in the analysis and applications of these problems. Also, only up to some extent the analytical solutions are obtained because it is really difficult to get an explicit form except only in very few cases. For example, analytical method is obtained in solving the Riccati differential equation with constant coefficients, [3].
This is the situation where numeric techniques or approximate approaches to get the solution nearly similar to the analytical solution. Once the numerical solutions are obtained, they get verified and compare with the other numerical methods. In the recent investigations, solution of Riccati differential equation has been obtained by Adomian's decomposition method in [4]. Moreover, a special Riccati differential equation i.e., the quadratic Riccati differential equation is solved by Abbasbandy using He's VIM, the homotopy perturbation method (HPM) and the iterated He's HPM, [5–7]. In this investigation, the accuracy of solution of Riccati Differential equation has been compared with the Adomian's decomposition method. Furthermore, piecewise VIM has been introduced by Geng for solving Riccati differential equation, [8]. Originally, He has proposed the VIM [9-14]. Also, several mathematicians have suggested the VIM to be an important tool to solve different linear and nonlinear problems, [15-25].

In this paper, we present the Fractional Homotopy Perturbation method to solve Riccati Differential equation. Our results will show the method FHPTM is more reliable and accurate in comparison to VIM. The method FHPTM will reduce the burden of computation work for sure.

2. Preliminaries

Definition 2.1 Consider a real function h (χ), χ > 0. It is called in space $C_\varsigma, \varsigma \in R$ if ∋ a real no. b (> ζ), s.t. h (χ) = χ^b $h_1(\chi)$, $h_1 \in C[0,\infty]$. It is clear that $C_\varsigma \subset C_\gamma$ if γ ≤ ζ.

Definition 2.2 Consider a function h (χ), χ > 0. It is called in space $C_\varsigma^m, m \in N \cup \{0\}$, if $h^{(m)} \in C_\varsigma$

Definition 2.3 The (left sided) Riemann–Liouville integral of fractional order $\nu > 0$ of a function h, $h \in C_\varsigma, \varsigma \ge -1$ is defined as

$$I^\nu h(t) = \frac{1}{\lceil \nu}\int_0^t \frac{h(\tau)}{(t-\tau)^{1-\nu}}\ d\tau = \frac{1}{\lceil \nu+1}\int_0^t h(\tau)\ (d\tau)^\nu$$

$$I^0 h(t) = h(t).$$

Definition 2.4 The Caputo fractional derivative (left sided) of h, $h \in C_{-1}^m, m \in N \cup \{0\}$,

$$D_t^\beta h(t) = \begin{cases} [I^{m-\beta} h^{(m)}(t)], m-1<\beta<m, m \in N, \\ \dfrac{d^m}{dt^m} h(t), \beta = m \end{cases}$$

a. $I_t^\varsigma h(x,t) = \frac{1}{\lceil \varsigma}\int_0^t (t-s)^{\varsigma-1} h(x,s)\ ds;\ \varsigma, t > 0.$

b. $D_\tau^\nu V(x,\tau) = I_\tau^{m-\nu}\frac{\partial^m V(x,\tau)}{\partial t^m}, m-1<\nu\le m.$

c. $I_t^\varsigma D_t^\varsigma h(t) = h(t) - \sum_0^{m-1} h^k(0+)\frac{t^k}{\lfloor k}$

d. $I^\nu t^\varsigma = \frac{\lceil \varsigma+1}{\lceil \nu+\varsigma+1} t^{\nu+\varsigma}.$

Definition 2.5 Mittag-Leffler function is demarcated by the given series representation, valid in entire complex plane:

$$E_\varsigma(z) = \sum_{m=0}^\infty \frac{z^m}{\lceil 1+\varsigma m}, \varsigma > 0, z \in C.$$

Definition 2.6 Laplace transform of a (piecewise) continuous function g (t) in [0,∞) is given as:

$$G(p) = L[g(t)] = \int_0^\infty e^{-pt} g(t) dt$$

Definition 2.7

a. Laplace transform of (Riemann–Liouville) fractional integral is given as:

$$L[I^\alpha f(t)] = p^{-\alpha} F(p)$$

b. Laplace transform of (Caputo) fractional derivative is given as:

$$L[D^\alpha g(t)] = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} g^{(k)}(0), n-1<\alpha\le n$$

3. The proposed FHPTM for the system of nonlinear fractional PDEs

To illustrate the process of solution of the FHPTM, we ponder over the system of nonlinear time-fractional PDEs :

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + S(u,v) + Q(u,v) = g(x,t), \tag{2}$$

with initial values $u(x,0) = h(x)$, (3)

Here, $\frac{\partial^\alpha}{\partial t^\alpha}$ is (Caputo) fractional derivative of order α, S and Q are operators, linear & nonlinear respectively; g_1 and g_2 are the source terms. Also, $0 < \alpha \le 1$.

The method comprises of taking Laplace transform on both sides of Eq. (2) and Eq. (3), as

$$L[D_t^\alpha u(x,t)] + L[S(u,v)] + L[Q(u,v)] = L[g(x,t)] \quad (4)$$

By differentiation property of Laplace transform,

$$L[u(x,t)] = p^{-1}h(x) - p^{-\alpha}L[g(x,t)] + p^{-\alpha}L[S(u,v) + Q(u,v)] \quad (5)$$

Taking inverse transform in Eqs. (5), we get

$$u(x,t) = G(x,t) + L^{-1}[p^{-\alpha}L\{S(u,v) + Q(u,v)\}] \quad (6)$$

Here G(x, t) are the terms coming from the source term and initial values.

Applying FHPTM, it is assumed that the result may be articulated as a power series,

$$u_n(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad (7)$$

Here, p is reflected as a small parameter $(p \in [0,1])$ called homotopy parameter.
The non-linear term is decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \quad (8)$$

where H_n is He's polynomials of $u_0, u_1, u_2, u_{3,} ......, u_n$ respectively. They are calculated by the given formulae:

$$H_n(u_0, u_1, u_2, ...) = \frac{1}{\lfloor n} \frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0,1,2,3,... \quad (9)$$

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G_1(x,t) + pL^{-1}[p^{-\alpha}L\{S_1(u,v) + Q_1(u,v)\}] \quad (10)$$

This is a pairing of FHPTM and transform of Laplace using He's polynomials.

Equating coefficients of the identical powers on both the sides, we get,

$$p^0 : u_0(x,t) = G(x,t)$$

Continuing in this way, the enduring components can completely be achieved also. Thus the series solution is fully calculated. At last, the analytical solution is approximated by the series,

$$u(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x,t)$$

The above solutions in series converge very rapidly, in general. Cherruault and Abbaoui [36] proved the convergence of this above kind of series.

4. Test Examples.

In this segment, we will apply the proposed method to some test problems.

Example 1. Consider a inhomogeneous linear system of time-fractional equations as,

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = 1 + 2u(x) - u^{2}(x), \tag{11}$$

Subject to the initial values

u (x, 0) = 0,

Taking Laplace transform on both sides of Eq. (11),

$$u(p,t) = \frac{0}{p} + \frac{1}{p^{\alpha+1}} + p^{-\alpha} L\left(1 + 2u(x) - u^{2}(x)\right) \tag{12}$$

Taking inverse transform on Eq. (12),

$$u(x,t) = 0 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + L^{-1}\left[p^{-\alpha} L\left(1 + 2u(x) - u^{2}(x)\right)\right] \tag{13}$$

By applying HPM,

$$\sum_{n=0}^{\infty} p^{n} u_{n}(x,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + L^{-1}\left[p^{-\alpha} L\left(1 + 2u(x) - u^{2}(x)\right)\right] \tag{14}$$

Equating coefficients of the identical powers of p in Eqs. (14) respectively,

$$p^{0} : u_{0}(x,t) = 0,$$

$$p^{1} : u_{1}(x,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)},$$

$$p^{2} : u_{2}(x,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{2\alpha}\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^{2}\Gamma(1+3\alpha)},$$ and so on…….

Exact Solution can be easily determined to be,

$$u(x,t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}x + \frac{\log\left(\frac{-1+\sqrt{2}}{1+\sqrt{2}}\right)}{2}\right).$$

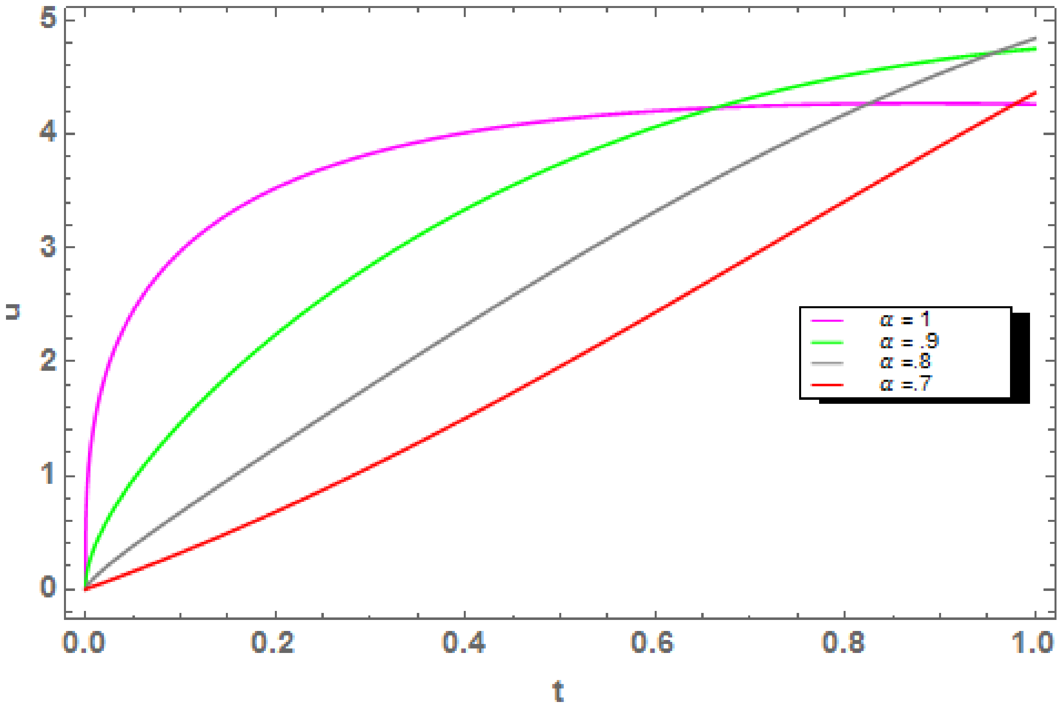

Fig 1. Comparison between different values of α

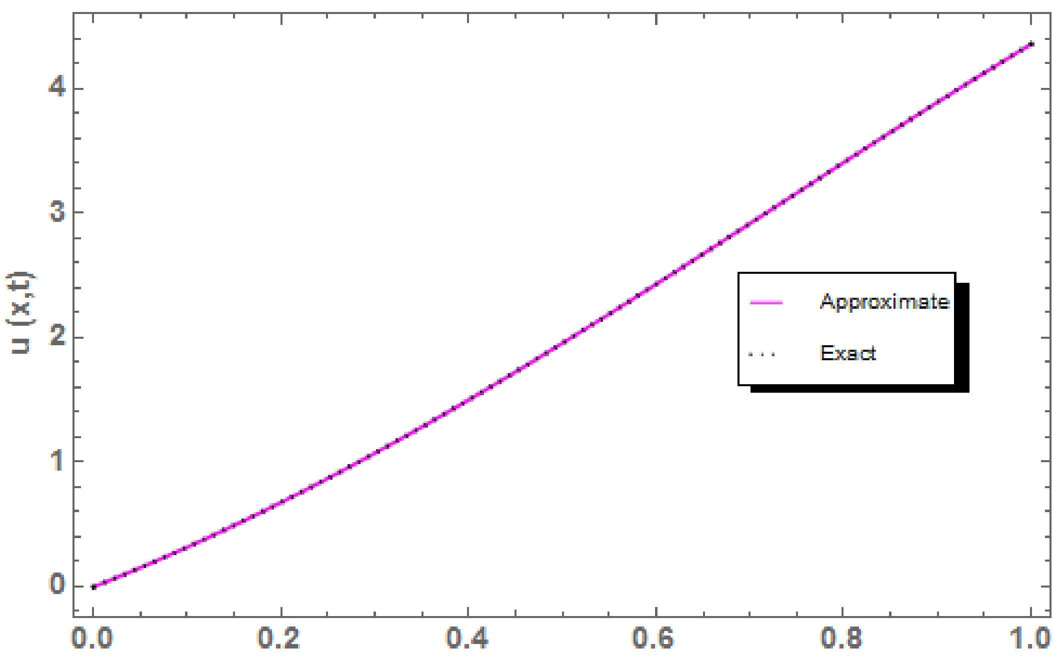

Fig. 2. Comparison between exact and approx. solution

u (x, t) at x = 0.01 and α = 1 for different values of t

x	*Exact Solution* $u(x)$	*FHPTM*	*MVIM*	*VIM*
0.4	0.567812	0.567812	0.513543	0.538667
1.2	1.95136	1.95136	1.90195	2.064
2.0	2.35777	2.35777	2.41229	3.33333
2.8	2.40823	2.40823	2.30603	3.32267
3.6	2.41359	2.41359	2.40026	1.008
4.0	2.41401	2.41401	2.50735	-1.33333

Table 1. It shows approximate and exact solution u(x, t) numeration using FHPTM, MVIM and VIM for example 1

Example 2. Consider a inhomogeneous linear system of time-fractional equations as,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = 1 + x^2 - u^2(x), \qquad (15)$$

Subject to the initial values

u (x, 0) = 1,

Taking Laplace transform on both sides of Eq. (15),

$$u(p,t) = \frac{1}{p} + \frac{1}{p^{\alpha+1}} + p^{-\alpha} L\left(1 + x^2 - u^2(x)\right) \qquad (16)$$

Taking inverse transform on Eq. (16),

$$u(p,t) = \frac{1}{p} + \frac{t^\alpha}{p^{\alpha+1}} + p^{-\alpha} L\left(1 + x^2 - u^2(x)\right) \qquad (17)$$

By applying HPM,

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \frac{t^\alpha}{p^{\alpha+1}} + L^{-1}[p^{-\alpha} L\left(1 + x^2 - u^2(x)\right)] \qquad (18)$$

Equating coefficients of the identical powers of p in Eqs. (18) respectively,

$$p^0 : u_0(x,t) = 1,$$

$$p^1 : u_1(x,t) = \frac{x^2 t^\alpha}{\Gamma(1+\alpha)},$$

$$p^2 : u_2(x,t) = \frac{t^\alpha \left(1 + x^2\right)\Gamma(1+\alpha)}{\Gamma(1+\alpha)^2} - \frac{t^{3\alpha} x^4 \Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2 \Gamma(1+3\alpha)}, \qquad \text{and so on…….}$$

Exact Solution can be easily determined to be, $u(x,t) = x + \frac{e^{-x^2}}{1 + \int_0^x e^{-t^2} dt}$

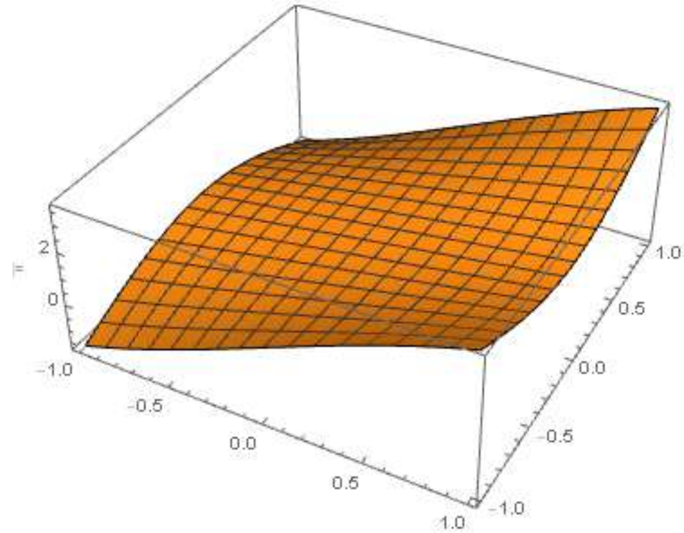

Fig. 3 The space -time graph of exact solution
u (x, t) at $0 \le t \le 1,\ 0 \le x \le 1$ and $\alpha = 1$

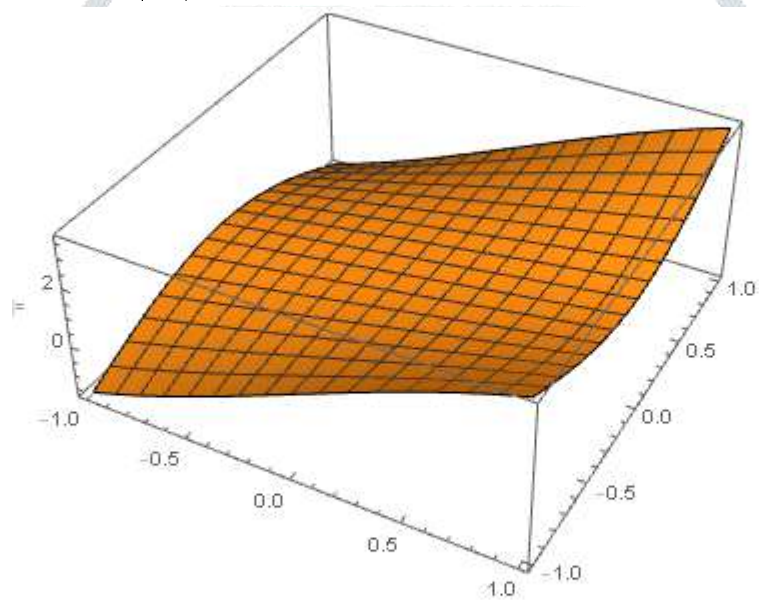

Fig. 4 The space -time graph of approximate solution
u (x, t) at $0 \le t \le 1,\ 0 \le x \le 1$ and $\alpha = 1$

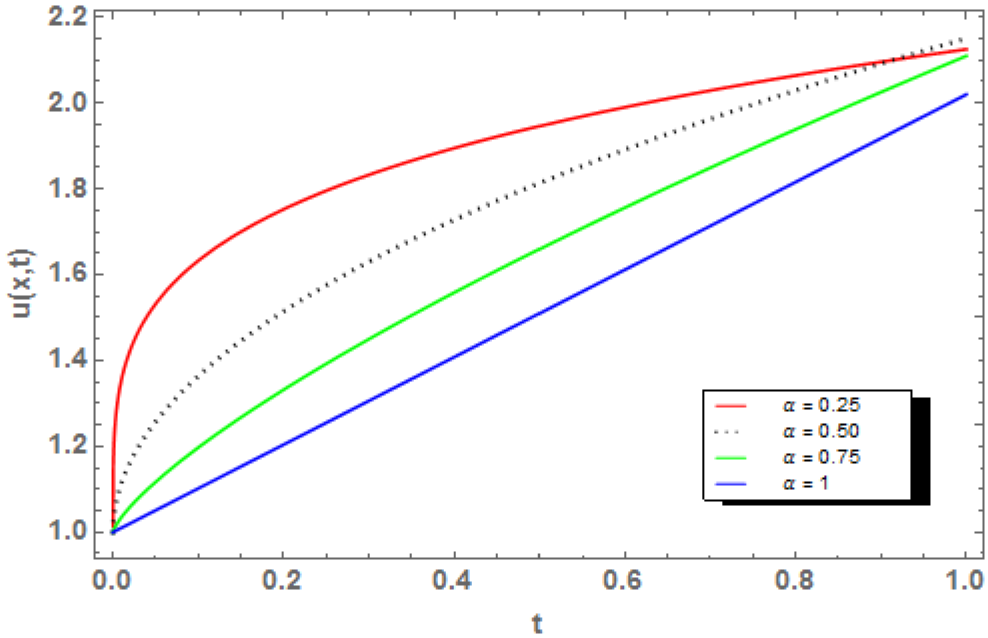

Fig 5. Approximate solution for u(t) at different value of α

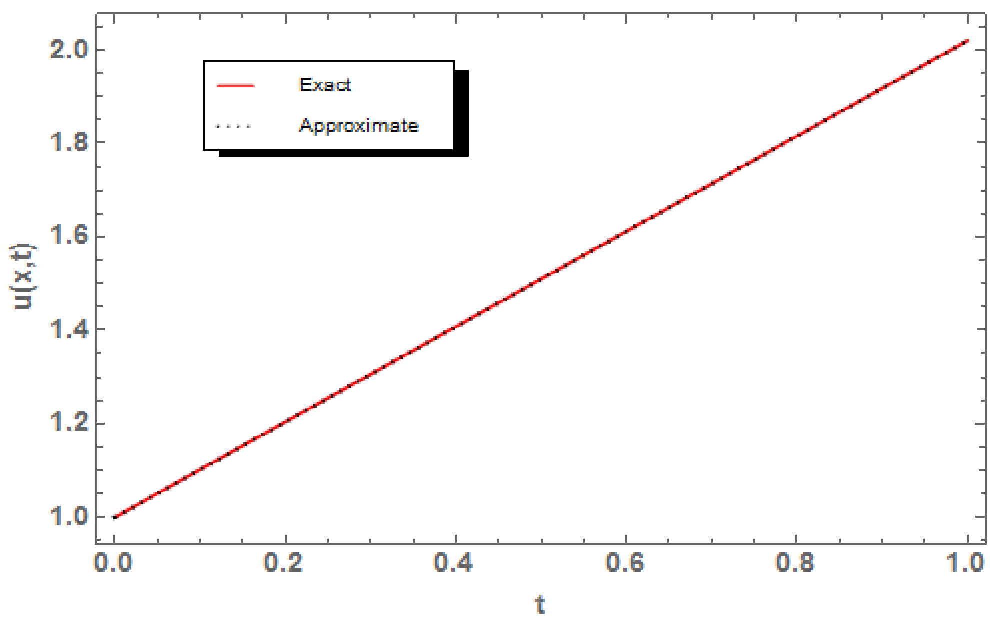

Fig. 6. Comparison between exact and approx. solution

(x, t) at x = 0.01 and α = 1 for different values of t

x	*Exact Solution* $u(x)$	*FHPTM*	*MVIM*	*VIM*
0.4	1.01765	1.01765	1.0153	1.01704
1.2	1.11809	1.11809	1.10893	1.09907
2.0	1.33114	1.33114	1.32233	1.17352
2.8	2.00973	2.00973	2.04175	-1.03075
3.6	2.80021	2.80021	2.76833	-23.3443
4.0	4.00000	4.00000	3.56075	-135.829

Table 2. It shows approximate and exact solution u(x, t) numeration

using FHPTM, MVIM and VIM for example 2

5. Conclusion

In this paper, FHPTM is employed successfully to solve very famous Riccati differential equation. Numerical results suggest that solution of Riccati equation has a rapid convergence to approximate numerical solutions. These numerical results obtained are compared with those approximations, which are obtained from MVIM and VIM. Further, FHPTM has the ability to reduce the computation cost significantly. Thus, it can be easily used for big and small parameters in differential equations. In wide class of differential equations involving highly nonlinear terms, FHPTM can also be applied which will give solution exactly similar to the solution obtained by analytical methods. The best part of this method is that there is no requirement of Adomian polynomials. Without linearization, FHPTM can be applied. This shows that FHPTM is not restrictive. Hence FHPTM is more convenient and accurate than other existing methods.

References

[1] W.T. Reid, Riccati Differential Equations, Academic Press, New York, 1972.

[2] J.F. Carinena, G. Marmo, A.M. Perelomov, M.F. Ranada, Related operators and exact solutions of Schrodinger equations, International Journal of Modern Physics A 13 (1998) 4913–4929.

[3] M.R. Scott, Invariant Imbedding and its Applications to Ordinary Differential Equations, Addison-Wesley, 1973.

[4] M.A. El-Tawil, A.A. Bahnasawi, A. Abdel-Naby, Solving Riccati differential equation using Adomian's decomposition method, Applied Mathematics and Computation 157 (2004) 503–514.

[5] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method, Applied Mathematics and Computation 172 (2006) 485–490.

[6] S. Abbasbandy, A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, Journal of Computational and Applied Mathematics 207 (2007) 59–63.

[7] S. Abbasbandy, Iterated He's homotopy perturbation method for quadratic Riccati differential equation, Applied Mathematics and Computation 175 (2006) 581–589.

[8] F.Z. Geng, Y.Z. Lin, M.G. Cui, A piecewise variational iteration method for Riccati differential equations, Computers and Mathematics with Applications 58 (2009) 2518–2522.

[9] J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics 34 (1999) 699–708.

[10] J.H. He, Variational iteration method for autonomous ordinary differential system, Applied Mathematics and Computation 114 (2000) 115–123.

[11] J.H. He, Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics B 20 (2006) 1141–1199.

[12] J.H. He, X.H. Wu, Variational iteration method: new development and applications, Computers and Mathematics with Applications 54 (2007) 881–894.

[13] J.H. He, Variational iteration method—some recent results and new interpretations, Journal of Computational and Applied Mathematics 207 (2007) 3–17.

[14] J.H. He, G.C. Wu, F. Austin, The variational iteration method which should be followed, Nonlinear Science Letters A 1 (1) (2010) 1–30.

[15] J.F. Lu, Variational iteration method for solving two-point boundary value problems, Journal of Computational and Applied Mathematics 207 (2007) 92–95.

[16] A.A. Soliman, M.A. Abdou, Numerical solutions of nonlinear evolution equations using variational iteration method, Journal of Computational and Applied Mathematics 207 (2007) 111–120.

[17] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burgers and coupled Burgers equations, Journal of Computational and Applied Mathematics 181 (2005) 245–251.

[18] D.D. Ganji, A. Sadighi, Application of He’s homotopy-perturbation method to nonlinear coupled systems of reaction–diffusion equations, International Journal of Nonlinear Sciences and Numerical Simulation 7 (4) (2006) 411–418.

[19] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, International Journal of Nonlinear Sciences and Numerical Simulation 7 (2006) 27–34.

[20] D.H. Shou, J.H. He, Application of parameter-expanding method to strongly nonlinear oscillators, International Journal of Nonlinear Sciences and Numerical Simulation 8 (1) (2007) 121–124.

[21] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 7 (1) (2006) 65–70.

[22] M. Tatari, M. Dehghan, On the convergence of He’s variational iteration method, Journal of Computational and Applied Mathematics 207 (2007) 121–128.

[23] J. Biazar, H. Ghazvini, He’s variational iteration method for solving hyperbolic differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (3) (2007) 311–314.

[24] H. Ozer, Application of the variational iteration method to the boundary value problems with jump discontinuities arising in solid mechanics, International Journal of Nonlinear Sciences and Numerical Simulation 8 (4) (2007) 513–518.

[25] E. Yusufoglu, Variational iteration method for construction of some compact and noncompact structures of Klein–Gordon equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 153–158.