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ON (1, 2)* η-OPEN SETS IN BITOPOLOGICAL SPACES

Hamant Kumar
Department of Mathematics
Veerangana Avantibai Government Degree College, Atrauli-Aligarh, U. P. (India)

Abstract: In this paper, we introduce (1, 2)*-η-open sets and obtain the relationships among some existing open sets like (1, 2)*-semi-open, (1, 2)*-α-open and $\Im_{1,2}$-open sets. Also we study some of basic properties of (1, 2)*-η-open sets. Further, we introduced (1, 2)*-η-neighbourhood and discuss some basic properties of (1, 2)*-η-neighbourhood.

Keywords : (1, 2)*-semi-open, (1, 2)*-α-open, (1, 2)*-η-open sets; (1, 2)*-η-closure and (1, 2)*-η-neighbourhood
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1. Introduction

The study of bitopological space was first intiated by Kelly [1] in the year 1963. By using the topological notions, namely, semi-open, regular-open, α-open and pre-open sets, many new bitopological sets are defined and studied by many topologists. In 2008, Ravi et al. [3] studied the notion of (1, 2)*-open sets in bitopological spaces. In 2004, Ravi and Thivagar [2] studied the concept of stronger from of (1, 2)*-quatient mapping in bitopological spaces and introduced the concepts of (1, 2)*-semi-open and (1, 2)*-α-open sets in bitopological spaces. In 2019, Subbulakshmi et al. [5] introduced the concept of η-open sets in topological spaces.

2. Preliminaries

Throughout the paper $(X, \Im_1, \Im_2)$, (Y, σ_1, σ_2) and $(Z, \wp_1, \wp_2)$ (or simply X, Y and Z) denote bitopological spaces.

Definition 2.1 [2]. Let S be a subset of X. Then S is said to be **$\Im_{1,2}$-open** if $S = A \cup B$, where $A \in \Im_1$ and $B \in \Im_2$. The complement of a $\Im_{1,2}$-open set is $\Im_{1,2}$-closed.

Definition 2.2 [2]. Let S be a subset of X. Then
(i) the **$\Im_{1,2}$-closure** of S, denoted by $\Im_{1,2}$-cl(S), is defined as $\cap$ {F : S $\subset$ F and F is $\Im_{1,2}$-closed};
(ii) the **$\Im_{1,2}$-interior** of S, denoted by $\Im_{1,2}$-int(S), is defined as $\cup$ {F : F $\subset$ S and F is $\Im_{1,2}$-open}.

Note 2.3 [2]. Notice that $\Im_{1,2}$-open sets need not necessarily form a topology.

Definition 2.4. A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) **regular (1, 2)*-open** [6] if $A = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(A))$,
(ii) **(1, 2)*-semi-open** [2] if $A \subset \Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int}(A))$,
(iii) **(1, 2)*-α-open** [2] if $A \subset \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int}(A)))$.

The complement of a $(1, 2)^*$-regular open (resp. $(1, 2)^*$-semi-open, $(1, 2)^*$-α-open) set is called $(1, 2)^*$-regular-closed (resp. $(1, 2)^*$-semi-closed, $(1, 2)^*$-α-closed).

The family of all $(1, 2)^*$-regular-open (resp. $(1, 2)^*$-semi-open, $(1, 2)^*$-α-open) sets in X is denoted by $(1, 2)^*$-RO(X) (resp. $(1, 2)^*$-SO(X), $(1, 2)^*$-αO(X)).

The $(1, 2)^*$-semi-closure (resp. $(1, 2)^*$-α-closure) of a subset A of X is denoted by $(1, 2)^*$-s-cl(A) (resp. $(1, 2)^*$-α-cl(A)), defined as the intersection of all $(1, 2)^*$-semi-closed. (resp. $(1, 2)^*$-semi-closed) sets containing A.

Remark 2.5. It is evident that any $\Im_{1,2}$-open set of X is a $(1, 2)^*$-α-open and each $(1, 2)^*$-α-open set of X is $(1, 2)^*$-semi-open. But the separate converses are not true.

3. (1, 2)*-η-open sets

Defination 3.1. A subset A of a bitopological space X is said to be **$(1, 2)^*$-η-open** if $A \subset \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int})(A)) \cup \Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int}((A))$ and $(1, 2)*$-η-closed if $A \supset \Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(A))) \cap \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(A))$.

The **$(1, 2)^*$-η-closure** of a subset A of X is denoted by $(1, 2)^*$-η-cl(A), defined as the intersection of all $(1, 2)^*$-η-closed sets containing A.

The family of all $(1, 2)^*$-η-open (resp. $(1, 2)^*$-η-closed) subsets of X is denoted by $(1, 2)^*$-ηO(X) (resp. $(1, 2)^*$-ηC(X)).

Proposition 3.2.
(i) Every $(1, 2)^*$-α-open set is $(1, 2)^*$-η-open.
(ii) Every $(1, 2)^*$-semi-open set is $(1, 2)^*$-η-open.
(iii) Every regular $(1, 2)^*$-open set is $(1, 2)^*$-η-open.
(iv) Every $\Im_{1,2}$-open set is a $(1, 2)^*$-η-open set.
Proof. (i)
Let A be a $(1, 2)^*$-α-open set, then
$A \subset \Im_1\Im_2\text{-int}(\Im_1\Im_{,2}\text{-cl}(\Im_1\Im_2\text{-int}(A)))$
$\Rightarrow A \subset \Im_1\Im_2\text{-int}(\Im_{1,}\Im_{,2}\text{-cl}(\Im_1\Im_2\text{-int}(A))) \cup \Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int})(A)))$
$\Rightarrow$ A is $(1, 2)^*$-η-open set.
Hence A is a $(1, 2)^*$-η-open set.

(ii) Let A be a $(1, 2)^*$-semi-open set, then
$A \subset \Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))$
$\Rightarrow A \subset \Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))) \cup \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A)))$
or
$A \subset \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))) \cup \Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))$
$\Rightarrow$ A is $(1, 2)^*$-η-open set.
Hence A is a $(1, 2)^*$-η-open set.

(iii) Let A be a $(1, 2)^*$-regular open set, then
$A = \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(A))$…………..(i)
$\Rightarrow \Im_1\Im_2\text{-int}(A) = \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(A)))$
$\Rightarrow \Im_1\Im_2\text{-int}(A) = \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-int}(A))$
Using (i)
$\Im_1\Im_2\text{-int}(A) = A$ ………………….. (ii)

Hence A is $\Im_1\Im_2$-open.
Now we know that
$A \subset \Im_1\Im_2\text{-cl}(A)$
$\Rightarrow \text{int}(A) \subset \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(A))$
Using (ii)
$A \subset \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(A))$
Again using (ii), we get
$A \subset \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A)))$
$\Rightarrow A \subset \Im_1\Im_2\text{-int}(\Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))) \cup \Im_1\Im_2\text{-cl}(\Im_1\Im_2\text{-int}(A))$
Hence, A is $(1, 2)^*$-η-open set.

(iv) East to prove.

Remark 3.3. We have the following implications for the properties of subsets:

$\Im_{1,2}$-open ⇒ $(1, 2)^*$-α-open ⇒ $(1, 2)^*$-semi-open ⇒ $(1, 2)^*$-η-open

Where none of the implications is reversible as can be seen from the following examples:

Example 3.4. Let X = {a, b, c, d}, $\Im_1$ = {ϕ, {a}, X} and $\Im_2$ = {ϕ, {b}, {a, b, c}, X}. Then
(i) The $\Im_{1,2}$-open sets are : ϕ, {a}, {b}, {a, b}, {a, b, c}, X.
(ii) The $(1, 2)^*$-semi open sets are : ϕ, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X.
(iii) The $(1, 2)^*$-α-open sets are : ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}, X
(iv) The$(1, 2)^*$-η-open sets are : ϕ, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, X.

Example 3.5. Let X = {a, b, c}, $\Im_1$ = {ϕ, {b}, X} and $\Im_2$ = {ϕ, {c}, X}. Then
(i) The $\Im_{1,2}$ -open sets are : ϕ, {b}, {c}, {b, c}, X}.
(ii) The $(1, 2)^*$-semi-open sets are : ϕ, X, {b}, {c}, {a, b}, {a, c}, {b, c}.
(iii) The $(1, 2)^*$-α-open sets are : ϕ, X, {b}, {c}, {b, c}.
(iv) The $(1, 2)^*$-η-open sets are : ϕ, {b}, {c}, {a, b}, {a, c}, {b. c}, X.

Example 3.6. Let X = {a, b, c,}, $\Im_1$ = {ϕ, {a}, X} and $\Im_2$ = {ϕ, {b}, X}. Then
(i) The $\Im_1\Im_2$-open sets are : ϕ, {a}, {b}, {a, b}, X.
(ii) The $(1, 2)^*$-semi open sets are : ϕ, {a}, {b}, {a, b}, {a, c}, {b, c}, X.
(iii) The $(1, 2)^*$-α-open sets are : ϕ, {a}, {b}, {a, b}, X.
(iv) The $(1, 2)^*$-η-open sets are : ϕ, {a}, {b}, {a, b}, {a, c}, {b, c}, X.

Example 3.7. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then
(i) The $\Im_1\Im_2$ -open sets are : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {a, c, d}, {b, c, d}.
(ii)The $(1, 2)^*$-semi open sets : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(iii) The $(1, 2)^*$-α-open sets are : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}, {a, c, d}, {b, c, d}.
(iv) The $(1, 2)^*$-η-open sets are : ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.

Theorem 3.8. Intersection of two $(1, 2)^*$-η-open sets need not be a $(1, 2)^*$-η-open set.
Proof. East to verify.

Example 3.9. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then $(1, 2)^*$-ηO(X) = {ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here A={c, d} and B = {a, b, d} are $(1, 2)^*$-η-open sets, but A ∩ B = {d} is not a $(1, 2)^*$-η-open set.

Theorem 3.9. The finite union of $(1, 2)^*$-η open sets is a $(1, 2)^*$-η open set.
Proof. East to verify.

4. $(1, 2)^*$-η-neighborhood and their basic properties

In this section, we introduce $(1, 2)^*$-η-neighborhood (shortly $(1, 2)^*$-η-nbhd in bitopological spaces by using the notion of $(1, 2)^*$-η-open sets. We prove that every nbhd of x in (X, $\Im_1$, $\Im_2$) is $(1, 2)^*$-η-nbhd of x but not conversely.

Definition 4.1. Let (X, $\Im_1$, $\Im_2$) be a bitopological space and let x ∈ (X, $\Im_1$, $\Im_2$). A subset N of (X, $\Im_1$, $\Im_2$) is said to be a **$(1, 2)^*$-η-nbhd** of x iff there exists a $(1, 2)^*$-η-open set G such that x ∈ G ⊂ N.

Definition 4.2. A subset N of a bitopological space (X, $\Im_1$, $\Im_2$), is called a **$(1, 2)^*$-η-nbhd** of A ⊂ (X, $\Im_1$, $\Im_2$) iff there exists a (1, 2)*-η-open set G such that A ⊂ G ⊂ N.

Remark 4.3. The $(1, 2)^*$-η-nbhd N of x ∈ (X, $\Im_1$, $\Im_2$) need not be a $(1, 2)^*$-η-open in (X, $\Im_1$, $\Im_2$).

Example 4.4. Let X = {a, b, c, d} and $\Im_1$= {ϕ, {a}, {b}, {a, b}, {b, c, d}, X} and $\Im_2$ = {ϕ, {a, c, d}, X}. Then $(1, 2)^*$-ηO(X) = {ϕ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Note that {a, d} is not a $(1, 2)^*$-η-open set, but it is a $(1, 2)^*$-η-nbhd of a, since {a} is a $(1, 2)^*$-η-open set such that a ∈ {a} ⊂ {a, d}.

Theorem 4.5. Every nbhd N of x ∈ (X, $\Im_1$, $\Im_2$) is a $(1, 2)^*$-η-nbhd of (X, $\Im_1$, $\Im_2$).
Proof. Let N be a nbhd of point x ∈ (X, $\Im_1$, $\Im_2$). To prove that N is a $(1, 2)^*$-η-nbhd of x. By definition of nbhd, there exists an open set G such that x ∈ G ⊂ N. As every open set is $(1, 2)^*$-η-open set so, G is $(1, 2)^*$-η-open set such that x ∈ G ⊂ N. Hence N is $(1, 2)^*$-η-nbhd of x.

Remark 4.6. In general, a $(1, 2)^*$-η-nbhd N of x ∈ (X, $\Im_1$, $\Im_2$) need not be a nbhd of x in (X, $\Im_1$, $\Im_2$), as seen from the following example.

Example 4.7. Let X = {a, b, c, d} with topology $\Im_1$ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X} and $\Im_2$ = {ϕ, {a, b, d}, X} Then $(1, 2)^*$-ηO(X) = {X, ϕ, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. The set {a, c} is $(1, 2)^*$-η-nbhd of the point c, since the $(1, 2)^*$-η-open set {a, c} is such that c ∈ {a, c} ⊂ {a, c}. However, the set {a, c} is not a nbhd of the point c, since no open set G exists such that c ∈ G ⊂ {a, c}.

Theorem 4.8. If a subset N of a space (X, $\Im_1$, $\Im_2$) is $(1, 2)^*$-η-open, then N is a $(1, 2)^*$-η-nbhd of each of its points.
Proof. Suppose N is $(1, 2)^*$-η-open. Let x ∈ N. We claim that N is $(1, 2)^*$-η-nbhd of x. For N is a $(1, 2)^*$-η-open set such that x ∈ N ⊂ N. Since x is an arbitrary point of N, it follows that N is a $(1, 2)^*$-η-nbhd of each of its points.

Definition 4.9. Let x be a point in a space (X, $\Im_1$, $\Im_2$). The set of all $(1, 2)^*$-η-nbhd of x is called the **$(1, 2)^*$-η-nbhd system** at x, and is denoted by $(1, 2)^*$-η-N(x).

Theorem 4.10. Let (X, $\Im_1$, $\Im_2$) be a bitopological space and for each x ∈ (X, $\Im_1$, $\Im_2$). Let $(1, 2)^*$- η-N(x) be the collection of all $(1, 2)^*$-η-nbhds of x. Then we have the following results.
(i) ∨ x ∈ (X, $\Im_1$, $\Im_2$), $(1, 2)^*$-η-N(x) ≠ ϕ.

(ii) N $\in$ $(1, 2)^*$-η-N(x) $\Rightarrow$ x$\in$ N.
(iii) N $\in$ $(1, 2)^*$-η-N(x), M $\supset$ N $\Rightarrow$ M $\in$ $(1, 2)^*$-η-N(x).
(iv) N $\in$ $(1, 2)^*$-η-N(x), M $\in$ $(1, 2)^*$- η-N(x) $\Rightarrow$ N $\cap$ M $\in$ $(1, 2)^*$-η-N(x).
(v) N $\in$ $(1, 2)^*$-η-N(x) $\Rightarrow$ there exists M $\in$ $(1, 2)^*$-η-N(x) such that M $\subset$ N and M $\in$ $(1, 2)^*$-η-N(y) for every y $\in$ M.

Proof. (i) Since (X, $\Im_1$, $\Im_2$) is a $(1, 2)^*$-η-open set, it is a $(1, 2)^*$-η-nbhd of every x $\in$ (X, $\Im_1$, $\Im_2$). Hence there exists at least one $(1, 2)^*$-η-nbhd (namely - (X, $\Im_1$, $\Im_2$)) for each x $\in$ (X, $\Im_1$, $\Im_2$). Hence$(1, 2)^*$-η-N(x) $\neq \phi$ for every x $\in$ (X, $\Im_1$, $\Im_2$).

(ii) If N $\in$ $(1, 2)^*$-η-N(x), then N is a (1, 2)*-η-nbhd of x. So by definition of $(1, 2)^*$-η-nbhd, x $\in$ N.

(iii) Let N $\in$ $(1, 2)^*$-η-N(x) and M $\supset$ N. Then there is a $(1, 2)^*$-η-open set G such that x $\in$ G $\subset$ N. Since N $\subset$ M, x $\in$ G $\subset$ M and so M is $(1, 2)^*$-η-nbhd of x. Hence M $\in$ $(1, 2)^*$-η-N (x).

(iv) Let N $\in$ $(1, 2)^*$-η-N(x) and M $\in$ (1, 2)*-η-N(x). Then by definition of $(1, 2)^*$-η-nbhd. Hence x $\in$ $G_1 \cap G_2$ $\subset$ N $\cap$ M $\Rightarrow$ (1). Since $G_1 \cap G_2$ is a $(1, 2)^*$-η-open set, (being the intersection of two (1, 2)*-η-open sets), it follows from (1) that N $\cap$ M is a $(1, 2)^*$-η-nbhd of x. Hence N $\cap$ M $\in$ $(1, 2)^*$- η-N(x).

(v) If N $\in$ $(1, 2)^*$-η-N(x), then there exists a $(1, 2)^*$-η-open set M such that x $\in$ M $\subset$ N. Since M is a $(1, 2)^*$-η-open set, it is $(1, 2)^*$-η-nbhd of each of its points. Therefore M$\in$ $(1, 2)^*$-η-N(y) for every y$\in$ M.

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