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DYNAMIC RESPONSE OF HIGHWAY BRIDGES TO MOVING VEHICLES CONSIDERING HIGHER MODES

[1] Asst Prof. S. M. Gajbhiye ,[2]Mr. Javed Aziz Siddiqui
[1]Assistant Professor, M. Tech Student
[1]Structural Engineering Department
[1]Civil Department, Guru Nanak Institute of Technology, Nagpur (MS), India

Abstract **:** The impact factor included in the design code formula to account for the dynamic effect of traffic is a function of only the span length with no reference to other factors affecting the bridge response. In this paper the dynamic response of highway bridges is studied to evaluate the various parameters affecting the bridge dynamic. The bridge is modeled as a one dimensional beam element. The vehicle model is ranged from moving force, moving mass and lumped mass supported by springs and damping elements. Closed form expressions to the bridge vibration modes and dynamic response to moving loads are obtained using Galerkin's approach. This gives more insight into the nature of the problem and helps identify the significant parameters affecting the bridge response. The developed approach allows for the consideration of higher modes of vibrations in the dynamic response of bridges to moving vehicles. Different bridge models are analyzed for different vehicle speed and different vehicle weight considering the contribution of the various individual modes. The important effect of higher modes on the bridge response is highlighted. Depending on the dynamic characteristics of the bridge and the speed of the vehicle higher modes contribution can become too significant to be neglected.

IndexTerms - Truck impact, bridge vibration, moving loads, Galerkin's approach

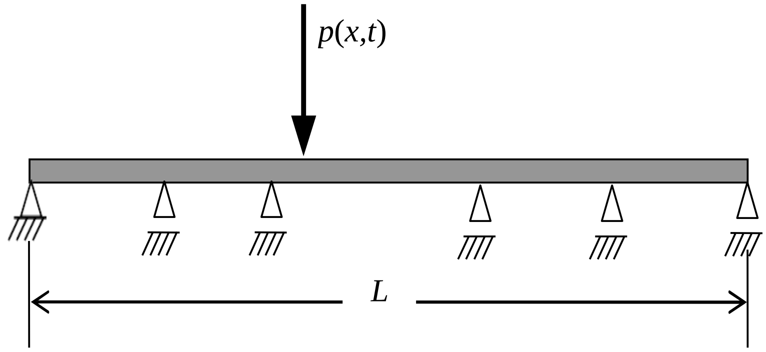
I. INTRODUCTION

Vibration of bridge structures under the passage of vehicles is a significant problem for bridge engineers where stresses are increased above those due to static-loads. This is usually accounted for by the code specified impact factor in bridge design. The investigation of dynamic behavior of bridge structures under been the subject of many previous research work. Different mathematical models have been used to simulate both the moving vehicle and the bridge dynamics. The vehicle simulation could be modeled as moving load [1-4], moving mass [5, 6], and lumped mass supported by springs and damping element [7-12]. Moving load vehicle simulation assumption is accepted for long span bridges; however this simulation becomes less accurate when the mass ratio between vehicle and bridge becomes large. To study the effect of heavy vehicle on bridges, moving mass simulation could be considered. Mass-sprung vehicle simulation is used to study the effect of vehicle dynamic resulting from the suspension and tire system particularly on rough surfaces.

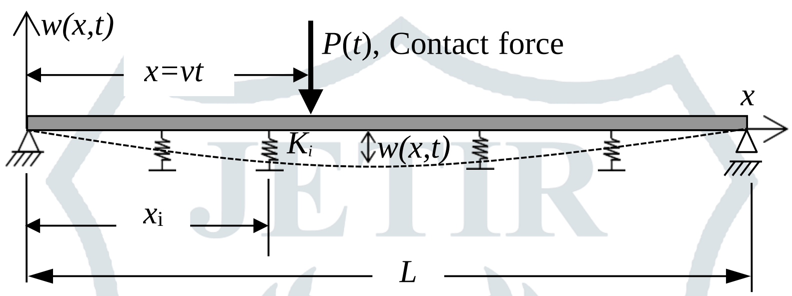
A beam that is simply supported at both ends is the most widely used bridge model that has been adopted in studying bridge response to moving vehicles. Various types of bridge models have been considered in studying the vehicle induced vibrations, including simple span bridge [1, 8, 13], multispan uniform or nonuniform bridges [14], girder or multi-girder bridges [15].

The results of those studies identified several parameters that affect the bridge dynamic response including: vehicle characteristic, bridge characteristic, vehicle speed and deck profile and roughness. Because of the complexity of bridge dynamics, establishing a clear correlation between these parameters and bridge response has not been easy, even when sophisticated models were used in analytical studies. In fact, the use of simplified models may be more effective in identifying such correlation [7].

A clear understanding of the dynamics of the bridge-vehicle system is necessary for building safe and economic bridges that resist vibration; and identifying vehicle characteristics to reduce bridge damage. Hence, the objective of this paper is to analyze the highway bridge dynamic response using simple bridge model, namely simple and continuous beams while the vehicle model may include moving load, moving mass, and lumped mass supported by springs and damping elements. Closed form mathematical expressions for the mode shapes of vibration was obtained using Galerkin's approach [16]. Then the bridge dynamic response to moving vehicle is obtained using mode superposition and Duhamel integral techniques. A parametric study is carried out considering various simple and continuous span bridges of different length to study the factors controlling the response problem. The considered parameters include: the bridge span, vehicle weight, vehicle speed, and the effect of higher modes. The derived closed form expressions for the vibration mode shapes and the bridge dynamic response help give more insight to the nature of the problem.



a) Multi-span continuous bridge model on rigid supports.



b) Idealized bridge model as elastically supported flexible beam

Fig. 1. Bridge model.

II. BRIDGE AND VEHICLE MODELING

Bridge Model

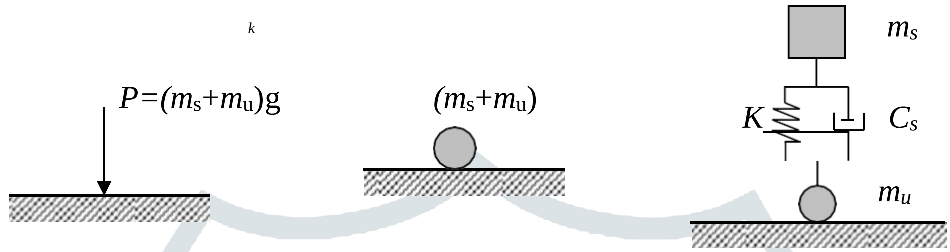
For the present study a multi-span continuous bridge is modeled as a single simply supported span on resting on intermediate linear springs with infinitely large stiffness. The beam is considered as undamped linear elastic Bernoulli-Euler beam of length L, mass per unit length m, bending stiffness EI, and supported by nk elastic spring of infinitely large stiffness Ki at locations of the supports xi, i=1, nk as shown in Fig. 1. The equation of motion and the boundary conditions, respectively, under the action of moving load p(x,t) can be written as

$$EIw^{iv}(x,t)+\sum_{i=1} K_i w(x_i,t)\delta(x-x_i)+m\ddot{w}(x,t)=P(t)\delta(x-vt), \qquad (1)$$

$$w'(0)=w'(L)=w''(0)=w''(L)=0 \qquad (2)$$

in which w(x,t) represents the displacement at any point x at any time t, δ(x-xi) is the Dirac delta function which is equal to zero everywhere except for x = xi, and the primes denotes differentiation with respect to x while the dots denotes differentiation with respect to t. The vehicle load is P(t) and v is the vehicle speed assumed constant. The following assumptions are used in the formulation of the equation of motion.

- The bridge is slender and therefore the effect of rotary inertia and shear deformation are neglected.
- Only flexural vibration of the bridge is considered. The torsional behavior caused by eccentric loading of the bridge deck is disregarded in this study.
- Bridge damping is small and therefore neglected.



(a) Moving load (b) Moving mass (c) Sprung mass SDOF

Fig. 2. Idealized vehicle models

Vehicle Model

Heavy vehicle consists of several major components such as tractor, trailers and suspension systems. It can be modeled by a set of lumped masses, springs and dampers. Figure 2 presents the vehicle models used in this study ranging from a moving constant force model, a moving mass model, to a sprung mass model.

The equation of motion of the vehicle models of Fig.2. can be written as:

- Moving force model

$$p(x,t) = (m_s + m_u)g \times \delta(x - vt) \tag{4}$$

- Moving mass model

$$p(x,t) = (m_s + m_u)(g + \ddot{w}_u) \times \delta(x - vt) \tag{5}$$

- Sprung mass model

$$-(m_s + m_u)g - m_u \ddot{w}_u + K_s(w_s - w_u) + C_s(\dot{w}_s - \dot{w}_u) + P(t) = 0 \tag{6}$$

$$m_s \ddot{w}_s + K_s(w_s - w_u) + C_s(\dot{w}_s - \dot{w}_u) = 0$$

Equation (5) and (6) are the dynamic equilibrium equation for the unsprung mass and sprung mass respectively, where wu and ws are the vertical displacement of the vehicle un-sprung mass mu and the vehicle sprung mass ms, respectively; Ks the stiffness of the linear spring connecting the two masses; Cs the damping coefficient of the viscous damper; v the vehicle velocity assumed constant; P(t) the contact force and g the acceleration of gravity. From Eqs. (5) and (6) the contact force may be expressed as:

$$p(x,t) = P(t)\delta(x - vt) = \{m_s \ddot{w}_s + m_u \ddot{w}_u + (m_s + m_u)g\}\delta(x - vt) \tag{7}$$

Vehicle-bridge interaction

Assuming that the vehicle never loses contact with the bridge (i.e. P(t)>0), the following coupling equations for the point of contact, x(t) can be expressed:

$$w_u(t) = w(x(t),t) \tag{8}$$

$$\dot{w}_u(t) = \frac{\partial w}{\partial x}v + \frac{\partial w}{\partial t} \tag{9}$$

$$\ddot{w}_u(t) = \frac{\partial^2 w}{\partial x^2}v^2 + \frac{\partial^2 w}{\partial t \partial x}2v + \frac{\partial w}{\partial x}a + \frac{\partial^2 w}{\partial t^2} \tag{10}$$

where $\dot{w}_u(t)$ and $\ddot{w}_u(t)$ denote the unsprung vertical velocity and acceleration, respectively, and v and a the vehicle velocity and acceleration in the longitudinal direction. Since the vehicle moves at constant velocity Eq. (10) reduces to:

$$\ddot{w}_u(t) = \frac{\partial^2 w}{\partial x^2}v^2 + \frac{\partial^2 w}{\partial t \partial x}2v + \frac{\partial^2 w}{\partial t^2} \tag{11}$$

It can be seen that the interaction force, $P(t)$, between the moving vehicle and the bridge depends on the velocity and acceleration of the vehicle, and the vibration of the bridge structure.

III. FREE VIBRATION ANALYSIS OF BRIDGE MODEL

Equation of motion for the undamped free vibration and the boundary conditions ofthe beam shown in Fig. 1. read

$$EIw^{iv}(x,t) + \sum_{i=1}^{n} K_k w(x_i,t)\delta(x - x_i) + m\ddot{w}(x_i,t) = 0, \tag{12}$$

$$w(0,t) = w'(0,t) = w(L,t) = w'(L,t) = 0 \tag{13}$$

Now, let the following product solution be assumed [17]:

$$w(x,t) = Y(x)e^{iwt} \tag{14}$$

Substitute Eq.(14) in Eqs. (12) and (13) yields the following eigenvalue problem

$$EIY^{iv}(x) + \sum_{i=1}^{n} K_k Y(x_i)\delta(x - x_i) - \omega^2 mY(x_i) = 0, \tag{15}$$

$$Y'(0) = Y'(L) = Y''(0) = Y''(L) = 0 \tag{16}$$

Galerkin's approach is used to obtain approximate mathematical expressions for the vibration modes $Y_k(x)$, k=1, 2, 3, … and the corresponding natural frequencies

ω_k. Let the deformed shape $Y(x)$ in (15) and (16) be perturbed by a small arbitrary change $\delta Y(x)$. The Extended principle of virtual work reads [16]

$$\int_0^l \left\{ EIY^{iv}(x) + \sum_{i=1}^{nk} K_i Y(x_i)\delta(x - x_i) - \omega^2 mY(x) \right\} \delta Y(x)dx$$
$$+ \left[EIY'(x)\delta Y'(x) - EIY''(x)\delta Y(x)\right]_0^l = 0, \tag{17}$$

The second term in (17) contains the contribution of the unsatisfied natural boundary conditions. In order to determine the vibration modes and natural frequencies let the following series be used

$$Y(x) = \sum_{j=1}^{\infty} a_j \Phi_j(x). \tag{18}$$

where a_j' are arbitrary constants. The shape functions $\Phi_j(x)$, j=1,2,3,… must satisfy the essential geometric boundary conditions but may violate any or all the natural boundary conditions. Now let $\Phi_j(x)$ be chosen as:

$$\Phi_j(x) = \sqrt{\frac{2}{L}} \sin\frac{j\pi x}{L}. \qquad j = 1,2,\ldots. \tag{19}$$

Since $\Phi_j(x)$, given by Eq. (19), satisfy all boundary conditions, the second term in (17) vanishes. The infinite series in Eq. (18) is truncated at a finite set of dimension n, thus yielding an approximate solution. The accuracy of the obtained solution depends on the dimension n of the assumed set of functions as well as on the dynamic characteristics of the considered structure. Substituting (18) and (19) in (17) and noting that a_j are arbitrary yields the following set of homogeneous algebraic equations:

$$\sum_{j=1}^{n} a_j \rho_{rj} = 0 \quad , \qquad r = 1,n \tag{20}$$

Where $$\varrho_{rj} = j^4\pi^4\varrho_{rj} + \sum_{i=1}^{nk} 2\eta_i \sin\left(\frac{j\pi x_i}{L}\right)\sin\left(\frac{r\pi x_i}{L}\right) - \lambda\varrho_{rj} \tag{21}$$

in which δ_{rj} is the Kronecker's symbol,

$\lambda = \omega^2 mL^4/EI$ is dimensionless frequency parameter, and

$\eta_i = K_i L^3/EI$ is dimensionless stiffness parameter of the i^{th} spring.

Equations (20) have a non trivial solution only if

$$\det \rho_{rj} = 0 \tag{22}$$

which yields the frequency parameters λ_K and consequently the natural frequencies ω_K, k=1 ,2, 3, … . Using λ_K in (20) yields the required coefficient a_K up to a common arbitrary factor, for the mode shapes Y_K, k=1, 2, 3, … . The stiffness parameters were given large values to simulate the rigid support. Then (10) was used to obtain the mathematical expression for the vibration mode shapes of the bridge.

A computer code using MATLAB program was developed to solve Eq. (22) to obtain the required natural frequencies and the vibration mode shapes. The obtained approximate closed form solution enables the analyst to identify the effect of various modes, especially higher modes, on the dynamic response. Furthermore, these closed form expression are used to obtain the straining actions of the bridge in terms of bending moment and shear force at any point and at any time due to moving vehicles. The calculated mode shapes are normalized to satisfy the orthonormality condition

$$\int^{l} mY_j(x)Y_k(x)dx = \delta_{jk}, \tag{23}$$

IV. DYNAMIC RESPONSE OF BRIDGE TO MOVING VEHICLE

Now let the following series expansion be used:

$$w(x,t) = \sum_i Y_i(x) z_i(t) \tag{24}$$

In which the coordinate functions $Y_i(x)$ are the previously calculated normalized mode shapes thus satisfying all boundary conditions, and $z_i(t)$ are the corresponding time response functions, $i = 1, n$. Substituting Eq. (24) in Eq. (1) yields

$$\sum_k \left\{ EIz_k(t) Y_k^{iv}(x) + z_k(t) \sum_{i=1}^{nk} K_i Y_k(x_i) \delta(x - x_i) + mY_k(x) \ddot{z}_k(t) \right\} = P(t) \delta(x - vt) \tag{25}$$

Multiplying both side of Eq.(25) by $Y_j(x)$ and integrating along the beam length, considering the orthonormality of the modes, and by virtue of (15) yields the following set of decoupled equations of motion for $z_k(t)$, k=1,n

$$Z_k(t) + \omega k^2{}_k Z_k(t) = \frac{1}{m} \int_0^L Yk(x) P(t) \varrho(x - vt) dx \qquad k=1,n \tag{26}$$

Making use of the properties of the Dirac delta function Eq. (26) reduces to

$$\ddot{z}_k(t) + \omega_k^2 z_k(t) = \frac{P(t)}{m} Y_k(vt), \qquad k = 1,n \tag{27}$$

Duhamel's integral is used to obtain a closed form solution for $zk(t)$ then the solution was utilized in (24) to get the displacement response of the bridge to vehicle load at any time and at any point on the bridge. The obtained mathematical expression of the vibration modes $Y_k(x)$ enables the analyst to calculate time response of internal

forces at different locations of the bridge. Thus the bending moment $M(x,t)$ and theshear force $V(x,t)$ at any point x and time t are obtained as

$$M(x,t) = -EIw'(x,t) \tag{28}$$
$$V(x,t) = -EIw''(x,t) \tag{29}$$

The interaction between vehicles and bridges is calculated through an iterativemethod as follows:

(i) Calculate the initial vehicle load from the vehicle dynamic response equations.
(ii) Determine the bridge response based on the calculated vehicle load.
(iii) Using the bridge response as a new input for the vehicle model, calculate anew vehicle load.
(iv) Calculate the bridge response based on vehicle load from (iii) and repeat untilconvergence is achieved.

V. MODEL VALIDATION AND NUMERICAL RESULTS

Numerical examples of simple span and continuous bridges are presented in this section for verification and validation of the developed models. Results of the present model are compared with those of previous research as well as finite element modeling using SAP2000 v. 10.0.1 [18]. In the following examples, dynamic amplification factors for displacements (*DI*), for bending moments (*MI*) and for shear (*VI*) are presented. The definition for *DI*, *MI*, and *VI* adopted in this study is the ratio of the absolute maximum live (moving) load dynamic response to the absolute maximum live (moving) load static response.

Simply supported bridge subjected to moving load vehicle idealization

The simply supported bridge subjected to a constant moving load, *P*, moving at constant speed, *v*, is the most fundamental problem that should be considered in thestudy of vehicle-induced vibrations on bridges. It belongs to the very few moving load problems that can be solved analytically, see [1]. Adopting the moving load model means that the influence of the inertia of the vehicle mass and the bridge-vehicle interaction are neglected. The dynamic effects are then caused by the varying position of the load. Still this vehicle idealization is good approximation for low values of the vehicle mass to bridge mass ratio and for long span bridges.

For the present purpose, let us consider a simple beam bridge of length L=34 m, mass per unit length m = 11400 kg/m, flexural rigidity $EI = 9.92\times10^{10}$ Nm^2 and vehicle load 350 kN. This bridge is solved using the exact analytical model of [1], the proposed method using Galerkin's approach, and the SAP2000 model. For this relatively simple case, there was perfect match in the natural frequencies and vibration modes derived from the three methods mentioned above. The vertical mid-span displacement, the mid-span moment, and the shear at the right end are presented in Figures 3-5, respectively, normalized by dividing by the maximum static values.

It is clear from Figs. 3-5 that the results of the present model are in very good agreement with those obtained using SAP2000 and the exact expressions. The maximum normalized displacement (i.e. DI) is calculated as 1.258. The corresponding value for the bending moment is 1.09 and for shear at right end is 1.16. Furthermore, the obtained value of DI is in good agreement with the values given in [1].

The present analysis allows for studying the individual modes contribution to the bridge response. Figures 6-8 show the effect of higher modes on the dynamic response of the bridge to moving load. While their effect on the displacement response is negligible, higher modes have significant effect on the maximum moment and shear values and they should be included in bridge analysis and design.

Two span continuous bridge response to moving vehicle

The dynamic response of two span continuous beams to moving vehicle, shown in Fig.9, and previously analyzed for moving load in [19], is presented. The bridge is of total length 81 m; mass per unit length 11400 kg/m, bending stiffness EI 9.92×10^{10} N/m^2. Two values for the vehicle load are considered, 300 kN and 600 kN, as well as two speeds 100 and 150 km/hr.

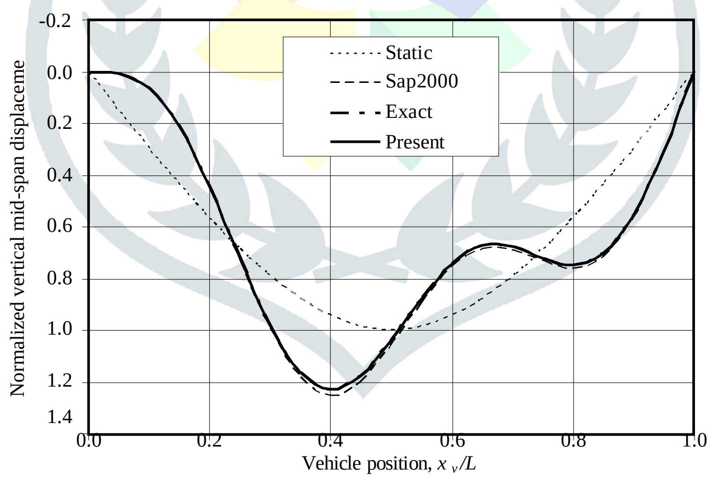


Fig. 3. Normalized vertical displacements at mid-span versus vehicle position

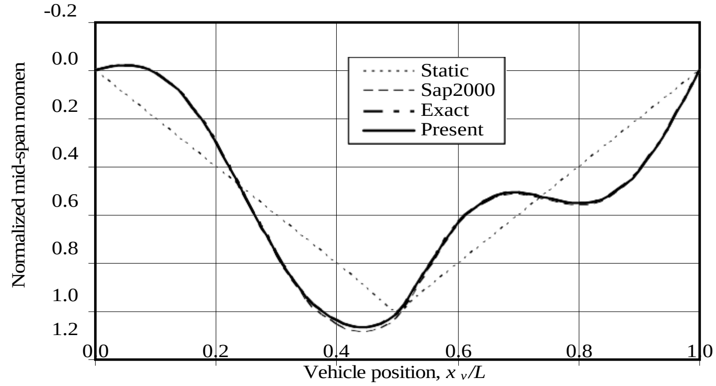


Fig. 4. Normalized mid-span moment versus vehicle position

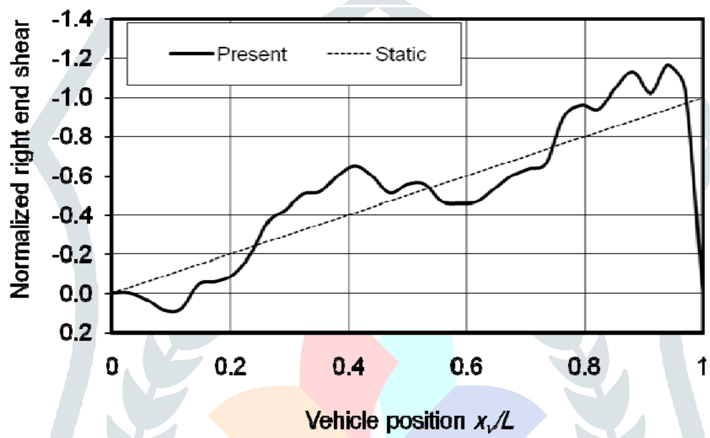


Fig. 5. Normalized shear at right versus vehicle position

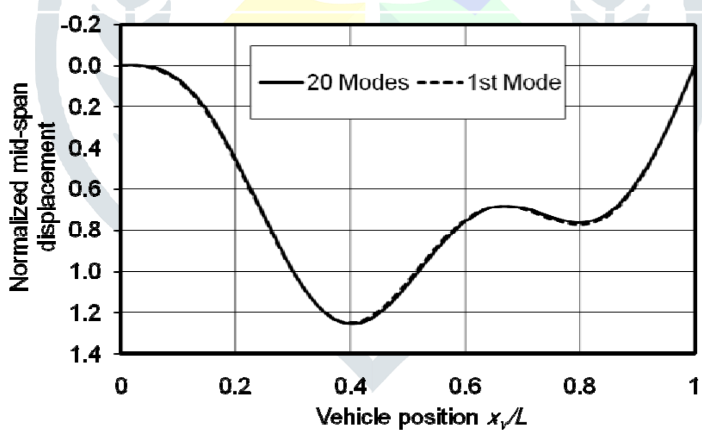


Fig. 6. The contribution of the 1^{st} and higher modes in the mid-span displacement

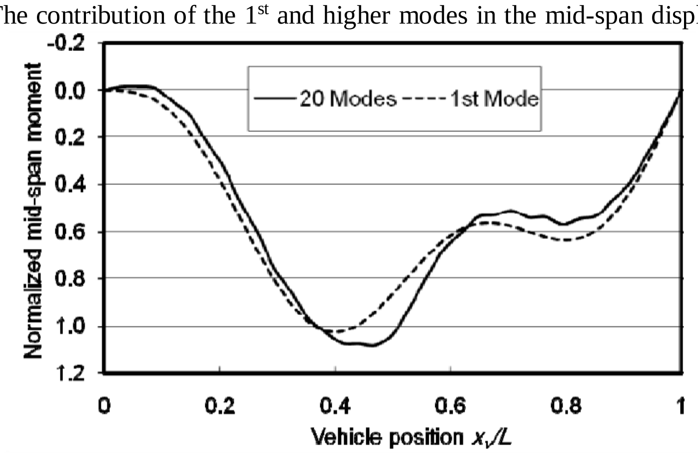


Fig. 7. The contribution of the 1^{st} mode and higher modes in to the mid-span moment

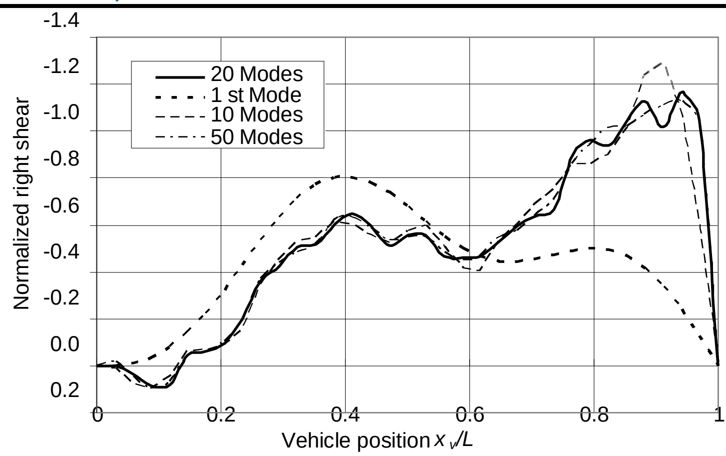


Fig. 8. The contribution of the 1st mode and higher mode in the right end shear.

The free vibration of the continuous bridge is analyzed using the outlined Galerkin's approach where the natural frequencies and mathematical expressions for the vibration mode shapes are obtained. The good correlation between the calculated natural frequencies and the results of finite element analysis is clear from Table 1. The derived first 6 modes of vibration are shown in Fig. 10.

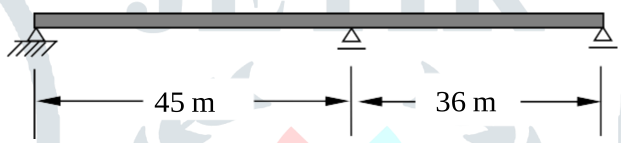


Fig. 9. Two-spans continuous bridge model.

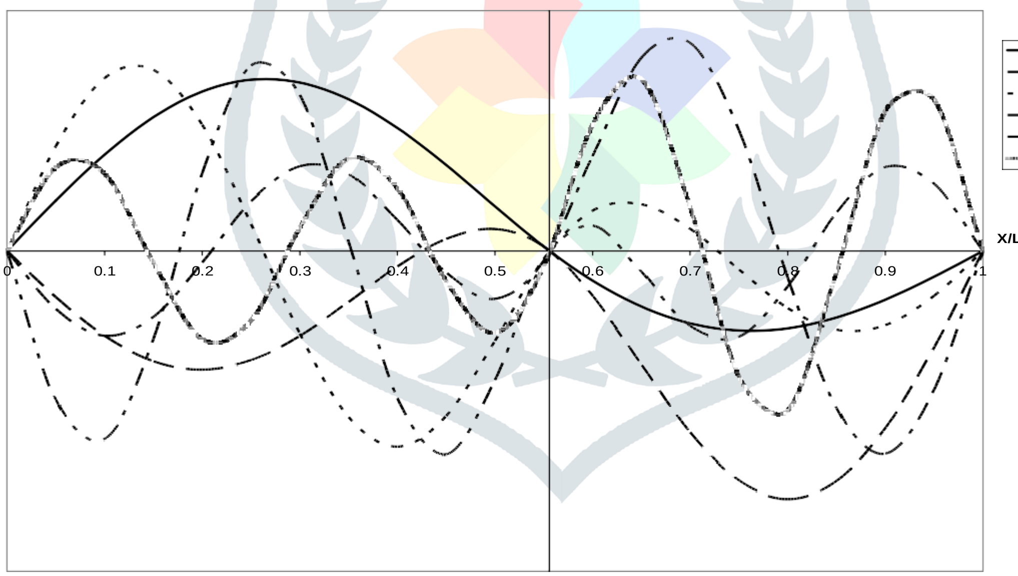


Fig. 10. The first 6 derived modes of vibration for the bridge model of Fig.9.

Table 1. First 6 calculated natural frequencies for the bridge model of Fig. 9.

Mode No.	Present Study (Galerkin's approach) Hz	Finite Element (Sap2000) Hz	% Difference
1	2.644	2.644	0
2	4.807	4.799	0.17%
3	10.212	10.222	-0.09%
4	16.108	16.077	0.19%
5	22.621	22.623	0.01%
6	33.945	33.897	0.14%

The two span continuous bridge is analyzed due to moving vehicle model. Figures 11 and 12 present the normalized vertical displacement of the mid-point of the first span and the second span for 300 kN vehicle moving at different vehicle speeds. Thenormalized displacement at the mid-point of the first span is 1.17 and 1.24 for vehicle velocity of 100 km/hr and 150 km/hr respectively. The increase in the response for higher speed is more pronounced for the last span, being 1.05 and 1.65 for speed 100 km/hr and 150 km/hr respectively. This can be explained by examining the vibration mode shapes displayed in Fig. 10. Higher speed on the short span excite the higher modes with larger amplitude in the second span. This effect of higher modes on the dynamic response is explicitly shown in Figs. 13 and 14. The increasing in the vehicle weight for the considered bridge has a negligible effect on the normalized displacement as shown in Fig. 15. The effect of higher modes on the bridge responseis more pronounced on the calculated bending moment as shown in Figs. 16 and 17.

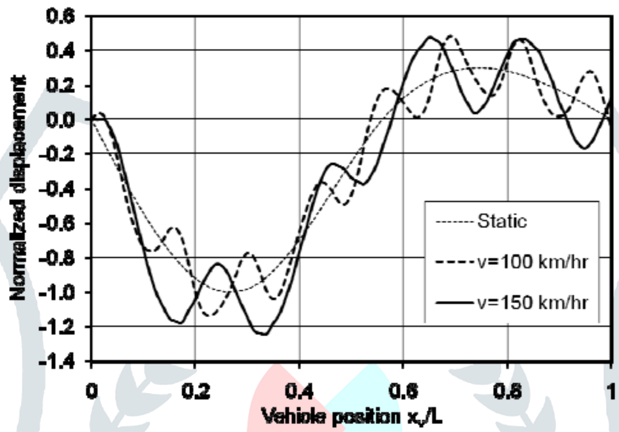


Fig.11. Normalized displacements at middle point of first span for 300 kN vehicle

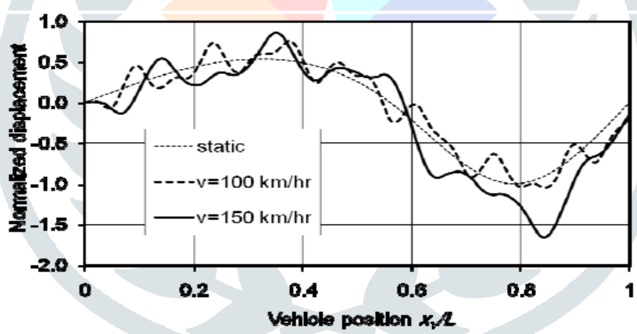


Fig. 12. Normalized displacements at middle point of second span 300 kN vehicle

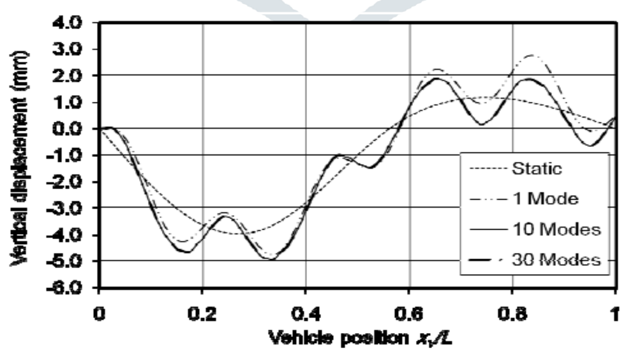


Fig. 13. Effect of higher modes on the displacement response at first span mid-point

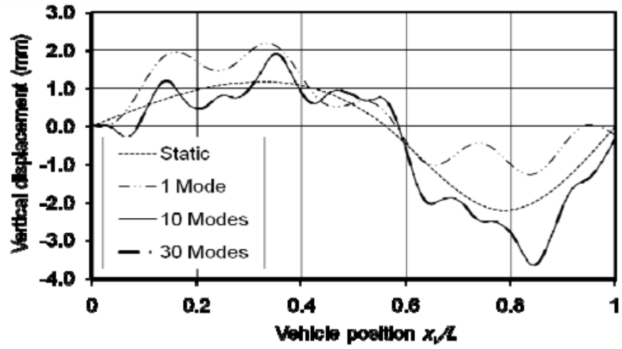


Fig. 14 Effect of higher modes on the displacement response at second span mid-point

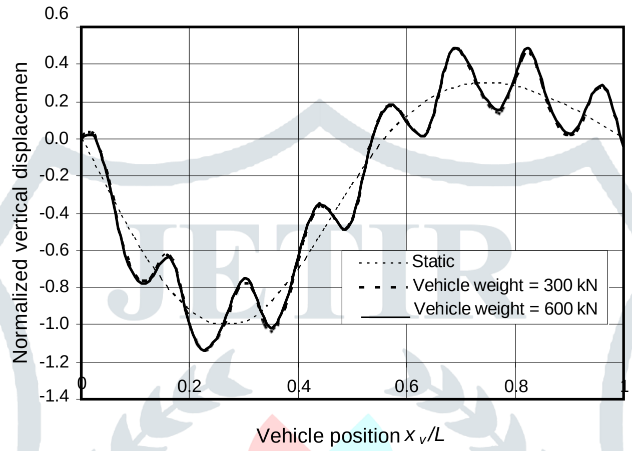


Fig. 15. Effect of vehicle weight on response at first span mid-point (v =100 km/hr.)

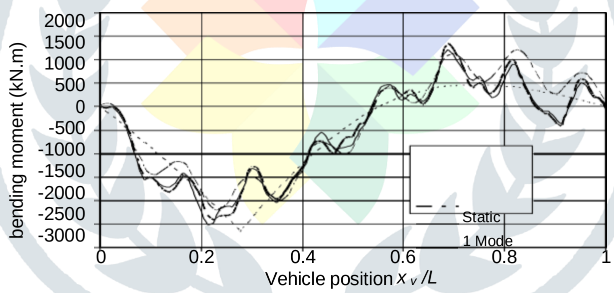


Fig. 16. Effect higher modes on the bending moment at middle of the first span.

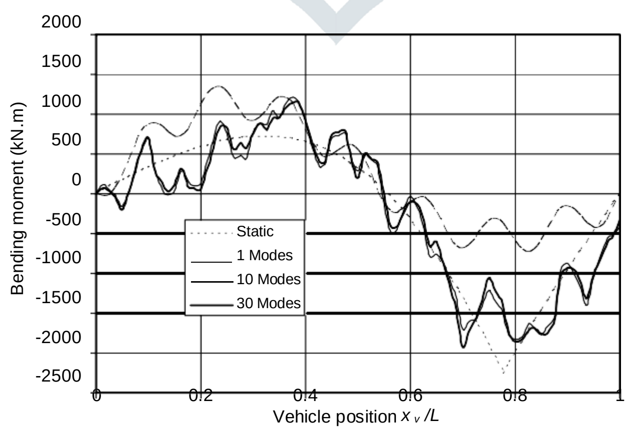


Fig. 17. Effect of higher modes on the bending moment at middle of the last span.

VI. CONCLUSIONS

The dynamic response of bridges to moving vehicles is presented in this paper. Different models of vehicles and bridge vehicle interaction are presented. Galerkin approach is used to obtain closed form mathematical expressions for the vibrationmode shapes of the bridge. Modal superposition and Duhamel's integration techniques are used to calculate bridge response. Due to the bridge vehicle interaction, the bridge response to moving vehicle is obtained through iterative approach. A computer code is developed using the MATLAB program to calculate the response. The efficiency and the validity of the proposed models and method of analysis are verified against previous research work as well as numerical finite element results using SAP2000.

The present approach, through the derived closed form solutions, enables the analyst to study the effect of individual natural modes of vibrations of the bridge on its overall dynamic response to moving vehicle. In general higher modes have more pronounced effect on the maximum moment and shear values than they have on the displacement response and, hence, they should be included in bridge analysis and design. Depending on the shape of the considered natural modes of vibrations, higher vehicle speed may cause higher dynamic response on some shorter bridge spans by exciting higher modes. For smooth bridge surface, increasing the vehicle weight has a negligible effect on the normalized displacement.

REFERENCES

1. Biggs, J. M., "Introduction to Structural Dynamics." McGraw-Hill Book Company, New York, 1964.
2. Tan, C. P., and Shore, S., "Response of Horizontally Curved Bridge to Moving Load", Journal of Structural Engineering, ASCE, 94(9), pp. 2135-2151, 1967.
3. Zhu, X. Q. and Law, S. S., "Moving Load Identification on Multi-span Continuous Bridges with Elastic Bearings." Mechanical Systems and Signal Processing, Vol. 20, pp. 1759-1782, 2005.
4. Garinei, A., "Vibration of Simple Beam-Like Modeled Bridge under Harmonic Moving Loads", Int. Journal of Engineering Science, Vol.84, pp.759-769, 2006.
5. Akin, J. E. and Mofid, M., "Numerical Solution for Response of Beams with Moving Mass", Journal of Structural Engineering, ASCE, 115(1), 120-131, 1989.
6. Bilello, C., Bergman, L. A., and Kuchma, D., "Experimental Investigation of a Small-Scale Bridge Model under a Moving Mass", Journal of Structural Engineering, ASCE, 130(5), pp 799-804, 2004.
7. Humar, J. L. and Kashif, A.H., "Dynamic Response Analysis of Slab-Type Bridges" Journal of Structural Engineering, ASCE, 121(2), pp 272-282, 1995.
8. Yang, Y. B., Liao, S. S., and Lin, B. H., "Impact Formulas for Vehicle Moving over Simple and Continuous Beams" Journal of Structural Engineering, ASCE, 121(11), pp 1644-1650, 1995.
9. Green, M. F., Cebon, D., "Dynamic Interaction between Heavy Vehicles and Highway Bridges." J. of Computers & Structures, 62 (2), pp 253-264, 1997.
10. Esmailzadeh, E., Jalili, N., "Vehicle-Passenger-Structure Interaction of Uniform Bridges Traversed by Moving Vehicles." Journal of Sound and Vibrations, Vol. 230, pp. 611-635, 2003.
11. Li, Y., Obrien, E., and Gonzalez, A., "The Development of a Dynamic Amplification Estimator for Bridges with Good Road Profiles." Journal of Sound and Vibration, Vol. 293, pp 125-137, 2006.
12. Yang , Y.B. and Yau, J.D., "Vehicle-Bridge Interaction Element for Dynamic Analysis" J. of Structural Engineering, ASCE, 123(11), pp. 1512-1518, 1997.
13. Yang, Y.B. and Lin, C.W. "Vehicle-Bridge Interaction Dynamics and Potential Applications" Journal of Sound and Vibration, Vol. 284, pp. 205-226, 2005
14. Yang, Y. B., Liao, S. S., and Lin, B. H., "Vehicle-Bridge Interaction Analysisby Dynamic Condensation" J. of Structural Engineering, ASCE, 121(11), pp. 1636-1643, 1995.
15. Hwang, E. and Nowak, A. S., "Simulation of Dynamic Loads for Bridges." Journal of Structural Engineering, ASCE, 117(5), pp. 1413-1434, 1991.
16. Leipholz, H. H., "Use of Galerkin's Method for Vibration Problems." The Shock and Vibration Digest, Vol. 8, pp. 3-18, 1976.
17. Abdel-Mooty, M. and Roorda J., "Free vibrations of continuous beams with singularities in inertia and stiffness distribution", Dynamics and Stability of Systems, Vol 6, no. 1, pp. 1-16, 1991.
18. SAP2000, "Integrated Finite Element Analysis and Design of Structures." Computers and Structures Inc., Berkley, California, USA, 1997
19. Yang, Y.B., Yau, J.D, and Wu, Y.S. "Vehicle Bridge Interaction Dynamics" World Scientific company, 2004