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Jβ-REGULAR AND Jβ-NORMAL SPACES IN TOPOLOGICAL SPACES

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Abstract:- The aim of this paper is to introduce and study of some new class of spaces namely Jβ-regular and Jβ-normal spaces in topological spaces by using Jβ-open sets. Moreover, we investigated the relationship among Jβ-T_0, Jβ-T_1, Jβ-T_2, Jβ-T_3, Jβ-T_4 separation axioms and Jβ-regular, α-regular, β-regular, α-normal, β-normal and Jβ-normal spaces, also some example & counter example are given to verify these relationships and its converse. Also we defined some function related to Jβ-regular and Jβ-normal spaces namely Jβ-open, Jβ-closed, gJβ-closed, Jβg-closed, Jβ-gJβ closed, quasi Jβ-closed, and Jβ-gJβ-continuous function. Besides it, we obtain some basic characterizations, properties and preservation theorems of Jβ-regular and Jβ-normal spaces.

Keyword:- Jβ-open set, Jβ-closed set, Jβ-regular spaces, Jβ-normal spaces, Jβ-T_3, Jβ-T_4 axioms, Jβ-irresolute, Jβ-neighborhood, gJβ-closed, Jβg-closed, Jβ-gJβ closed, quasi Jβ-closed, and Jβ-gJβ-continuous function etc.

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1. INTRODUCTION

In 1937, M. Stone **[14]** introduced the notation of regular open sets. In 1965, O. Njasted **[12]** introduced and defined α-open sets. In 1970, N. Levine **[7]** generalized the concept of closed sets to generalized closed set. In 1983, Abd-El-Monsef et al. **[2]** initiated the concept of β-open sets. Abd-El-Monsef et al. **[3]** defined the concept of β-regular space in 1985. In 1990, R. A. Mohmoud and M. E. Abd-El-Monsef **[10]** defined β normal space. R. Devi et al. **[4]** defined α-regular spaces in 1998. In 2000, G. B. Navalagi **[11]** defined α-normal spaces.In 2016, S. P. Missier and M. Annalakshmi **[13]** introduced the notation of regular star open sets. In 2019, P. L. Meenakshi **[8]** initiated the notation of η*-open sets. In 2019, Amir A. Mohammed and S. Beyda Abdullah **[1]** introduced the notation of ii-open sets. In 2019, P. L. Meenakshi and K. Sivakamasundari **[9]** intoduced the concept of J-open sets. In 2022, Hamant Kumar **[6]** initiated the concept of Jβ-open sets. Recently in 2022, Anuj kumar and B. S. Sharma **[5]** introduced a new class of separation axioms namely Jβ-T_0, Jβ-T_1 and Jβ-T_2 separation axioms.

2. PRELIMINARIES

Throughout this paper, spaces (X, ℑ), (Y, σ), and (Z, γ) (or simply X, Y and Z) always mean topological spaces. Let f: X→Y (or simply f) always denote map. Let G be a subset of a space X. The closure of G, interior of G and complement of G are denoted by cl(G), int(G) and G^C (or X-G) respectively.

Definition 2.1 A subset G of a topological space (X, ℑ) is said to be
- (i) **regular open [14]** if G = int(cl(G)).
- (ii) **α-open [12]** if G ⊂ int(cl(int(G))).
- (iii) **β-open [2]** if G ⊂ cl(int(cl(G))).
- (iv) **ii-open [1]** if there exist an open set A such that

a). A≠ ∅, X
b). G ⊂ cl(G∩ A)
c). intG= A.

(v) **generalized closed** (briefly **g-closed**) **[7]** if cl(G) ⊂ A whenever G ⊂ A and A ∈ ℑ.

The complement of a regular open (resp. α-open, β-open, ii-open and g-closed) set is called **regular-closed** (resp. **α- closed β-closed, ii-closed** and **g-open)**. The intersection of all regular closed (resp. α-closed, β-closed, ii-closed and g-closed) sets containing G, is called **regular-closure** (resp. **α-closure, β-closure, ii-closure** and **generalized-closure**) of G, and is denoted by **r-cl(G)** (resp. **α-cl(B), β-clG), ii-cl(G)** and **cl*(G)**). The set of all regular open (resp. α-open, β-open, ii-open and g-open) in X is denoted by **r-o(X)** (resp. **α-o(X), β-o(X), ii-o(X),** and **g-o(X)**). The set of all regular closed (resp. α-closed, β-closed, ii-closed and g-closed) in X is denoted by **r-c(X)** (resp. **α-c(X), β-c(X), ii-c(X),** and **g-c(X)**). A subset G of a topological space (X, ℑ) is said to be **clopen** if it is both open and closed in (X, ℑ).

Definition 2.2 A subset G of a topological space (X, ℑ) is said to be

(i) **regular*-open** (or **r*-open) [13]** if G = int(cl*(G)).

(ii) **η*-open [8]** if it is a union of regular*-open sets (r*-open sets).

(iii)**J-closed [9]** if cl(G) ⊂ A whenever G⊂ A and A is η*-open in (X, ℑ).

(iv)**Jβ-closed [6]** if β-cl(G) ⊂ A whenever G⊂ A and A is η*-open in (X, ℑ).

The complement of a regular*-open (resp. η*-open, J-closed and Jβ-closed) set is called **regular*-closed (**resp**. η*-closed, J-open** and **Jβ-open)**. The union of all regular*-open (resp. η*-open, J-open and Jβ-open) sets of X contained in G is called **regular*-interior** (resp. **η*- interior, J- interior** and **Jβ- interior**) of G and is denoted by **r*-int(G)** (resp. **η*-int(G), J- int(G)** and **Jβ- int(G)**). The intersection of all regular*-closed (resp. η*-closed, J-closed and Jβ-closed) sets of X containing G is called **regular*-closure** (resp. **η*- closure, J-closure** and **Jβ- closure**) of G is denoted by **r*-cl(G)** (resp. **η*-cl(G), J- cl(G)** and **Jβ- cl(G)**). The set of all r*-closed (resp. r*-open, η*-closed, J-closed, Jβ-closed, η*-open, J-open and Jβ-open) set in X is denoted by r*c(X) (resp. r*o(X), η*- c(X), J- c(X), Jβ- c(X), η*-o(X), J- o(X) and Jβ- o(X)).

2.3 Lemma. Let G be a subset of a space X and g ∈ X. The following properties hold for Jβ-cl(G) :

(i) g∈ Jβ-c1(G) if and only if G ∩ M = ∅ for every M ∈ Jβ-o(X) containing g.

(ii) G is Jβ-closed if and only if G = Jβ-cl(G) .

(iii)Jβ-c1(G) ⊂ Jβ-c1(H) if G ⊂ H.

(iv)Jβ-c1(Jβ-c1(G)) = jβ-c1(G).

(v) Jβ-c1(G) is Jβ-closed.

Proposition 2.4 Every regular open set is r*-open set.

Proof. Let G be a regular open set then G=int(cl(G)). Since every regular open set is clopen so G is closed, also every closed set is generalized closed set. Hence G is g-closed. So we get cl(G)=cl*(G). By the property of regularity we get, G=int(cl*(G)). Hence G is r*-open set.

Proposition 2.5 Every r*-open set is η*-open set.

Proof. By definition, since every η*-open set is union of r*-open sets, it is obvious that every r*-open set is η*-open set.

Proposition 2.6 Every η*-open set is open set.

Proof. Let G be η*-open set. Let p ∈ G then p ∈ ∪C_i where C_i are r*-open set. Now every r*-open set viz. C is open as (C=int(cl*(C)) ⇒ intC = int(int(cl*(C))) = int(cl*(C))=C). Hence p ∈ D where D is open set. Hence every η*-open set is open set.

Proposition 2.7 Every open set is α-open set.
Proof. Let G is open set then G = int(G). Also G ⊂ cl(G) ⇒ int(G) ⊂ int(cl(G)) ⇒ G ⊂ int(cl(G)) = int(cl(int(G))) ⇒ G⊂ int(cl(int(G))). Hence G is α-open set.

Proposition 2.8 Every α-open set is ii-open set.
Proof. Let G is α-open set, then if G ⊂ int(cl(int(G))) ⊂ cl(int(G)). So, there exist an open set, say, A≠ $\emptyset, X$ satisfying int(G) ⊂ A, it follows that int(G) ⊂ A∩ G. Therefore G ⊂ cl(G∩ A). Now we shall prove that int(G) = A. Note that if int(G) ≠ A, for all A ∈ $o(X)$, then cl(int(G)) ≠ cl(A). From above inclusions we conclude that G ⊂ cl(int(G) ∩ G ∩ A). This implies that G ⊄ cl(A). That is a contradiction. Therefore, G is $ii -$ open set.

Proposition 2.9 Every ii-open set is β-open set.
Proof. Let G is ii-open set, then there exist A∈ $o(X)$ such that A ≠ $\emptyset, X$ and G ⊂ cl(G∩ A) and int(G)= A. Since G ⊂ cl(G∩ A) ⊂ cl(A), also G ⊂ cl(G) ⇒ int(G) ⊂ int(cl(G)) ⇒ A ⊂ int(cl(G)) ⇒ cl(A) ⊂ cl(int(cl(G))). But above we find G ⊂ cl(A), so G ⊂ cl(int(cl(G))). Hence G is β-open set.

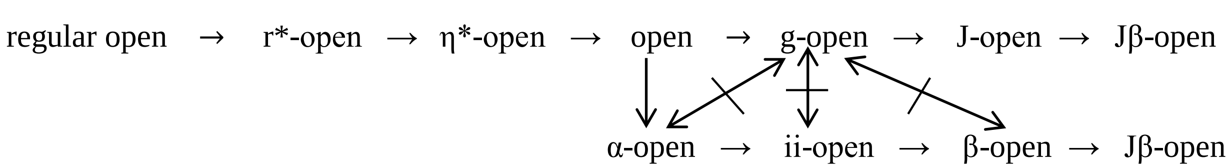
Proposition 2.10 Every β-open set is Jβ-open set.
Proof. Let A is β-open set then X-A=G (say is) β-closed set ⇒ β-cl(G)=G. Now by definition G is Jβ-closed set. Hence A is Jβ-open set.

Proposition 2.11 Every open set is g-open set.
Proof. Let A is open set then X-A=G(say) is closed set ⇒ Cl(G)=G. Hence by the definition of g-closed set G is G-closed set. Hence A is g-open set.

Proposition 2.12 Every g-open set is J-open set.
Proof. . Let A is g-open set then X-A=G(say) is g-closed set then cl(G) ⊂ H whenever G ⊂ H and H be any open set in X. Since every η*-open set is open, then cl(G) ⊂ H whenever G ⊂ H and H be η*-open set in X, which implies that G is J-closed. Hence A is J-open set.

Proposition 2.13 Every J-open set is Jβ-open set.
Proof. Let A is J-open set then X-A=G(say) is J-closed set then cl(G) ⊂ H whenever G ⊂ H and H be any η*-open set in X. since β-cl(G) ⊂ cl(G) then β-cl(G) ⊂ H whenever G ⊂ H and H be η*-open set in X, which implies that G is Jβ-closed. Hence A is Jβ-open set.

Remark 2.14 From the above definitions, theorems and results, the relationship among Jβ-open sets and some other existing weaker and stronger forms of open sets are given in the following diagram:



Where none of the implications is reversible as can be seen from the following counter examples:
Here A ⟷ B stand for neither A imply B nor B imply A.

Example 2.15 Let X= {g, h, i} and ℑ = {ϕ, {g}, X} then:
r-o(X) = {ϕ, X}
r*-o(X) = η*-o(X) = {ϕ, {g}, X}
α-o(X) = ii-o(X) = β-o(X) = {ϕ, {g},{g, h}, {g, i}, X}

g-o(X) = J-o(X) = Jβ-o(X) = {ϕ, {g}, {h}, {i},{g, h}, {g, i}, X}
it is clear that {g} is r*-open set but not regular open set, {g, h} is α-open set but not open set and {h} is g-open set but not α-open, ii-open, β-open and open set. {i} is Jβ-open set but not β-open set.

Example 2.16 Let X= {g, h, i} and ℑ = {ϕ, {g, h}, X} then:
r-o(X) = r*-o(X) = η*-o(X) = {ϕ, X}
α-o(X) = ii-o(X) = {ϕ, {g, h}, X}
β-o(X) = {ϕ, {g}, {h}, {g, h}, {g, i}, {h, i}, X}
g-o(X) = {ϕ, {g}, {h}, {g, h}, X}
J-o(X) = Jβ-o(X) = {ϕ, {g}, {h}, {i}, {g, h}, {g, i}, {h, i}, X}
{h} is β-open set but not α-open and ii-open set, also{g, i} is β-open set but not g-open set and {i} is Jβ-open and J-open set but not β-open and g-open set.

Example 2.17 Let X= {g, h, i} and ℑ = {ϕ, {g}, {g, h}, X} then:
r-o(X) = {ϕ, X}
r*-o(X) = η*-o(X) = {ϕ, {g}, X}
α-o(X) = ii-o(X) = β-o(X) = {ϕ, {g},{g, h}, {g, i}, X}
g-o(X) = {ϕ, {g}, {h}, {g, h}, X}
J-o(X) = Jβ-o(X) = {ϕ, {g}, {h}, {i}, {g, h}, {g, i}, X}
Here {g, h} is open set but not η*-open set. {g, i} is α-open, β-open and ii-open set but not g-open, and {h} is g-open set but not α-open, β-open and ii-open set. Also {i} is J-open and Jβ-open set but not g-open and β-open set.

Example 2.18 Let X= {g, h, i, j} and ℑ = {ϕ, {g}, {h, i}, {g, h, i}, X} then:
r-o(X) = r*-o(X) = {ϕ, {g}, {h, i}, X}
η*-o(X) = α-o(X) = {ϕ, {g}, {h, i}, {g, h, i }, X}
ii-o(X) = {ϕ, {g}, {g, j}, {h, i}, {g, h, i}, {h, i, j}, X}
β-o(X) = Jβ-o(X) = {ϕ, {g}, {h}, {i}, {g, h}, {g, i}, {g, j}, {h, i}, {h, j}, {i, j}, {g, h, i}, {g, h, j}, {g ,i ,j}, {h, i, j}, X}
g-o(X) = J-o(X) = {ϕ, {g}, {h}, {i},{g ,h}, {g ,i}, {h, i}, {g, h, i}, X}
Here {g, h, i} is η*-open set but not r*-open set. {g, j} is ii-open set but not α-open set. {i, j} is β-open and Jβ-open set but not ii-open set. {g, j} is β-open and Jβ-open set but not g-open, and J-open set. {i} is g-open set but not open set.

3. Jβ-REGULAR SPACE

Definition 3.1 A topological space X is said to be **Jβ-regular space** (resp. **α-regular** [4], **β-regular [3]**) if for every closed set G and a point h ∉ G, there exist disjoint Jβ-open (resp. α-open, β-open) sets J and K of X such that G ⊂ J and h ∈ K.

Theorem 3.2 Every regular space is Jβ–regular space.
Proof. Since every open set is jβ-open set, so proof is obvious.

Remark 3.3 By the definition stated above, we conclude some implication

regular space → α-regular space → β-regular space → Jβ-regular space

Where none of the implications is reversible as can be seen from the following counter examples:

Example 3.4 Let X= {g, h, i} and ℑ = {ϕ, {g}, {g ,h}, X} then X is Jβ-regular but neither β-regular nor α-regular. As closed set {i} and h ∈ X, there not exist disjoint β-open, α-open sets J and K such that {i} ⊂ J and h ∈ K. For Jβ-regular, there exist disjoint Jβ-open sets {i} and {h} such that {i} ⊂ {i} and h ∈ {h}.

Example 3.5 Let X= {g, h, i, j} and ℑ = {ϕ, {g}, {h, i}, {g, h, i}, X} then X is β-regular as well as Jβ-regular but neither regular nor α-regular. As closed set {j} and h ∈ X, there not exist disjoint open, α-open sets J and K such that {j} ⊂ J and h ∈ K. For Jβ-regular and β-regular, there exist disjoint Jβ-open and β-open sets {g, j} and {h} such that {j} ⊂ {g, j} and h ∈ {h}.

Example 3.6 Let X = {g, h, i} and $\mathfrak{I}$ = {ϕ, {g}, {h, i}, X} then X is regular space as well as α-regular space, β-regular space and Jβ-regular space

Theorem.3.7 The following properties are equivalent for a space X:

1. X is Jβ-regular.
2. For each h ∈ X and each open set J of X containing h, there exists K ∈ Jβ-o(X) such that h ∈ K ⊂ Jβ-cl(K) ⊂ J.
3. For each closed set G of X, ∩{Jβ-cl(K) : G ⊂ K ∈ Jβ-o(X)} = G.
4. For each subset D of X and each open set J of X such that D ∩ J ≠ ∅, there exists K∈ Jβ-o(X) such that D ∩ K ≠ ∅ and Jβ-cl(K) ⊂ J.
5. For each non empty subset D of X and each closed subset G of X such that D ∩ G = ∅, there exist K, L ∈ Jβ-o(X) such that D ∩ K ≠ ∅, G ⊂ L and K ∩ L ≠ ∅.

Proof. (1) ⟹ (2). Let J be an open set containing h, then X − J is closed in X and h ∈ X − J. By (a), there exist L, K ∈ Jβ-o(X) such that h ∈ K, X − J ⊂ K and K ∩ L = ϕ .By **Lemma 2.3**, we get Jβ-cl(K) ∩ L = ϕ and hence h ∈ K ⊂ Jβ-cl(K) ⊂ J.

(2) ⟹ (3). Let G be a closed set of X. If G ⊂ K, then from **Lemma 2.3 (iii)**, Jβ-cl(G) ⊂ Jβ-cl(K) which gives G ⊂ Jβ-cl(K) as G ⊂ Jβ-cl(G). Therefore, ∩{Jβ-cl(K) : G ⊂ K ∈ Jβ-o(X)} ⊃ G.
Conversely, let h ∈G. Then X − G is an open set containing h. By (b), there exists J ∈ Jβ-o(X) such that h ∈ J ⊂ Jβ-cl(J) ⊂ X − G. Put K = X − Jβ-cl(J). By **Lemma 2.3,** G ⊂ K ⊂ Jβ-o(X) and h ∈ Jβ-cl(K). This state that ∩{Jβ-cl(K) : G ⊂ K ∈ Jβ-o(X)}⊂ G.
Hence ∩{Jβ-cl(K) : G ⊂ K ∈ Jβ-o(X)} = G.

(3) ⟹ (4). Let D be a subset of X and let J be open in X such that D ∩ J ≠ ∅. Let h ∈ D ∩ J, then X − J is a closed set not containing h. By (c), there exists L ∈ Jβ-o(X) such that X – J ⊂ W and h ∉ Jβ-cl(L). Put K = X − Jβ-cl(L). Then K ⊂ X − L. Also h ∈ K ∩ D. From **Lemma 2.3**, we have K ∈ Jβ-o(X), and Jβ-cl(K) ⊂ Jβ-cl(X − L) = X − L ⊂ J.

(4) ⟹ (5). Let D be a subset of X and let G be a closed set in X such that D ∩ G =∅, where D is non empty. Since X − G is open in X and D is non empty, by (d), there exists K ∈ Jβ-o(X) such that D ∩ K ≠ ∅ and Jβ-c1(K) ⊂ X − G. Put L = X − Jβ-c1(K), then G ⊂ L. Also, K ∩ L ≠ ∅. By **Lemma 2.3**, L ∈ Jβ-o(X).

(5) ⟹ (1). Proof is obvious.

Theorem. 3.8 A topological space X is Jβ-regular if and only if for each closed set G of X and each h ∈ X – G, there exist Jβ-open sets J and K of X such that h ∈ J and G ⊂ K and Jβ-cl(J) ∩ Jβ-cl(K) = ϕ.
Proof: Let G be a closed set in Jβ-regular space X and h ∉ G. Then there exist Jβ-open sets J_h and K such that h ∈ J_h, G ⊂ K and J_h ∩ K = ϕ. This Implies that J_h ∩ Jβ-cl(K) = ϕ, as Jβ-cl(K) is Jβ closed and h ∉ Jβ -cl(K). Since X is Jβ -regular, there exist Jβ -open sets A and B of X such that h ∈ A, Jβ -cl(K) ⊂ B and A ∩ B = ϕ. This implies Jβ -cl(A) ∩ B = ϕ. Take J = J_h ∩ A. Then J and K are open sets of X such that h ∈ J and B ⊂ K and Jβ -cl(J) ∩ Jβ -cl(K) = ϕ, as Jβ-cl(J) ∩ Jβ-cl(K) ⊂ Jβ-cl(A) ∩ B = ϕ.
Conversely, let for each closed set G of X and each h ∈ X – G, there exist Jβ-open sets J and K of X such that h ∈ J, G ⊂ K and Jβ-cl(J) ∩ Jβ-cl(K) = ϕ. Now U ∩ V ⊂ Jβ-cl(J) ∩ Jβ-cl(K) = ϕ. Therefore J ∩ K = ϕ. Thus X is Jβ-regular.

Definition. 3.9 A space X is said to be **Jβ-T_3 space** if it is Jβ-regular as well as **Jβ-T_1 [5]** space.

Theorem. 3.10 Every Jβ-T_3 space is a Jβ-T_2 space.

Proof. Since X be Jβ-T_3, X is both Jβ-T_1 and Jβ-regular. Since X is Jβ-T_1 every singleton subset {h} of X is a Jβ-closed. Let {h} be a Jβ-closed subset of X and $g \in X - \{h\}$. Then we have $h \neq g$ since X is Jβ-regular, there exist two Jβ-open sets J and K such that $\{h\} \subset J$, $g \in K$, and such that $J \cap K = \phi$ i.e. J and K are disjoint Jβ-open sets containing g and h respectively. Since g and h are arbitrary, for every pair of distinct points, there exist disjoint Jβ-open sets. Hence X is Jβ-T_2 space.

Theorem.3.11 Every subspace of a Jβ-regular space is Jβ-regular, i.e. Jβ-regularity is a hereditary property.
Proof. Let X be a Jβ-regular space. Let Y be a subspace of X. Let $h \in Y$ and G be a closed set in Y such that $h \notin G$. Then there is a closed set C of X with $G = Y \cap C$ and $h \notin C$. Since X is Jβ-regular, there exist two disjoint Jβ -open sets J and K such that $h \in J$ and $C \subset K$. Note that $Y \cap J$ and $Y \cap K$ are Jβ-open sets in Y. Also $h \in J$ and $h \in Y$, which implies $h \in Y \cap J$ and since $C \subset K$ and $G = Y \cap C \Rightarrow Y \cap C \subset Y \cap K$ i.e. $G \subset Y \cap K$. Also, $(Y \cap J) \cap (Y \cap K) = \phi$. Hence Y is J$\beta$-regular space.

Theorem.3.12 Every Jβ-compact Hausdorff space is a Jβ-T_3 space and hence a Jβ-regular.
Proof. Suppose X be a Jβ-compact Hausdorff space, i.e. X is a Jβ-T_2 space. But every Jβ-T_2 space is Jβ-T_1. To prove that it is Jβ-T_3 space it is sufficient to prove that it is Jβ-regular. Let G be a closed subset of X, and $h \notin G$, that is $h \in X - G$,so that any point $g \in G$ is a point of X, that is g and h are distinct. Now X is a Jβ -T_2 space and g, h be two distinct element of X. there exists two Jβ-open sets J_h and K_g such that $J_h \cap K_g = \phi$ where $g \in K_g$ and $h \in J_h$. Now let relative topology of topology $\Im$, is denoted by $\Im^*$ so that the collection $A^* = \{G \cap K_g : g \in G\}$ is a Jβ-$\Im^*$ open cover of G. But G is closed and also X is Jβ-compact (G, $\Im^*$) is also Jβ-compact. Hence G has finite subcover, there exists points $g_1, g_2, \ldots . g_n$ in G such that $A_i^* = \{G \cap K_{gi} : i = 1, 2, \ldots ..n\}$ are finite subcover for G. Now $G = \cup \{G \cap K_{gi} : i = 1, 2, \ldots ..n\}$ or $G = G \cap \{\cup \{ K_{gi} : i = 1, 2, \ldots ..n\}\}$, this implies that $G \subset \cup \{ K_{gi} : i = 1, 2, \ldots ..n\}$, hence $G \subset K$ where $K = \cup \{K_{gi} : i = 1, 2, \ldots ..n\}$ is Jβ-open set containing G, as K is the union of Jβ-open sets. Again $\{J_{hi} : i = 1, 2, 3, \ldots ..n\}$ is collection of Jβ-open sets containing h and hence $J = \cap \{J_{hi} : i = 1, 2, \ldots ..n\}$ is also a Jβ-open set containing h. Also $J \cap K = \phi$, otherwise $J_{hi} \cap K_{gi} \neq \phi$ for some i. Hence for each closed set G and an element h in X – G we obtain two disjoimt Jβ-open sets J and K such that $h \in j$, $G \subset K$. Hence (X, $\Im$) is Jβ-regular. Also X is Jβ-T_2 so it is Jβ-T_1 and hence X is Jβ-T_3.

4. Jβ-NORMAL SPACE

Definition 4.1. A space X is termed as **Jβ-normal** (resp. **α-normal [11], β-normal [10]**) if for any pair of disjoint closed sets G and H, there exist disjoint Jβ-open (resp. α-open, β-open) sets J and K such that $G \subset J$ and $H \subset K$.

Theorem 4.2 Every normal space is Jβ–normal space.
Proof. Since every open set is jβ-open set, so proof is obvious.

Remark 4.3 By the definition stated above, the following implications holds for X

normal space → α-normal space → β-normal space → Jβ-normal space

Where the converse of either of these implications is not be true, as can be seen from the following counter examples:

Example.4.4 Let X = {g, h, i, j} and $\Im$ = {ϕ, {g}, {h}, {g, h}, {g, h, i}, {g, h, j}, X}. Then the space (X, $\Im$) is β-normal, but it is neither α-normal nor normal space as:
C(X) = {ϕ, {i}, {j}, {i, j}, {g, i, j}, {h, i, j}, X}
α-o(X) = $\Im$ = {ϕ, {g}, {h}, {g, h}, {g, h, i}, {g, h, j}, X}
β-o(X) = {ϕ, {g}, {h}, {g, i}, {g, h}, {g, j}, {h, i}, {h, j}, {g, h, i}, {g, h, j}, {h, i, j}, X}
Let G = {i} and H = {j} be disjoint closed sets in X, there do not exist disjoint open and α-open sets J and K such that $G \subset J$ and $H \subset K$, but for β- normal, take J ={g, i} and K = {h, j} as J and K are Jβ-open set.

Example.4.5 Let X = {g, h, i} and $\Im$ = {ϕ, {g}, {g, h}, {g, i}, X}. Then the space (X, $\Im$) is Jβ-normal, but it is neither β-normal nor α-normal as:
C(X) = {ϕ, {h}, {i}, {h, i}, X}
α-o(X) = β-o(X) = $\Im$ = {ϕ, {g}, {g, h}, {g, i}, X}.
Jβ-o(X) = {ϕ, {g}, {h}, {i}, {g, h}, {g, i},{h, i}, X}
For disjoint closed set {h} and {i} there do not exist disjoint open, α-open and β-open sets J and K such that {h}⊂ J and {i} ⊂ K, but for Jβ- normal, take J ={h} and K = {i} as J and K are Jβ-open set.

Example 4.6. Let X = {g, h, i} and $\Im$ = {ϕ, {g}, {h}, {g, h}, {h, i}, X}. Then the space (X, $\Im$) is normal as well as Jβ-normal, since:
C(X) = {ϕ, {g}, {i}, {g, i}, {h, i}, X}
For disjoint closed set G = {g} and H = {i} (or {h, i}) there exist disjoint open sets J = {g} and K = {h, i} such that G ⊂ J and H ⊂ K.

Theorem. 4.7 For a space X the following are equivalent:

(1) X is Jβ-normal.
(2) For every pair of open sets J and K as J ∪ K = X, there exist Jβ-closed sets G and H such that G ⊂ J, H ⊂ K and G ∪ H = X,
(3) For every closed set F and every open set L containing F, there exists a Jβ-open set J such that F ⊂ J ⊂ Jβ-cl(J) ⊂ L.

Proof: **(1) ⇒ (2)** Let J and K be a pair of open sets in a Jβ-normal space X such that X = J ∪ K. Then X – J and X − K are disjoint closed sets. Since X is Jβ-normal, there exist disjoint Jβ-open sets J_1 and K_1 such that X − J ⊂ J_1 and X − K ⊂ K_1. Let G = X − J_1, H = X − K_1. Then G and H are Jβ-closed sets such that G ⊂ J, H ⊂ K and G ∪ H = X.

(2) ⇒ (3) Let F be a closed set and L be an open set containing F. Then X − F and L are open sets whose union is X. Then by (2), there exist Jβ-closed sets A_1 and A_2 such that A_1 ⊂ X − F and A_2 ⊂ L and A_1 ∪ A_2 = X. Then F ⊂ X − A_1, X − L ⊂ X − A_2 and (X − A_1) ∩ (X – A_2) = ϕ. Let J = X − A_1 and K = X − A_2. Then J and K are disjoint Jβ-open sets such that F ⊂ J ⊂ X − K ⊂ L. As X − K is Jβ-closed set, we have Jβ-cl(J) ⊂ X − K and F ⊂ J ⊂ Jβ-cl(J) ⊂ L.

(3) ⇒ (1) Let F_1 and F_2 be any two disjoint closed sets of X. Put L = X − F_2, then F_2 ∩ L = ϕ. F_1 ⊂ L, where L is an open set. Then by (3), there exists a Jβ-open set J of X such that F_1 ⊂ J ⊂ Jβ-cl(J) ⊂ L. It follows that F_2 ⊂ X − Jβ-cl(J) = K , say, then K is Jβ-open and J ∩ K = ϕ. Hence F_1 and F_2 are separated by Jβ-open sets J and K. Therefore X is Jβ-normal.

Definition 4.8 A space X is said to be **Jβ-T_4 space** if it is Jβ-normal as well as **Jβ-T_1 [5]** space.

Theorem 4.9 Every Jβ-T_4 space is a Jβ-T_3 space.
Proof. Since X be Jβ-T_4, X is both Jβ-T_1 and Jβ-normal. So for X is Jβ-T_3 it is sufficient to prove that X is Jβ-regular. Let G be a closed subset of X and h is an element of X-G. Since X is Jβ-T_1 so every singleton subset of X is a Jβ-closed, so {h} be a Jβ-closed subset of X. since X is Jβ-normal, then there exist two disjoint open sets J and K such that G ⊂ J and {h} ⊂ K i. e. h ∈ K. hence X is Jβ-regular and X is T_1 also. Hence X is Jβ-T_3 space.

Remark 4.10 by the definitions and theorems, we conclude that:

$$J\beta\text{-}T_4 \Rightarrow J\beta\text{-}T_3 \Rightarrow J\beta\text{-}T_2 \Rightarrow J\beta\text{-}T_1 \Rightarrow J\beta\text{-}T_0$$

Remark 4.11 Neither Jβ-regular implies Jβ-normal space, nor Jβ-normal space implies Jβ-regular spaces:

Example 4.12 Let X= {g, h, i} and $\mathfrak{I}$ = {ϕ, {g}, {g, h}, X} then the space (X, $\mathfrak{I}$) is Jβ-normal from **ex. 2.27,** but not Jβ- regular as, for closed set {h, i} and j ∉ {h, i} there do not exist disjoint Jβ-open sets J and K such that {h, i} ⊂ J ad j ∈ K.

5. Some functions related with Jβ- regular and normal spaces

Definition 5.1 A subset G of a space (X, $\mathfrak{I}$) is said to be
(i) **generalized Jβ-closed** (briefly **gJβ-closed) set** if Jβ-cl(G) ⊂ A whenever G ⊂ A and A is open.
(ii) **Jβ-generalized closed** (briefly **Jβg-closed**) **set** if Jβ-cl(G) ⊂ A whenever G ⊂ A and A is Jβ-open.

Definition 5.2 A function f : X → Y is said to be
(i) **Jβ-open [5]** if the image of each open set of X is Jβ-open set in Y.
(ii) J**β closed [5]** if the image of each closed set of X is Jβ-closed set in Y.
(iii) **generalized Jβ-closed** (briefly **gJβ-closed**) if the image of each closed set of X is gJβ-closed in Y.
(iv) **Jβ generalized closed (**briefly **Jβg-closed)** if for image of each closed set of X is Jβg-closed in Y.
(v) **quasi Jβ-closed** if the image of each Jβ-closed set of X is closed in Y.
(vi) **Jβ-gJβ-closed** if the image of each Jβ-closed set of X is gJβ-closed in Y.
(vii) **Jβ-Jβg closed** if the image of each Jβ-closed set of X is Jβg-closed in Y.

Definition 5.3 Let X be a topological space. A subset N ⊂ X is called a **Jβ-neighbourhood [5]** (briefly **Jβ-nhd**) of a point h ∈ X if there exist a Jβ-open set J such that h ∈ J ⊂ N.

Definition 5.4 A function f : X → Y is said to be **Jβ-gJβ-continuous** if the inverse image of a Jβ-closed set of Y is gJβ-closed set in X.
.

Definition 5.5 A function f : X → Y is said to be **Jβ-irresolute [5]** if the inverse image of a Jβ-open set of Y is Jβ-open set in X.

Definition 5.6 A function f : X → Y is called
(i) **pre Jβ-open** if f(J) ∈ Jβ-o(Y) for each J ∈ Jβ-o(X),
(ii) **pre Jβ-closed** if f(J) ∈ Jβ-cY) for each J ∈ Jβ-c(X),
(iii) **almost Jβ-irresolute** if for each h in X and each Jβ-neighbourhood K of f(h), Jβ-cl(f^{-1}(K)) is a Jβ-neighbourhood of h.

Remark 5.7 Every closed function is Jβ-closed but not conversely. Also, every Jβ-closed function is gJβ-closed because every Jβ-cosed set is gJβ-closed. Also it is obvious that Jβ-closed function and Jβ-gJβ-closed function imply gJβ –closed function.

Theorem 5.8 A surjective function f : X → Y is gJβ-closed (resp. Jβ-gJβ-closed) if and only if for each subset F of Y and each open (resp. Jβ-open) set J of X containing f^{-1}(F) , there exists a gJβ-open set K of Y such that F ⊂ K and f^{-1}(K) ⊂ J.
Proof. let f is gJβ-closed (resp. Jβ-gJβ-closed). Let F be any subset of Y and J be open (resp Jβ-open) set of X containing f^{-1}(F). Put K = Y – f(X – J). Then the complement K^c of K is given as K^c = Y – K = f(X – J). Since X – J is closed (resp. Jβ-closed) in X and f is gJβ-closed (resp. Jβ- gJβ-closed), f(X – J) = K^c is gJβ-closed. Therefore, K is gJβ-open in Y. It is easy to see that F ⊂ K and f^{-1}(K) ⊂ J.
Conversely, let G be a closed (resp. gJβ-closed) set of X. Put F = Y − f(G), then we have f^{-1}(F) ⊂ X −G and X −G is open (resp. Jβ-open) in X. Then by assumption, there exists a gJβ -open set K of Y such that F = Y − f(G) ⊂ K and f^{-1}(K) ⊂ X − G. Now f^{-1}(K) ⊂ X − G implies K ⊂ Y − f (G) =F. Also F ⊂ K and so F =K. Therefore, we obtain f(G) = Y − K and hence f (G) is gJβ-closed in Y. This shows that f is gJβ-closed (resp Jβ-gJβ-closed) function.

Remark 5.9 We can prove the necessity part of the above theorem by replacing each set to closed set in the form of the proposition given below:

Proposition 5.10 If a surjective function $f : X \to Y$ is gJβ-closed (resp. Jβ-gJβ-closed) then for a closed set G of Y and for any open (resp. Jβ-open) set J of X containing $f^{-1}(G)$, there exists a Jβ-open set K of Y such that $G \subset K$ and $f^{-1}(K) \subset J$.

Proof. By previous theorem, there exists a gJβ-open set L of Y such that $G \subset L$ and $f^{-1}(L) \subset G$. Since G is closed, then we have $G \subset$ Jβ-int (L). Put K = Jβ-int(L). Then $K \in$ Jβ-o(Y), $G \subset K$ and $f^{-1}(K) \subset J$.

Proposition 5.11 If $f : X \to Y$ is continuous and Jβ-gJβ-closed function and G is gJβ-closed set in X, then f(G) is gJβ-closed in Y.

Proof. Let K be an open set of Y containing f(G), then $G \subset f^{-1}(K)$. As f is continuous $f^{-1}(K)$ is open in X. Since G is gJβ-closed in X, by a definition, we get Jβ-c1(G) $\subset f^{-1}(K)$ and hence f(Jβ-c1(G)) $\subset$ K. Since f is Jβ-gJβ-closed function and Jβ-c1(G) is Jβ-closed set in X, f(Jβ-c1(G)) is gJβ-closed in Y and hence we get Jβ-c1(f(Jβ-c1(G))) $\subset$ K. By definition of the Jβ-closure of a set, $G \subset$ Jβ-c1(G) which implies f(G) $\subset$ f(Jβ-c1(G)) and we know that**,** Jβ-c1(f(G)) $\subset$ Jβ-c1(f(Jβ-c1(G))) $\subset$ J. Hence Jβ-c1(f(G)) $\subset$ J. That is f (G) is gJβ-closed in Y .

Proposition 5.12 If $f : X \to Y$ is an open Jβ-irresolute bijection and G is gJβ-closed set in Y, then $f^{-1}(G)$ is gJβ-closed in X .

Proof. Let J be an open set of X containing $f^{-1}(G)$. Then $G \subset f(J)$ and f(J) is open in Y. Since G is gJβ-closed in Y, Jβ-c1(G) $\subset$ f(J) and hence we have f^{-1}(Jβ-c1(G)) $\subset$ J. Since f is Jβ-irresolute, f^{-1}(Jβ-c1(G)) is Jβ-closed in X, we have Jβ-c1(f^{-1}(G)) $\subset f^{-1}$(Jβ-c1(G) $\subset$ J. hence prove that $f^{-1}(G)$ is gJβ-closed in X.

Theorem 5.13 Let $f : X \to Y$ and $g : Y \to Z$ be the two functions, then

(i) If $gof : X \to Z$ is gJβ-closed and if $f : X \to Y$ is a continuous surjection, then $g : Y \to Z$ is gJβ-closed.

(ii) If $f : X \to Y$ is gJβ-closed with $g : Y \to Z$ is continuous and Jβ-gJβ-closed, then $gof : X \to Z$ is gJβ-closed.

(iii) If $f : X \to Y$ is closed and $g : Y \to Z$ is gJβ-closed, then $gof : X \to Z$ is gJβ-closed.

Proof. (i) Let G be a closed set of Y. Then $f^{-1}(G)$ is closed in X since f is continuous. By hypothesis gof(f^{-1}(G)) is gJβ-closed in Z. Hence G is gJβ-closed.

(ii) Proof comes from **Proposition 5.11**

(iii)The proof is obvious from definitions.

Theorem 5.14 The following properties are equivalent for a space X :

(a) X is Jβ-regular.

(b) For each closed set G and each point h from complement of G, there exists a Jβ-open set J and a gJβ-open set K such that $h \in J$ and $G \subset K$ and $J \cap K = \phi$.

(c) For each $B \subset X$ and each closed set G such that $B \cap G = \phi$, there exist a Jβ-open set J and a gJβ-open set K such that $B \cap J \neq \phi$, $G \subset K$ and $J \cap K = \phi$.

(d) For each closed set H of X, $H = \cap$ {Jβ-c1(K) : $H \subset K$ and K is gJβ-open.

Proof. (a) $\Rightarrow$ (b). The proof is obvious since every Jβ-open set is gJβ-open.

(b) $\Rightarrow$ (c). Let $B \subset X$ and let G be a closed set in X such that $B \cap G = \phi$. For a point $h \in B$ then h is contained in X – G and hence there exists $J \in$ Jβ-o(X) and a gJβ-open set K such that $h \in J$ and $G \subset K$ and $J \cap K = \phi$. Also $h \in B$ and $h \in J$ implies $h \in B \cap J$. So $B \cap J \neq \phi$.

(c) $\Rightarrow$ (a). Let G be a closed set in X and let $h \in X - G$. Then, $\{h\} \cap G = \phi$ and there exist $J \in$ Jβ-o(X) and a gJβ-open set L such that $h \in J$, $G \subset L$ and $J \cap L = \phi$. Put K = Jβ-int(L), then by the definition of gJβ-open set, we have $G \subset K$, K is Jβ-open set and $J \cap K = \phi$. Therefore X is Jβ-regular.

(a) $\Rightarrow$ (d). For a closed set F of X, by **Theorem 3.7**, we obtain

$G \subset \cap$ {Jβ-c1(K) : $G \subset K$ and K is gJβ-open} $\subset \cap$ {Jβ-c1(K) : $G \subset K$ and K is Jβ-open} =G Therefore, $G = \cap$ { Jβ-c1(K) : $G \subset K$ and K is gJβ-open}.

(d) $\Rightarrow$ (a). Let G be a closed set of X and h $\in$ X – G. by (d), there exists a gJβ-open set L of X such that G $\subset$ L and h $\in$ X – Jβ-c1(L). Since G is closed, G $\subset$ Jβ-int(L) by the definition of gJβ-open set. Put K = Jβ-int(L), then G $\subset$ K and K $\in$ Jβ-o(X). Since h $\in$ X – Jβ-c1(L), h $\in$ X – Jβ-c1(K). Put J = X – Jβ-c1(K) then, h $\in$ J and J is Jβ-open and J $\cap$ K = ϕ. This shows that X is Jβ-regular.

Theorem 5.15 If f : X $\to$ Y is a continuous Jβ-open gJβ-closed surjection and X is regular, then Y is Jβ-regular.
Proof. Let k $\in$ Y and let K be an open set of Y and k $\in$ K. Let h be a point of X such that k = f(h). By the regularity of X, there exists an open set J of X such that h $\in$ J $\subset$ c1(J) $\subset$ f^{-1}(K). We have k $\in$ f(J) $\subset$ f(cl(J)) $\subset$ K. Since f is Jβ-open and gJβ-closed, f(J) is Jβ-open and f(c1(J)) is gJβ-closed in Y. So, we obtain, k $\in$ f(J) $\subset$ Jβ-c1(f(J)) $\subset$ Jβ-cl(f(c1(J))) $\subset$ K. Now by the **Theorem 5.14**, Y is Jβ-regular.

Theorem 5.16 If f : X $\to$ Y is a continuous pre Jβ-open, Jβ-gJβ-closed surjection and X is Jβ-regular, then Y is Jβ-regular.
Proof. Let G $\in$ c(Y) and h $\in$ Y – G. Then f^{-1}(h) $\cap$ f^{-1}(G) = ϕ and f^{-1}(G) is closed in X. Since X is Jβ-regular, for a point g $\in$ f^{-1}(h), there exist J, K be two open set in X such that g $\in$ J, f^{-1}(G) $\subset$ K and J $\cap$ K = ϕ. Since G is closed in Y, by **Proposition 5.10**, there exist a Jβ-open set L such that G $\subset$ L and f^{-1}(L) $\subset$ K. Since f pre Jβ-open, we have h = f (g) $\in$ f(J) and f(J) $\in$ Jβ-o(Y). Since J $\cap$ K = ϕ, f^{-1}(L) $\cap$ J = ϕ and hence L $\cap$ f(J) = ϕ. Hence Y is Jβ-regular.

Theorem 5.17 A function f : X $\to$ Y is pre Jβ-closed if and only if for each subset F in Y and for each Jβ-open set J in X containing f^{-1}(F), there exists a Jβ-open set K containing F such that f^{-1}(K) $\subset$ J.
Proof. Assume that f is pre Jβ-closed. Let F be a subset of Y and J $\in$ Jβ-o(X) containing f^{-1}(F). Now take K = Y − f(X − J), then K is a Jβ-open set of Y such that F $\subset$ K and f^{-1}(K) $\subset$ J.
Converse: suppose that G be any Jβ-closed set of X. Then f^{-1}(Y − f(G)) $\subset$ X − G and X – G is Jβ-open set in X. There exists a Jβ-open set K of Y such that Y − f(G) $\subset$ K and f^{-1}(K) $\subset$ X − G. Therefore, we have f(G) $\supset$ Y − K and G $\subset$ f^{-1}(Y − K). Hence, we get f(G) = Y − K and f(G) is Jβ-closed in Y . This proves that f is pre Jβ-closed.

Lemma 5.18 Let f : X $\to$ Y define a function from X to Y , then following are equivalent:
(1) f is almost Jβ-irresolute,
(2) f^{-1}(K) $\subset$ Jβ-int(Jβ-cl(f^{-1} (K))) for every K $\in$ Jβ-o(Y).

Theorem5.19 A function f : X $\to$ Y is almost Jβ-irresolute if and only if f(Jβ-cl(J)) $\subset$ Jβ-cl(f(J)) for every Jβ-open set J in X.
Proof. Assume that J be a Jβ open set in X. Suppose k $\notin$ Jβ-cl(f(J)). Then there exists a Jβ-open set K in Y such that K ∩ f(J) = ϕ. Hence, f^{-1}(K) ∩ J = ϕ. Since J be Jβ-o(X), we have Jβ-int(Jβ-cl(f^{-1}(K))) ∩ Jβ-cl(J) = ϕ. Then by **Lemma 5.18**, f^{-1} (K) ∩ Jβ-cl(J) = ϕ and hence K ∩ f(Jβ-cl(J)) = ϕ. This implies that k $\notin$ f(Jβ-cl(J)).
Converse: If K be a Jβ-open set in Y, then A = X − Jβ-cl(f^{-1}(K)) $\in$ Jβ-o(X). By hypothesis, f(Jβ-cl(A)) $\subset$ Jβ-cl(f(A)) and hence X − Jβ-int(Jβ-cl(f^{-1}(A))) = Jβ-cl(A) $\subset$ f^{-1}(Jβ-cl(f(A))) $\subset$ f^{-1}(Jβ-cl(f(X − f^{-1}(K)))) $\subset$ f^{-1}(Jβ-cl(Y − K)) = f^{-1} (Y − K) = X − f^{-1}(K). Hence, f^{-1}(K) $\subset$ Jβ-int(Jβ-cl(f^{-1}(K))). By **Lemma 5.18**, f is almost Jβ-irresolute.

Theorem5.20 If f : X $\to$ Y is a pre Jβ-open continuous almost Jβ-irresolute function from a Jβ-normal space X onto a space Y , then Y is Jβ-normal.
Proof: Let G be a closed subset of Y and J be an open set containing G. Then by continuity of f, f^{-1}(G) is closed and f^{-1}(J) is an open set of X such that f^{-1}(G) $\subset$ f^{-1}(J). As X is Jβ-normal, there exists a Jβ-open set K in X such that f^{-1}(G) $\subset$ K $\subset$ Jβ-cl(K) $\subset$ f^{-1}(J) by using **Theorem 4.7** Then, f(f^{-1}(G)) $\subset$ f(K) $\subset$ f(Jβ-cl(K)) $\subset$ f(f^{-1}(J)). Since f is pre Jβ-open almost Jβ-irresolute surjection, we obtain G $\subset$ f(K) $\subset$ Jβ-cl(f(K)) $\subset$ J. Then again by **Theorem 4.7** the space Y is Jβ-normal.

Theorem 5.21 If f : X → Y is a pre Jβ-closed continuous function from a Jβ-normal space X onto a space Y , then Y is Jβ-normal.
Proof: Let G_1 and G_2 be disjoint closed sets in Y. Then $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are closed sets in X. Since X is Jβ-normal, then there exist two disjoint Jβ-open sets J and K such that $f^{-1}(G_1) \subset J$ and $f^{-1}(G_2) \subset K$. By **Theorem 5.17**, there exist Jβ-open sets L and M such that $G_1 \subset L$, $G_2 \subset M$, $f^{-1}(L) \subset J$ and $f^{-1}(M) \subset K$. Also, L and M are disjoint. Hence, Y is Jβ-normal.

Theorem 5.22 Let f : X → Y and g : Y → Z be functions. Then
(i) if f is Jβ-gJβ-closed and g is continuous Jβ-gJβ-closed then the composition gof : X → Z is Jβ-gJβ-closed.
(ii) if f is pre Jβ-closed and g is Jβ-gJβ-closed then the composition gof : X → Z is Jβ-gJβ-closed.
(iii) if f is quasi Jβ-closed and g is gJβ-closed then the composition gof : X → Z is Jβ-gJβ-closed.

Theorem 5.23 Let f : X → Y and g : Y → Z be functions and let the composition gof : X → Z be Jβ-gJβ-closed. If f is a Jβ-irresolute surjection, then g is Jβ-gJβ-closed.
Proof: Let H ∈ Jβ-c(Y). Since f is Jβ-irresolute and surjective, $f^{-1}(H)$ ∈ Jβ-c(X) and (gof)($f^{-1}(H)$) = g(H). Hence, g(H) is gJβ-closed in Z and hence g is Jβ-gJβ-closed.

Remark 5.24 Every Jβ-irresolute function is Jβ-gJβ-continuous but not conversely.

Theorem 5.25 A function f : X → Y is Jβ-gJβ-continuous if and only if $f^{-1}(K)$ is gJβ-open in X for every K ∈ Jβ-o(Y).

Theorem 5.26 If f : X → Y is closed Jβ-gJβ-continuous, then $f^{-1}(H)$ is gJβ-closed in X for each gJβ-closed set H of Y .
Proof: Let H be a gJβ-closed set of Y and J be an open set of X containing $f^{-1}(H)$. Put K = Y − f(X−J), then K is open in Y , H ⊂ K , and $f^{-1}(K) \subset J$. Therefore, we have Jβ-cl(H) ⊂ K and hence $f^{-1}(H) \subset f^{-1}$(Jβ-cl(H)) ⊂ $f^{-1}(K)$ ⊂ J. Also, f is Jβ-gJβ-continuous, f^{-1}(Jβ-cl(H)) is gJβ-closed in X and hence Jβ-cl($f^{-1}(H)$) ⊂ Jβ-cl(f^{-1}(Jβ-cl(H))) ⊂ J. This proves that $f^{-1}(H)$ is gJβ-closed in X.

Theorem 5.27 If f : X → Y is an open Jβ-gJβ-continuous bijection, then $f^{-1}(H)$ is gJβ-closed in X for every gJβ-closed set H of Y .
Proof: Let H be a gJβ-closed set of Y and J be an open set of X containing $f^{-1}(H)$. Since f is an open surjective, H = f(f^{-1} (H)) ⊂ f(J) and f(J) is open. Therefore, Jβ-cl(H) ⊂ f(J). Since f is injective, $f^{-1}(H) \subset f^{-1}$(Jβ-cl(H)) ⊂ f^{-1}(f(J)) = J. Since f is Jβ-gJβ-continuous, f^{-1}(Jβ-cl(H)) is gJβ-closed in X and hence Jβ-cl($f^{-1}(H)$) ⊂ Jβ-cl(f^{-1}(Jβ-cl(H))) ⊂ J. Hence $f^{-1}(H)$ is gJβ-closed in X.

Theorem 5.28 Let f : X → Y be a function from X to Y and g : Y → Z be an open Jβ-gJβ continuous bijection from Y to Z and let the composition gof : X → Z be Jβ-gJβ-closed then f is Jβ-gJβ-closed.
Proof: Let H be a Jβ-closed set of X. Then (gof)(H) is gJβ-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.27**, f(H) is gJβ-closed in Y and hence f is Jβ-gJβ-closed.

Theorem 5.29. Let f : X → Y be a function from X to Y and g : Y → Z is a closed Jβ-gJβ-continuous injection from Y to Z and let the composition gof : X → Z be Jβ-gJβ-closed then f is Jβ-gJβ-closed.
Proof: Let H be a closed set in X. Then (gof)(H) is gJβ-closed in Z and g^{-1}((gof)(H)) = f(H). By **Theorem 5.26**, f(H) is gJβ-closed in Y and hence f is Jβ-gJβ-closed.

6. Preservation theorems and other characterizations of Jβ-normal spaces

Theorem 6.1 For a topological space X, the following are equivalent:
(a) X is Jβ-normal,
(b) for any pair of disjoint closed sets G and H of X, there exist disjoint gJβ-open sets J and K of X such that G ⊂ J and H ⊂ K ,

(c) for each closed set G and each open set K containing G, there exists a gJβ-open set J such that cl(G) ⊂ J ⊂ Jβ-cl(J) ⊂ K,
(d) for each closed set G and each g-open set L containing G, there exists a Jβ-open set M such that G ⊂ M ⊂ Jβ-cl(M) ⊂ int(L),
(e) for each closed set G and each g-open set L containing G, there exists a gJβ-open set J such that G ⊂ J ⊂ Jβ-cl(J) ⊂ int(L),
(f) for each g-closed set H and each open set N containing H, there exists a Jβ-open set M such that cl(H) ⊂ M ⊂ Jβ-cl(M) ⊂ N,
(g) for each g-closed set H and each open set N containing H, there exists a gJβ-open set J such that cl(H) ⊂ J ⊂ Jβ-cl(J) ⊂ N.

Proof: (a) ⇔ (b) ⇔ (c) : Since every Jβ-open set is gJβ-open, then proof is obvious.

(d) ⇒ (e) ⇒ (c) and (f) ⇒ (g) ⇒ (c) : Since every closed (resp. open) set is g-closed (resp. g-open), then proof is obvious.

(c) ⇒ (e) : Let G be a closed subset of X and L be a g-open set such that G ⊂ L. Since L is g-open and G is closed, G ⊂ int(L). Then, there exists a gJβ-open set J such that G ⊂ J ⊂ jβ-cl(J) ⊂ int(L).

(e) ⇒ (d) : Let G be any closed subset of X and L be a g-open set containing G. Then there exists a gJβ-open set J such that G ⊂ J ⊂ Jβ-cl(J) ⊂ int(L). Since J is gJβ-open, G ⊂ Jβ-int(J). Put M = Jβ-int(J), then M is Jβ-open and G ⊂ M ⊂ Jβ-cl(M) ⊂ int(L).

(c) ⇒ (g) : Let H be any g-closed subset of X and N be an open set such that H ⊂ N. Then cl(H) ⊂ N. Therefore, there exists a gJβ-open set J such that cl(H) ⊂ J ⊂ Jβ-cl(J) ⊂ N.

(g) ⇒ (f) : Let H be any g-closed subset of X and N be an open set containing H. Then there exists a gJβ-open set J such that cl(H) ⊂ J ⊂ Jβ-cl(J) ⊂ N. Since J is gJβ-open and cl(H) ⊂ J, we have cl(H) ⊂ Jβ-int(J), take M = Jβ-int(J), then M is Jβ-open and cl(H) ⊂ M ⊂ Jβ-cl(M) ⊂ N.

Theorem 6.2 If f : X → Y is a continuous quasi Jβ-closed surjection and X is Jβ-normal, then Y is normal.
Proof: Let G_1 and G_2 be any disjoint closed sets of Y. Since f is continuous surjection, $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are disjoint closed sets of X. Since X is Jβ-normal, there exist disjoint Jβ-open sets J_1, J_2 such that $f^{-1}(G_1) \subset J_1$ and $f^{-1}(G_2) \subset J_2$. Put $K_1 = Y - f(X - G_1)$ and $K_2 = Y - f(X - G_2)$, then K_1 and K_2 are open in Y, $G_i \subset K_i$ and $f^{-1}(K_i) \subset G_i$ for i = 1, 2. Since $J_1 \cap J_2 = \phi$ and f is surjective; we have $K_1 \cap K_2 = \phi$. This shows that Y is normal.

Lemma 6.3 A subset J of a space X is gJβ-open if and only if G ⊂ Jβ-int(J) whenever G is closed and G ⊂ J.

Theorem 6.4 Let f : X → Y be a closed Jβ-gJβ-continuous injection. If Y is Jβ-normal, then X is Jβ-normal.
Proof: Let G_1 and G_2 be disjoint closed sets of X, Since f is a closed injection, $f(G_1)$ and $f(G_2)$ are disjoint closed sets of Y . By the Jβ-normality of Y , there exist two disjoint Jβ-open sets $K_1$and K_2 in Y such that $f(G_1) \subset K_1$ and $f(G_2) \subset K_2$. Since f is Jβ-gJβ-continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are disjoint gJβ-open sets of X and $G_i \subset f^{-1}(K_i)$ for i = 1, 2. Now, put J_i = Jβ-int($f^{-1}(K_i)$) for i = 1, 2. Then, $J_i \in$ Jβ-o(X), $G_i \subset J_i$ and $J_1 \cap J_2 = \phi$. This shows that X is Jβ-normal.

Corollary 6.5 If f : X → Y is a closed Jβ-irresolute injection and Y is Jβ-normal, then X is Jβ-normal.
Proof: Since every Jβ-irresolute function is Jβ-gJβ-continuous the proof is same as previous theorem.

Lemma 6.6 A function f : X → Y is almost gJβ-closed if and only if for each subset A of Y and each regular open set J of X containing $f^{-1}(A)$, there exists a gJβ-open set K of Y such that A ⊂ K and $f^{-1}(K) \subset J$.

Lemma6.7 If f : X → Y is almost gJβ-closed, then for each closed set H of Y and each regular open set J in X containing $f^{-1}(H)$, there exists a open set K in Y such that H ⊂ K and $f^{-1}(K) \subset J$.

Theorem 6.8 Let f : X → Y be a continuous almost gJβ-closed surjection. If X is normal, then Y is Jβ-normal.
Proof: Let H_1 and H_2 be any disjoint closed sets of Y. Since f is continuous, $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are disjoint closed sets of X. By the normality of X, there exist disjoint open sets J_1 and J_2 such that $f^{-1}(H_i) \subset J_i$, where i = 1, 2. Now, put $L_i = int(cl(J_i))$ for i = 1, 2, then L_i are regular open sets in X, $f^{-1}(H_i) \subset J_i \subset L_i$ and $L_1 \cap L_2 = \phi$. By **Lemma 6.7**, there exists two Jβ-open sets K_1 and K_2 such that $H_i \subset K_i$ and $f^{-1}(K_i) \subset L_i$, where i = 1, 2. Since $L_1 \cap L_2 = \phi$ and f is surjective, we have $K_1 \cap K_2 = \phi$. This shows that Y is Jβ-normal.

Corollary 6.9 If f : X → Y is a continuous Jβ-closed surjection and X is normal, then Y is Jβ-normal.

Conclusion

This paper is devoted to introduce some new weaker version of normality and regularity namely Jβ-regular and Jβ-normal spaces in topological spaces by using Jβ-open sets. Moreover, we investigated the relationship between these new spaces and some other topological spaces, also some example & counter example are given to verify these relationships and its converse. Also we defined some function related to Jβ-regular and Jβ-normal spaces. Besides it, we discussed some topological properties of Jβ-regular and Jβ-normal spaces. Of course, the entire content will be a successful tool for the researchers for finding the way to obtain the results in the context of such types of regular and normal spaces.

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