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The Generalization of the Exponential Curve Power Rayleigh Distribution: Its Properties and Estimations

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Abstract: In this paper, a generalization of the power Rayleigh distribution known as the Generalized Exponential curve Power Rayleigh (GEPR) distribution has been addressed using an exponential curve transformation. The newly proposed distribution's statistical properties include its mode, moments, survival function, hazard rate function, order statistic, and random number generation. Use estimation methods, including Maximum Likelihood Estimation (MLE), briefly discussed in order to estimate the parameters of this distribution.

Keywords: Probability distribution, Hazard rate function, Survival function, Maximum Likelihood Estimation, Power Rayleigh Distribution, Lifetime Distribution.

I. Introduction

Lifetime data modelling and analysis are essential in the fields of applied sciences like medicine, finance, engineering, etc. Numerous lifetime distributions, including the exponential, Weibull, gamma, and many others, are significant in this situation. The chosen probability model or distribution has a significant impact on the consistency and accuracy of statistical analysis. Due to this, creating new distributions has become a fundamental idea in statistical theory in recent years. Typically, this is done by adding a new parameter to the baseline distribution. Saeed, Ijaz, Khalil, and Ali (2021). One thing to keep in mind is that the generalization techniques mentioned above all include a few extra parameters in the initial model. In one sense, the additional parameter(s) allows the distribution to analyze complex data structures with more flexibility, but on the other hand, it makes parameter estimation and other inferential procedures more difficult. In light of these challenges, Kumar et al. propose a small number of new transformation techniques in which no additional parameters are added beyond those involved in the baseline distribution. By including a shape parameter ($\alpha > 0$) in the cumulative distribution function of the baseline distribution, Gupta, Gupta, and Gupta (1998) proposed the cumulative distribution function (CDF) of a new distribution. Further generalization techniques were used by Gupta and Kundu (2001), Seenoi, Supapakorn, and Bodhisuwan (2014), etc. to create more adaptable probability models. Using the Quadratic rank transmutation map (QRTM), as described by Shaw and Buckley (2009), is another well-known method for generalizing baseline distribution. For instance, Kumar, Singh, and Singh (2015a), Sine, Singh, and Singh (2015b), and Kumar, Singh, and Singh (2016) all discuss DUS transformation. The baseline distribution is always the exponential distribution. Kumar, Singh, Singh, and Mukherjee (2017) have published a paper using the Weibull distribution M transformation.

The following is how the paper is set up: Generalized Exponential Curve Power Rayleigh (GEPR) is developed and graphically displayed in section (II). Subsections after that discuss the survival function, the hazard function, the shape of the distribution, and Random number generation. In section (III), a number of mathematical and statistical properties of the new distribution are derived, including the raw moments, mode, quantiles, and order statistics. Section (IV) discusses the parameter estimation technique from a frequentist point of view. Finally, we wrap up the paper in section (V).

II. Generalized Exponential curve Power Rayleigh (GEPR) distribution:

Let X be a random variable from a Power Rayleigh distribution with parameters $\theta > 0, \lambda > 0$, if its probability density function (PDF) is given by:

$$g(x) = \frac{\theta}{\lambda^2} x^{2\theta-1} e^{\frac{-x^2\theta}{2\lambda^2}};\ \theta > 0, \lambda > 0, x > 0 \ \ldots\ldots\ldots (1)$$

and the corresponding cumulative distribution function is given by

$$G(x;\theta,\lambda) = 1\text{-}e^{\frac{-x^2\theta}{2\lambda^2}};\ \theta > 0, \lambda > 0, x > 0 \ \ldots\ldots\ldots (2)$$

The Power Rayleigh distribution is used as the original continuous distribution in the current study, and we use Exponential curve transformation to obtain a new lifetime distribution. If a baseline lifetime distribution's PDF and CDF are g(x) and G(x), respectively, the PDF of a new distribution lifetime distribution is defined as:

$$f(x) = \alpha\beta g(x)e^{\beta G(x)};\ \alpha > 0, \beta > 0, x > 0\ \ldots\ldots\ldots\ldots (3)$$

The corresponding CDF is given by

$$F(x) = \alpha e^{\beta G(x)};\ \alpha > 0, \beta > 0, x > 0\ \ldots\ldots\ldots\ldots (4)$$

The Survival function and Hazard rate function is given as:

$$S(x) = 1 - F(x);\ \ldots\ldots\ldots\ldots\ldots (5),$$

and

$$H(x) = \frac{f(x)}{1-F(x)}$$

$$H(x) = \frac{\alpha\beta g(x)e^{\beta G(x)}}{1-\alpha e^{\beta G(x)}};\ \ldots\ldots\ldots\ldots\ldots (6),$$

respectively.

A reminder that the GEPR is a Power Rayleigh distribution extension is necessary. Greater flexibility in analyzing complex datasets has been made possible by the newly acquired. Figure 1 and 2 illustrate the shape of the PDF and CDF of GEPR distribution respectively.

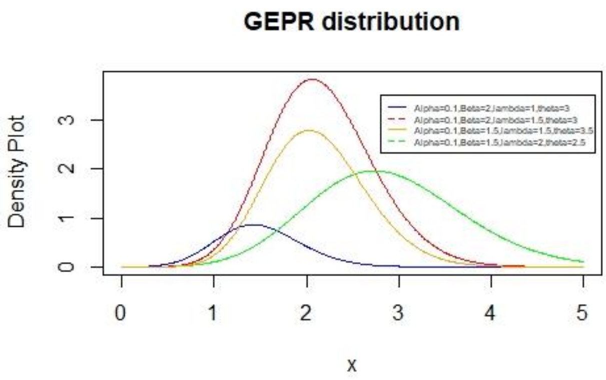

Figure 1: PDF of GEPR distribution for different parameter values

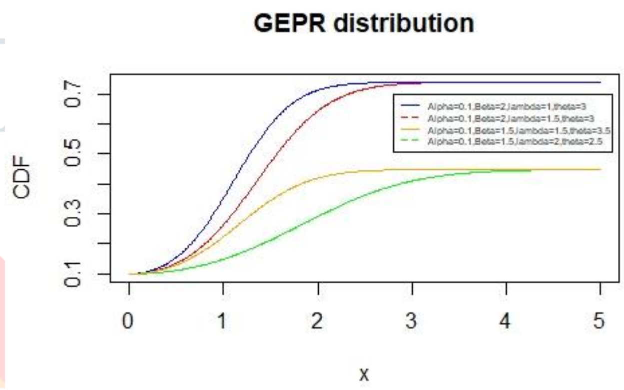

Figure 2: CDF of GEPR distribution for different parameter values

Hazard rate function and Survival function:

The Survival function and the Hazard rate function are the two most important inter-related probability measures for the lifetime distribution. Both measures are commonly used to describe and model the inherent characteristics of various survival data sets. The survival function is denoted as $S(x) = P(X > x) = 1 - F(x)$; Similarly, the Hazard rate function, $H(x)$ is given as:

$$H(x) = \frac{f(x;\Theta)}{S(x;\Theta)};\ S(x;\Theta) > 0$$

Where $\Theta = \{\alpha, \beta, \theta, \lambda\}$ is a parameter. The following expressions (9) and (10), respectively, give the survival and hazard rate function for the ECTPR distribution.

$$S(x;\Theta) = 1 - \alpha e^{\beta\left(1-e^{\frac{-x^2\theta}{2\lambda^2}}\right)};\ \ldots\ldots\ldots (9)$$

and

$$H(x;\Theta) = \frac{\frac{\alpha\beta\theta}{\lambda^2}x^{2\theta-1}e^{\frac{-x^2\theta}{2\lambda^2}}e^{\beta(1-e^{\frac{-x^2\theta}{2\lambda^2}})}}{1 - \alpha e^{\beta\left(1-e^{\frac{-x^2\theta}{2\lambda^2}}\right)}};\ \ldots\ldots\ldots (10)$$

As seen in Figure 3, the hazard rate function curve initially increases, then starts to decrease, and finally converges to some constant value. In survival analysis, it is well known that lifetime distributions with a hazard rate that first increases and then decreases are very helpful. This lifetime distribution, for instance, resembles the hazard rate curve of infant mortality rate. For more information, see Kotz and Nadarajah (2000), Rao and Mbwambo (2019), and others. The shapes of the Hazard rate and Survival functions of the GEPR distribution are shown in Figures 3 and 4, respectively.

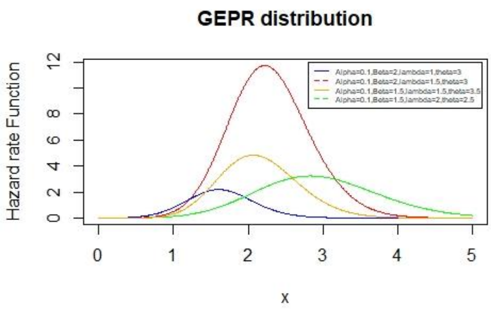

Figure 3: Hazard rate function for different parameter values

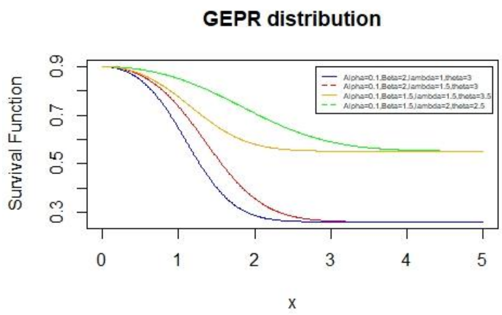

Figure 4: Survival function for different parameter values

Shape of GEPR distribution:

The shape of the distribution is significant because it provides information about the nature of the distribution. Glaser (1980) proposed a theorem for mathematically determining the shape of the hazard rate. According to the theorem, $\eta(x) = -\frac{f'(x)}{f(x)}$; where $f(x)$ is continuous and twice differentiable on the interval (0, ∞). If $\eta'(x) > 0 \ \forall \ x > 0$, then the hazard rate is increasing and if $\eta'(x) < 0 \ \forall \ x > 0$, then the hazard rate is decreasing. So, we have

$$\eta(x) = -\frac{(\theta x^2 - 2\lambda^2\theta + \lambda^2)e^{\frac{x^2\theta}{2\lambda^2}} - \beta\theta x^2}{x\lambda^2}$$

Therefore,

$$\eta'(x) = \frac{\beta\theta}{\lambda^2} - \frac{(\theta^2x^4 - 2\lambda^2(\theta-1)\theta x^2 + 2\lambda^4\theta - \lambda^4)e^{\frac{x^2\theta}{2\lambda^2}}}{\lambda^4x^2}; \ \ldots\ldots\ldots\ldots (11)$$

The last term of equation (11) attains the minimum value zero as $x \to \infty$. Therefore, it is clearly seen that $\eta'(x) < 0$ i.e., the distribution has decreasing hazard rate function.

Random number generation:

The inversion method or inverse transformation method is used to generate random numbers from the GEPR distribution. The algorithm generates a random number U from the Uniform (0, 1) distribution first, and then the equation $x = F^{-1}(U)$ generates a random number x from the GEPR distribution. Here we have,

$$U = \alpha e^{\beta\left(1 - e^{\frac{-x^2\theta}{2\lambda^2}}\right)}$$

Therefore,

$$x = \sqrt{\frac{-2\lambda^2 log\left(1 - \frac{logU - log\alpha}{\beta}\right)}{\theta}}; \ \ldots\ldots\ldots\ldots\ldots (12)$$

We can easily produce random numbers of size n from the GEPR distribution by using equation (12) for known values of the parameter values.

III. STATISTICAL PROPERTIES OF GEPR DISTRIBUTION:

This section derives and discusses some basic and significant statistical and mathematical measures of the Generalized Exponential curve Power Rayleigh (GEPR) distribution, such as moments, mode, and order statistics.

Moments:

The r^th order raw moment about origin, μ'_r of the proposed distribution with having PDF (7) is obtained as follows:

$$\mu'_r = E(X^r)$$

$$\mu'_r = \alpha\beta e^{\beta}2^{-\theta-\frac{r}{2}-1}x^{2\theta+r-2}\left(\frac{\theta x^2}{\lambda^2}\right)^{-\theta-\frac{r}{2}+1}\left(\Gamma\left(\frac{r}{2}+\theta, \frac{2\theta x^2}{\lambda^2}\right) - 2^{\theta+\frac{r}{2}}\Gamma\left(\frac{r}{2}+\theta, \frac{\theta x^2}{\lambda^2}\right)\right); \ldots\ldots. (13)$$

If we substitute r=1 in equation (13), the above equation is reduced to

$$\mu'_1 = \alpha\beta e^{\beta}2^{-\theta-\frac{3}{2}}x^{2\theta-1}\left(\frac{\theta x^2}{\lambda^2}\right)^{-\theta+\frac{1}{2}}\left(\Gamma\left(\frac{1}{2}+\theta, \frac{2\theta x^2}{\lambda^2}\right) - 2^{\theta+\frac{1}{2}}\Gamma\left(\frac{1}{2}+\theta, \frac{\theta x^2}{\lambda^2}\right)\right); \ldots. (14)$$

So, we have the mean μ'_1 our newly proposed GEPR distribution. From equation (13) it can be seen that, $r = 2, \mu'_2$ becomes undefined. As a result, because the gamma function is defined only for positive numbers, the variance and higher-order raw moments of this distribution cannot be calculated.

Mode:

For any distribution, the value with the highest probability area is the mode. Thus, the value for which the maximum value of $f(x;\Theta)$ equation (7) is obtained is the mode for the GEPR distribution. Mode is the result of $f'(x;\Theta)=0$, and $f''(x;\Theta)<0$, respectively. So, differentiating equation (7) with respect to x and equating to zero, we get

$$-\frac{\alpha\beta\theta x^{2\theta-2}e^{\frac{-x^2\theta}{2\lambda^2}}(((\theta x^2-2\lambda^2\theta+\lambda^2)e^{\frac{x^2\theta}{2\lambda^2}}-\beta\theta x^2)e^{\beta\left(1-e^{\frac{-x^2\theta}{2\lambda^2}}\right)}}{\lambda^4}=0;\ \dots\dots..(15)$$

It is obvious that equation (15) cannot be solved analytically. As a result, some numerical iteration techniques can be used to solve equation (15) numerically. We prefer the Newton-Raphson method in this case in particular.

Order Statistics:

The order statistic is a crucial tool in reliability theory and quality control testing for estimating the time until failure of a particular item by taking into account a few early failures Mukherjee, Dey, and Raheem (2017). Let $X_1 < X_2 < X_2 < \dots < X_n$ be an ordered sample from a continuous distribution with CDF $F_X(x)$ and PDF $f_x(x)$. Then the PDF of r^{th} order statistic $X_{(r)}$ is given by:

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!\,(n-r)!}f_X(x)\big(F_X(x)\big)^{r-1}[1-F_X(x)]^{n-r};r=1,2,\dots,n$$

So, for the GEPR distribution PDF of the r^{th} order statistic is given as:

$$f_{X_{(r)}}(x)=\frac{n!}{(r-1)!\,(n-r)!}\frac{\alpha\beta\theta}{\lambda^2}x^{2\theta-1}e^{\frac{-x^2\theta}{2\lambda^2}}e^{\beta\left(1-e^{\frac{-x^2\theta}{2\lambda^2}}\right)}(\alpha e^{\beta(1-e^{\frac{-x^2\theta}{2\lambda^2}})})^{r-1}[1-\alpha e^{\beta(1-e^{\frac{-x^2\theta}{2\lambda^2}})}]^{n-r};\dots\dots..(16)$$

The smallest order statistic is always the sample's minimum, i.e., $X_{(1)}=\min(X_1,X_2,\dots,X_n)$, while the largest order statistic is the sample's maximum, i.e., $X_{(n)}=\max(X_1,X_2,\dots,X_n)$.. By substituting r = 1 and n in equation (16), the expressions for the smallest and largest order statistics are obtained. The corresponding CDF of the r^{th} order statistic is obtained as follows:

$$F_{X_{(r)}}(x)=\sum_{i=r}^{n}\binom{n}{i}F_X^i(x)[1-F_X(x)]^{n-i}$$

$$F_{X_{(r)}}(x)=\sum_{i=r}^{n}\binom{n}{i}(\alpha e^{\beta(1-e^{\frac{-x^2\theta}{2\lambda^2}})})^i[1-\alpha e^{\beta\left(1-e^{\frac{-x^2\theta}{2\lambda^2}}\right)}]^{n-i};\ \dots\dots\dots(17)$$

IV. Statistical inference:

Estimating the unknown parameter(s) for the given sample is an important step in fully understanding the probabilistic model in statistics. Several estimation procedures under classical and Bayesian paradigms are available in the literature; for more information, see Louzada, Ramos, and Perdona' (2016), Dey, Dey, and Kundu (2014), Kundu and Raqab (2005), Mazucheli, Ghitany, and Louzada (2016), Fan (2015), and others. The goal of this study is to estimate the unknown parameters of the GERP distribution using frequentist methods. The Maximum likelihood method (MLE) is briefly described here.

Maximum likelihood estimation method:

The Maximum likelihood estimation method (MLE) satisfies a number of desirable properties for a good estimator, such as consistency, asymptotic efficiency, invariance property, and so on. As a result, the MLE is one of the most commonly used techniques for parameter estimation. Let $x_1,x_2,\dots,x_n$ be the sample of size n, drawn from the GERR distribution with PDF given in equation (7).

The likelihood function is given by:

$$L(\Theta;X)=(\frac{\alpha\beta\theta}{\lambda^2})^n\prod_{i=1}^{n}(x_i^{2\theta-1})\,e^{\beta\left(1-e^{\frac{-\theta\sum_{i=1}^n x_i^2}{2\lambda^2}}\right)}e^{\frac{-\theta\sum_{i=1}^n x_i^2}{2\lambda^2}};\dots\dots\dots(18)$$

From equation (18), the log-likelihood function given as:

$$logL(\Theta;X)=l=nlog\left(\frac{\alpha\beta\theta}{\lambda^2}\right)+\sum_{i=1}^{n}logx_i^{2\theta-1}+\beta\left(1-e^{\frac{-\theta\sum_{i=1}^n x_i^2}{2\lambda^2}}\right)-\frac{\theta\sum_{i=1}^n x_i^2}{2\lambda^2};\dots\dots.(19)$$

As a result, to maximize equation (18), we differentiate the log-likelihood function (19) above with respect to the parameters and equate it to zero, giving us the expressions shown below:

$$\frac{\partial l}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots.(20)$$

$$\frac{\partial l}{\partial \beta} = 0 \Rightarrow \frac{n}{\beta} - e^{\frac{-\theta \sum_{i=1}^{n} {x_i}^2}{2\lambda^2}} + 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots.(21)$$

$$\frac{\partial l}{\partial \lambda} = 0 \Rightarrow \frac{-\beta\theta \sum_{i=1}^{n} {x_i}^2 e^{\frac{-\theta \sum_{i=1}^{n} {x_i}^2}{2\lambda^2}}}{\lambda^3} - \frac{2n}{\lambda} + \frac{\theta \sum_{i=1}^{n} {x_i}^2}{\lambda^3} = 0 \ldots\ldots.(22)$$

$$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \frac{n}{\theta} - \frac{-\beta \sum_{i=1}^{n} {x_i}^2 e^{\frac{-\theta \sum_{i=1}^{n} {x_i}^2}{2\lambda^2}}}{2\lambda^2} - \frac{\sum_{i=1}^{n} {x_i}^2}{2\lambda^2} + 2\sum_{i=1}^{n} log x_i = 0 \ \ldots\ldots\ldots\ldots.(23)$$

It is obviously that equation (20) – (23) is not written explicitly and hence it cannot be solved analytically, we will illustrate numerically. So, to obtain the MLE's of the unknown parameters to solve the equation (20) – (23) non-linear equation numerically.

V. Conclusion

In this paper, we examine and introduce the four parameters of the Generalized Exponential curve Power Rayleigh (GEPR) distribution. We derive the newly introduced distribution's probability density function, cumulative distribution function, and its Survival function, and Hazard rate function. We compute some general properties and estimation techniques, like, raw moments, mode, order statistics, and Maximum likelihood estimation.

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