POSITIVE SEMIDEFINITE MATRICES WITH REFERENCE TO INDEFINITE INNER PRODUCT

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ABSTRACT: We discuss the characterization of positive semidefinite matrices and positive semidefinite block matrices in indefinite inner product space.

KEYWORDS: Indefinite matrix product; Indefinite inner product space; EP matrix; Range symmetric matrix.

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1. INTRODUCTION

An indefinite inner product in \( \mathbb{C}^n \) is a conjugate symmetric sesquilinear form \([x, y]\) together with the regularity condition that \([x, y] = 0, \forall y \in \mathbb{C}^n\) only when \(x = 0\). Any indefinite inner product is associated with a unique invertible complex matrix \(J\) (called a weight) such that \([x, y] = (x, Jy)\), where \((.,.)\) denotes the Euclidean inner product on \( \mathbb{C}^n \). We also make an additional assumption on \(J\), that is, \(J^2 = I\), to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors ([2],[6]). This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [6]. More precisely, the indefinite matrix product of two matrices \(A\) and \(B\) of sizes \(m \times n\) and \(n \times l\) complex matrices, respectively, is defined to be the matrix \(A \circ B = AJ_nB\). The adjoint of \(A\), denoted by \(A^*\), is defined to be the matrix \(J_mA^*J_m\), where \(J_m\) and \(J_n\) are weights.

Many properties of this product are similar to that of the usual matrix product ([6]). Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results in indefinite inner product spaces in a manner analogous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar also established in [6] that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate than the usual matrix product. Recall that the Moore-Penrose inverse exists if and only if \(\text{rank}(AA^*) = \text{rank}(A^*A) = \text{rank}(A)\). If we take \(A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), then \(AA^*\) and \(A^*A\) are both the zero matrix and so \(\text{rank}(AA^*) < \text{rank}(A)\), thereby proving that the Moore-Penrose inverse does not exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, \(\text{rank}(A \circ A^*) = \text{rank}(A^* \circ A) = \text{rank}(A)\). Thus, the Moore-Penrose \(J\)-inverse with real or complex entries exists over an indefinite inner product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinent to extend the study of generalized inverses to the setting of indefinite inner product.

In this paper we study the positive semidefinite matrices in \(\mathcal{A}\) We have also established the characterization theorems in \(\mathcal{A}\). Further, we have determined the properties of block matrices over indefinite inner product space. The following notations are used in this paper, \(\mathcal{A}\) denotes the indefinite inner product space, \(\mathbb{C}^{m \times n}, (R(A), N(A))\) denotes the class of \(m \times n\) matrices, range space, null space in Euclidean space and \(\mathbb{C}^{m \times n}_{J_mJ_n}\), \(Ra(A), Nu(A)\) denotes the class of \(m \times n\) matrices, range space, null space in indefinite inner product space respectively.

2. PRELIMINARIES

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. [5] Let \(A \in \mathbb{C}^{m \times n}_{J_mJ_n}, B \in \mathbb{C}^{n \times k}_{J_mJ_n}\). Let \(I_n\) be an arbitrary but fixed \(n \times n\) complex matrix such that \(J_n = I_n^{-1}\). The indefinite matrix product of \(A\) and \(B\) (relative to \(I_n\)) is defined by \(A \circ B = AJ_nB\).

Definition 2.2. [5] For \(A \in \mathbb{C}^{m \times n}_{J_mJ_n}\), \(A^* = I_nA^*J_n\) is the adjoint of \(A\) relative to \(I_n\) and \(J_m\).

Definition 3.3. [5] A matrix \(A \in \mathbb{C}^{n \times n}_{J_mJ_n}\) is said to be \(J\)-invertible if there exists \(X \in \mathbb{C}^{n \times n}_{J_mJ_n}\), such that \(A \circ X = X \circ A = I_n\) such an \(X\) is denoted by \(A^{-1} = I_{J_mJ_n}A^{-1}\).

Remark 2.4. [5] For the identity matrix \(I\), it reduces to a generalized inverse of \(A\) and \(A^* = A^{-1}\). It can be easily verified that \(X\) is a generalized inverse of \(A\) if and only if \(I_nX^{-1} J_m\) is a generalized inverse of \(A\) under the usual product of matrices. Hence \(A^{-1} = \{X : I_nX^{-1}J_m\} \) is a generalized inverse of \(A\).

Definition 2.5. [5] For \(A \in \mathbb{C}^{m \times n}_{J_mJ_n}, X \in \mathbb{C}^{n \times n}_{J_mJ_n}\) is called the Moore - Penrose \(J\)-inverse of \(A\) if it satisfies the following equations: (i) \(A \circ X \circ A = A\), (ii) \(X \circ A = X\), (iii) \(A \circ X \circ A = X\), (iv) \(A \circ X = X\). Such an \(X\) is denoted by \(A^{-1}\) and represented as \(A^{-1} = I_{J_mJ_n}A^{-1}\).

Definition 2.6. [5] The range space of \(A \in \mathbb{C}^{m \times n}_{J_mJ_n}\) is defined by \(Ra(A) = \{y = A \circ x \in \mathbb{C}^{m}; x \in \mathbb{C}^n\}\). The null space of \(A \in \mathbb{C}^{m \times n}_{J_mJ_n}\) is defined by \(Nu(A) = \{x \in \mathbb{C}^n; A \circ x = 0\}\).
Property 2.7. [5][7] Let $A \in \mathbb{C}^{m \times n}$. Then
(i) $(A^t)^t = A$.
(ii) $(A^t)^t = A$.
(iii) $(AB)^t = B^t A^t$.
(iv) $Ra(A^t) = Ra(A^t)$.
(v) $Ra(A^t) = Ra(A)$, $Ra(A^t) = Ra(A^t)$.
(vi) $Nu(A^t) = Nu(A^t)$, $Nu(A^t) = Nu(A^t)$.

Definition 2.8. [5][8] A is range symmetric in $J$ if and only if $Ra(A) = Ra(A^t)$ (or equivalently $Nu(A) = Nu(A^t)$).

Remark 2.9. [5][9] In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.

Theorem 2.10. [5][10] For $A \in \mathbb{C}^{n \times n}$, the following are equivalent:
(i) $A$ is range symmetric in $J$.
(ii) $JA$ is EP.
(iii) $JA$ is EP.
(iv) $Nu(A) = Nu(A^t)$.
(v) $A^t A^t = A^t A$.
(vi) $(A^t)^t = JA^t = AA^t$.
(vii) $A$ is J-EP.

Theorem 11. [1] Let $A, B \in \mathbb{C}^{n \times n}$ then the following equivalence hold:
(i) $R(A) \subseteq R(A^t) \iff N(A^t) \subseteq N(C) \iff C = CA^t A$ for every $A^t \in A(1)$
(ii) $R(B) \subseteq R(A) \iff N(A) \subseteq N(B^t) \iff B = AA^t B$ for every $A^t \in A(1)$
(iii) $CA^t B$ is invariant for every $A^t \in A(1)$

Lemma 122.12. [3] Let $A$ and $B$ be matrices in $J$ then $N(A^t) \subseteq N(B^t)$ if and only if $Nu(A^t) \subseteq Nu(B^t)$.

3. CHARACTERIZATION OF POSITIVE SEMIDEFINITE MATRICES

Definition 3.13 A matrix $A \in \mathbb{C}^{n \times n}$ is said to be Positive semidefinite(PSD) in $J$ denoted as $A \geq 0 \iff A$ is $J$–EP and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.

Theorem 3.214 For $A \in \mathbb{C}^{n \times n}$, $A \geq 0 \iff JA \geq 0 \iff AJ \geq 0$.

Proof. Suppose $A \geq 0$, then by Definition 3.1, $A$ is $J – EP$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.

From Theorem 2.10 it follows that both $JA$ and $AJ$ are EP.

$A \geq 0 \iff [Ax, x] \geq 0$ and $A$ is $J – EP$

$\iff (Ax, x) \geq 0$ and $A$ is $J – EP$

$\iff [Ax, x] \geq 0$ and $J A \geq 0$.

Similarly, $A \geq 0 \iff AJ \geq 0$ can be proved.

Corollary 3.315 For $A \in \mathbb{C}^{n \times n}$, $A \geq 0 \iff A = S^t JS$, for some $S \in \mathbb{C}^{n \times n}$.

Proof. $A \geq 0 \iff JA \geq 0 \iff JA = S^t JS \iff S = S^t JS \iff A = S^t JS$.

Corollary 3.416 For $A \in \mathbb{C}^{n \times n}$, $A \geq 0 \iff A^t \geq 0$

Proof. Since $A \geq 0$, by Definition (3.1), $A$ is $J – EP$ and hence $Ra(A) = Ra(A^t)$.

by Theorem 3.4 [4] and Property (2.7) we get $A^t$ is $J – EP$.

For any $x \in \mathbb{C}^n$, $[A^t]^t = [(S^t)^t S]^t x = [x, (S^t) S] x = [Sx, Sx] = (Sx, Sx) = [Sx]^t x \geq 0 \iff A^t x] \geq 0$. Thus $A \geq 0 \iff A^t \geq 0$. $A \geq 0 \iff (A^t)^t = A \geq 0 \iff A \geq 0 \iff (A^t)^t = A \geq 0$ follows from the above steps and using Property (2.7), we have $A \geq 0 \iff A \geq 0$.

Theorem 3.517 For $A \in \mathbb{C}^{n \times n}$, $A \geq 0 \iff$ for any $P \in \mathbb{C}^{n \times n}$, $AP \geq 0$.

Proof. Since $A \geq 0$, by Definition (3.1), $Ra(A) = Ra(A^t)$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.

Let $B = P^t AP$, $Ra(B^t) = Ra(P^t A^t P)$ and $Ra(P^t A^t P) = Ra(P^t A^t P) = Ra(P^t A^t P) = Ra(B)$, which implies $B$ is $J – EP$.

Now, for $x \in \mathbb{C}^n$, $[Bx, x] = [P^t AP x, x] = [APx, Px] = [Ay, y] \geq 0$, where $y = Px$, which implies $B = P^t AP \geq 0$.

Theorem 3.618 For $A \in \mathbb{C}^{n \times n}$, $A \geq 0 \iff A^{[t]} \geq I_0$.

Proof. $A \geq 0 \iff JA \geq 0$ \hspace{1cm} [By Theorem 3.2]

$\iff (JA)^t \geq 0$

$\iff A^{[t]} I \geq 0$

$\iff A^{[t]} \geq I_0$. 

Theorem 3.622 For $A \in \mathbb{C}^{n \times n}$, $A < 0 \iff A^{[t]} < I_0$.

Proof. $A < 0 \iff JA < 0$ \hspace{1cm} [By Theorem 3.2]

$\iff (JA)^t < 0$

$\iff A^{[t]} I < 0$

$\iff A^{[t]} < I_0$. 

4. POSITIVE SEMIDEFINITE BLOCK MATRICES IN \( \mathcal{J} \)

In this section, we derive equivalent conditions for a block matrix to be positive semidefinite (psd) in \( \mathcal{J} \) and establish formulae for \( \{1\}-\)inverse, \( \{1,2\}-\)inverse, \( \{1,3\}-\)inverse and \( \{1,4\}-\)inverse of positive semidefinite block matrices in \( \mathcal{J} \).

**Lemma 4.1** [1][19] Let \( M = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \) be a psd matrix where \( A, D \) are Hermitian. Then \( M \geq 0 \iff A \geq 0, \ A A^* B = B \) and \( D - B^* A B \geq 0 \).

**Theorem 4.220** Let \( M = \begin{pmatrix} A & X \\ X^* A & X \end{pmatrix} \) \( \in \mathbb{C}^{(m+n) \times (m+n)} \) then \( M \succeq 0 \iff A \succeq 0, \ A X = (AX) \iff A X = (AX)^* \).

**Proof.** \( M \succeq 0 \iff J M J = 0 \iff J M \succeq 0 \). \( M \succeq 0 \iff J M J \geq 0 \iff J M \succeq 0 \iff J M J \succeq 0 \) \( \iff J M J \succeq 0 \).

**Lemma 4.321** Let \( M = \begin{pmatrix} A & X \\ X^* A & X \end{pmatrix} \) with \( A \) is \( J - \)EP and \( X^* A = (AX) \) then \( \text{Nu}(A) \subseteq \text{Nu}((AX) \subseteq \text{Nu}(A) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \).

**Proof.** Since \( A \) is \( J - \)EP then \( \text{Nu}(A) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \).

**Theorem 4.4** Let \( M = \begin{pmatrix} A & X \\ X^* A & X \end{pmatrix} \) \( \in \mathbb{C}^{(m+n) \times (m+n)} \) then \( M \succeq 0 \iff A \succeq 0, \ A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \).

**Proof.** Since \( A \) is \( J - \)EP \( \iff \) then \( \text{Nu}(A) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \subseteq \text{Nu}(AX) \).

**Theorem 4.522** Let \( M = \begin{pmatrix} A & X \\ X^* A & X \end{pmatrix} \) be an \( (m+n) \times (m+n) \) positive semidefinite matrix with \( X^* A = (AX) \), the Schur complement of \( A \) in \( M \) is \( X^* A X - X^* A A X = S \) then the generalized inverse of \( M \) is given by \( M^{-1} = (A^2 + A B S^2 C A^2 + A B S^2 C A)^{-1} \).

**Proof.** By Theorem 4.2, since \( M \succeq 0 \iff A \succeq 0, \ A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \).

We claim: \( M^{-1} \) is a \( \{1\}-\)inverse of \( M \).

Using Theorem 4.2 and Lemma 4.2 for the above conditions we get \( C = CA^T A = B = AA^T B \).

**Theorem 4.6** Let \( M \succeq 0 \iff A \succeq 0, \ A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \).

**Proof.** By Theorem 4.2, since \( M \succeq 0 \iff A \succeq 0, \ A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \iff A X = (AX) \).

We claim: \( M^{-1} \) is a \( \{1\}-\)inverse of \( M \).
Theorem 4.623  Let $M = \begin{pmatrix} A & AX \\ X^\dagger A & X^\dagger AX \end{pmatrix}$ be an $(m+n) \times (m+n)$ positive semidefinite matrix. Let $M^{[a]}$ be the form (4.1), for $\alpha = (1,2)$, if $A^{(1,2)}$ and $S^{(1,2)}$ are $(1,2)$-inverse of $A$ and $S$ respectively then $M^{[1,2]}$ is a $(1,2)$-inverse of $M$ in $\mathcal{S}$.

Proof. One can easily verify $M^{[1,2]}$ is a $(1,2)$-inverse of $M$.

Theorem 4.724  Let $M = \begin{pmatrix} A & AX \\ X^\dagger A & X^\dagger AX \end{pmatrix}$ be an $(m+n) \times (m+n)$ positive semidefinite matrix. Let $M^{[a]}$ be the form (4.1), for $\alpha = (1,3)$, if $A^{(1,3)}$ and $S^{(1,3)}$ are $(1,3)$-inverse of $A$ and $S$ respectively with $S^{(1,3)}$ Hermitian and $rk(M) = rk(A) + rk(S)$. Then $M^{[1,3]}$ is a $(1,3)$-inverse of $M$ in $\mathcal{S}$.

Proof. By Theorem 4.2, since $M \geq 0 \iff A \geq 0, X^{[1]}A = (AX)^{[1]}, N(A^{[1]}) \subseteq N((AX)^{[1]})$ and $X^{[1]}AA^{\dagger}AX \geq X^{[1]}AX$. Since $A \geq 0, A$ is $J - EP$ from Definition 2.8 and Definition 3.1 it follows that $N(A) = N(A^{[1]})$. By using Lemma 4.3, we have $M \geq 0 \iff A \geq 0, N(A) \subseteq N((AX)^{[1]})$ and $X^{[1]}AA^{\dagger}AX \geq X^{[1]}AX$. $A^*$ is also a $(1,3)$-inverse of $A$. For $\alpha = 1$ in (4.1).

By using Theorem 2.11 and Lemma 2.12 for the above conditions we get $X^{[1]}A = X^{[1]}AA^{\dagger}A, AX = AA^{\dagger}AX$ and $X^{[1]}AA^{\dagger}AX \geq X^{[1]}AX$.

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