Structure in the zero – divisor graph of a non – commutative Γ –near ring

[1]R.Rajeswari, [2]N.Meenakumari
[1,2]PG & Research Department of Mathematics, A. P. C. Mahalaxmi College for Women Thoothukudi, India.
[1]rajeswarir30@yahoo.com,[2]meenakumari.n123@gmail.com

Abstract

Let M be a non – commutative gamma near ring with Nil(M) its ideal of nilpotent elements, $Z(M)$ its set of zero divisors respectively. In this paper, we introduce directed zero divisor graphs of M. We investigate various characteristics of this graph and obtain corresponding results related with the Γ- near – ring theoretic concepts.

Key words

non - commutative gamma near rings, zero – divisors, regular element, nilpotent elements, tournament.

1.Introduction

In [2], the zero – divisor graph of a commutative ring M is defined to be the undirected graph whose vertices are the non zero zero – divisors of M, The concept of a Gamma near – rings [8] was introduced by Sathyanarayana and the ideal theory in Gamma near- rings was studied by Bh.Satyanarayana and G.L.Booth. the concept of a commutative ring was introduced by David F.Anderson, Ayman Badawi[3].We assume throughout that all rings are non – commutative ring $Z(\Gamma(M)), Reg\ (\Gamma(M))$ are zero divisor and regular graphs respectively. For distinct vertices x and y of a Graph G, let d(x, y) be the length of the shortest path from x to y. The diameter of a connected graph is the supremum of the distances between vertices. For any graph G, the girth of G is the length of a shortest cycle in G and is denoted by gr(G). If G has no cycle, we define the girth of G to be infinite. A clique of a graph is a maximal complete sub graph and the number of graph vertices in the largest clique or graph G, denoted by ω(G) is called the clique number of G. A graph G is bipartite with vertex classes V_1, V_2 if the set of all vertices of G is $V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and edge of G joins a vertex from V_1 to a vertex of V_2. A complete bipartite graph is a bipartite graph containing all edges joining the vertices of V_1 $a$and $_2$. A complete bipartite graph on vertex sets of size graph on vertex sets of size m an n is denoted by $K_{m,n}$for any positive integer, $K_{1,n}$is called a star graph. Let G be a directed graph and let x be vertex of G. The number of edges of G of the form $y \to x$ is called the in degree of x, and the number of edges of G of the form $x \to y$ is called the out degree of x. The vertex x is a sink if its out degree is zero, and x is a source if its in degree is zero. A vertex that is both a sink and source is called isolated. The vertex x is called a proper sink (resp proper source) if it is a sink and its in degree (resp out degree) is non zero. An element which is not a left zero – divisor is called left regular, and an element which is not a right – zero divisor is called right regular. Any element that is both left and right regular is called regular. Thus, an element which is left regular and a right zero – divisor will be a sink, and an element which is right regular and a left zero –

divisor will be a source. A directed graph G is called tournament if for every two distinct vertices x and y of G, exactly one of $x \to y$ and $y \to x$ is an edge of G.In this Paper we study the directed zero – divisor graph in a non - commutative Γ −near ring and tournaments are discussed and also generalized. It is shown that the zero – divisor graph of a finite Γ −near ring cannot be a network. The final result proves that the zero – divisor graph of a finite Γ −near ring must have an even number of directed edges.

2. Directed Zero – divisor graphs

Definition 2.1

Let M be a non commutative Γ-near-ring with I and let Z(M) be its set of zero - divisors. The Zero - divisor graph of M, denoted by Γ(M) is the (directed) graph with vertices Z(M)* = Z(M) \{0}, the set of non zero zero - divisors of M and for distinct x,y ∈ Z(M)*, the vertices x and y are adjacent then $x\gamma y$=0 for all $\gamma \in \Gamma$.

Theorem 2.2

Let M be a Γ −near ring with no non – zero nilpotent elements and $\Gamma(M) \neq \emptyset$. If $x\gamma y = 0$ for some $x, y \in M$ and $\gamma \in \Gamma$, then $y\gamma x = 0, \gamma \in \Gamma$. In particular $\Gamma(M)$ is not a tournament.

Proof

By the hypothesis, $\Gamma(M)$ must have more than one vertex (since if $\Gamma(M)$ consists only of the vertex x, then $x^2 = 0$). Note that if $x, y \in M$ such that $x\gamma y = 0$, then $(y\gamma x)^2 = (y\gamma x)\gamma(y\gamma x) = y\gamma(x\gamma y)\gamma x$ implies $y\gamma x = 0$. Thus for each edge $x \to y$ of $\Gamma(M)$, $y \to x$ is also an edge (Note that this implies the set of right zero – divisors of M is equal to the set of left zero – divisors of M). Hence $\Gamma(M)$ cannot be a tournament.

Theorem 2.3

Let M be a Γ - near – ring such that $\Gamma(M) \neq \emptyset$. Then $\Gamma(M)$ is a tournament if and only if $\Gamma(M)$ consists of exactly one vertex.

Proof

If $\Gamma(M)$ consists of exactly one vertex, then it is trivially a tournament.Let M be a Γ - near – ring such that $\Gamma(M)$ consists of two or more vertices. Assume $\Gamma(M)$ is a tournament. By **Theorem 2.2** there is some $0 \neq x \in M$ such that $x^2 = 0$. Assume $\Gamma(M)$ consists only of two vertices, x and a, and one edge $x \to a$, that is, $x\gamma a = 0$ and $a\gamma x \neq 0, \gamma \in \Gamma$. Note that $2x = 0$, since $2x$ is a zero – divisor of M and $2x \neq x$ and $2x \neq a$ (otherwise, $a\gamma x = 2x^2 = 0$). Now a + x is a zero – divisor of M. since $x\gamma(a + x) = 0$. But clearly $a + x \neq a \; or \; x \; and \; a + x \neq 0$ since $x + x = 0$. Thus a + x must be a third vertex of $\Gamma(M)$, a contradiction. By an analogous argument, we get a contradiction if $\Gamma(M)$ consists only of the vertices x and b and the edge $b \to x$. So $\Gamma(M)$ must have at least three distinct vertices. Note that $2x = 0$ (otherwise $\Gamma(M)$ would contain both paths $-x \to x \; and \; x \to -x$). By the definition of a tournament, $\Gamma(M)$ must contain one of the three following path structures.

Case (1) x is a source. Then $x \to a$ and $x \to b$ are edges of $\Gamma(M)$ for distinct vertices a and b of $\Gamma(M)$. That is $x\gamma a = x\gamma b = 0, b\gamma x \neq 0 \; and \; a\gamma x \neq 0, \gamma \in \Gamma$. Note that $(b\gamma x)\gamma x = b\gamma x^2 = 0$. Since x is a source and $b\gamma x \neq 0$, we must have $b\gamma x = x$. Similarly $a\gamma x = x$. By the definition of a tournament either $a \to b \; or \; b \to a$ is a path in $\Gamma(M)$. That is, either $a\gamma b = 0 \; (or) b\gamma a = 0$. However this implies either $x = a\gamma x = a\gamma(b\gamma x) = (a\gamma b)\gamma x = 0$ (or) $x = b\gamma x = b\gamma(a\gamma x) = (b\gamma a)\gamma x = 0$, a contradiction in either case.

Case(2)x is a sink. Then $b \to x$ and $a \to x$ are edges of $\Gamma(M)$ for distinct vertices a and b of $\Gamma(M)$. As in case (1), $x\gamma a = x\gamma b = x$. Again either $a\gamma b = 0\ (or) b\gamma a = 0$. However this implies either $x = x\gamma b = (x\gamma a)\gamma b = x\gamma(a\gamma b) = 0\ (or) x = x\gamma a = (x\gamma b)\gamma a = x\gamma(b\gamma a) = 0$, a contradiction in either case.

Case (3) There is a path of the form $a \to x \to b$ in $\Gamma(M)$ for distinct vertices a and b of $\Gamma(M)$. That is $a\gamma x = x\gamma b = 0, x\gamma a \neq 0$ and $b\gamma x \neq 0$. Since $\Gamma(M)$ is a tournament, either $a\gamma b = 0\ or\ b\gamma a = 0$.

i) Suppose $b\gamma a = 0$. Then $a\gamma b \neq 0$. Note that $x\gamma(b + x) = 0$. Since $2x = 0, b + x \neq 0$. Also $b + x \neq a$(otherwise, $\Gamma(M)$ contains both edges $a \to x\ and\ x \to a$) and clearly $b + x \neq x$. Since $\Gamma(M)$ is a tournament, $\Gamma(M)$ must contain one of the edges $a \to b + x\ (or) b + x \to a$. However, $a\gamma(b + x) = a\gamma b + a\gamma x = a\gamma b \neq 0$ and $(b + x)\gamma a = b\gamma a + x\gamma a = x\gamma a \neq 0$, a contradiction.

ii) Suppose $a\gamma b = 0$. Then $b\gamma a \neq 0$. Note that $(a + x)\gamma x = a\gamma x + x^2 = 0$. Since $2x = 0, a + x \neq 0$. Also $a + x \neq b$ since $b\gamma x \neq 0\ and\ (a + x)\gamma x = 0$. Because $a + x \neq a, \Gamma(M)$ must contain one of the edges $a \to a + x$ (or) $a + x \to a$

sub case(i) Suppose $a\gamma(a + x) = 0$. Then $0 = a^2 + a\gamma x = a^2$. Thus0020$(x\gamma a)\gamma a = x\gamma(a^2) = 0$. Since $\Gamma(M)$ is a tournament, we must have $a = x\gamma a$(otherwise, $\Gamma(M)$ contains both the edges $a \to x\gamma a$ and $x\gamma a \to a$). But then $x\gamma a = x\gamma(x\gamma a) = x^2\gamma a = 0$, a contradiction

Sub case (ii) Suppose $(a + x)\gamma a = 0$, then $a^2 = -x\gamma a = x\gamma a \neq 0$. Note that $a^2 \neq x$(otherwise $x = x\gamma a$ and therefore both $a^4 = (a^2)^2 = x^2 = 0\ and\ a^4 = a^2\gamma a^2 = x\gamma a^2 = (x\gamma a)\gamma a = x\gamma a = x \neq 0$). But then $\Gamma(M)$ contains both the edges $a^2 \to x\ and\ x \to a^2$ (since $a^2\gamma x = a\gamma(a\gamma x) = 0\ and\ x\gamma a^2 = x\gamma(x\gamma a) = x^2\gamma a = 0$), this is a contradiction

We have a contradiction in all possible cases. Therefore $\Gamma(M)$ cannot be a tournament.

Theorem 2.4

If M is a finite Γ - near – ring, then $\Gamma(M)$ has an even number of directed edges.

Proof

The result is trivially true if $\Gamma(M) \neq \emptyset$. For any vertex x in $\Gamma(M)$, if $x \to y$(resp $y \to x$) is an edge, then $-x \to y$(resp $y \to -x$) is also an edge (thus, the result is true if the characteristics of M is odd). Therefore, one needs only verify that there are an even number of edges among those $x \in M$ such that $2\gamma x = 0$. Since these elements form a sub Γ - near – ring of M, it is enough to show that if M is of characteristic 2, then $\Gamma(M)$ has an even number of directed edges.

Suppose M has characteristic 2. It is the case that $x \to$ is an edge in $\Gamma(M)$ whenever $y \to x$ is an edge, then $\Gamma(M)$ will have an even number of edges. (Note that this implies the result is true if M is commutative).Therefore suppose $x \to$ is an edge in $\Gamma(M)$ whenever $y \to x$ is not an edge. That is, suppose $x\gamma y = 0\ and\ y\gamma x \neq 0$ for distinct non – zero $x, y \in M$. Letting E be the set if all such edges, one can view E in terms of ordered pairs in $M \times M: E = \{(x, y)/x\gamma y = 0, y\gamma x \neq 0, x \neq 0, y \neq 0, x \neq y\}$. It is enough to show that E has an even number of elements.

Let $A = \{(x, y) \in E/y is\ nilpotent\}$. Let $(x, y) \in A$ and m be the least positive integer such that $y^m = 0$. Then $(x +$

$y^{m-1}, y) \in A$ (note $x + y^{m-1} \neq 0$. Since $y\gamma x \neq 0$ *and* $y^m = 0$). Define a map $p: A \to A$ *by* $p(x, y) = (x + y^{m-1}, y)$ (Note that $x + y^{m-1} \neq y$. If this were so, then $y^2 = y\gamma(x + y^{m-1}) = 0$. But this implies m = 2 , and therefore $x + y = y$). Since $p(x + y^{m-1}, y) = (x + y^{m-1} + y^{m-1}, y) = (x, y)$. A must have an even number of elements. Let $B = \{(x, y) \in E / x \text{ is nilpotent}\}$. For $(x, y) \in B$ let n be the least positive integer such that $x^n = 0$. Defining $q: B \to B$ *by* $q(x, y) = (x, y + x^{n-1})$ shows B has an even number of elements.

Consider $(x, y) \in A \cap B$, let m, n be the least positive integers such that $x^n = y^m = 0$. Let $y^a\gamma x^b$ denote the first non – zero entry in the ordered list $\left\{ \begin{matrix} y^{m-1}\gamma x^{m-1}, y^{m-1}\gamma x^{n-2}, \dots, \\ y^{m-1}\gamma x, y^{m-2}\gamma x^{n-1}, y^{m-2}\gamma x^{n-2}, \dots y\gamma x \end{matrix} \right\}$. Note that such an element exists, because $y\gamma x \neq 0$. Also for any positive integer i, $y^{a+i}\gamma x^b = y^a\gamma x^{b+i} = 0$,and thus $(y + y^a\gamma x^b)^j = y^j$ for any integer $j \geq 2$. Note $y + y^a\gamma x^b \neq 0$ because $y\gamma x \neq 0$ and $(y^a\gamma x^b)\gamma x = 0$. Thus we can define a map $r: A \cap B \to A \cap B$ by $r(x, y) = (x, y + y^a\gamma x^b)$(Note that $x \neq y + y^a\gamma x^b$. If this were so, then $x^2 = 0 = y^2$, implying $y^a\gamma x^b = y\gamma x$. Thus $x + y = y\gamma x$. However $(y\gamma x)^2 = 0$ *and* $(x + y)^2 = y\gamma x \neq 0$). Since $r(x, y + y^a\gamma x^b) = (x, y + y^a\gamma x^b + (y + y^a\gamma x^b)\gamma x^b) = (x, y + y^a\gamma x^b + y^a\gamma x^b) = (x, y)$, $A \cap B$ must have an even number of elements. Let $P = E \setminus (A \cup B) = \{(x, y) \in E / x, y \text{ are not nilpotent}\}$. Let $S = \{(x, y) \in P / \text{there exists } m, n \text{ such that } y^m\gamma x^n = 0\}$. One may take m and n to be minimal (that is if $y^s\gamma x^t = 0$, then either $s > m$ *or* $s = m$ *and* $t \geq n$). Since $y\gamma x \neq 0$, either $m > 1$, *or* $m = 1$ *and* $n > 1$. Note that for $m > 1, y + y^{m-1}\gamma x^n \neq 0$ *and for* $m = 1, x + y\gamma x^{n-1} \neq 0$. For any positive integer $i \geq 2, (y + y^{m-1}\gamma x^n)^i = y^i$ *when* $m = 1$. Define a map $f: S \to S$ *by* $f(x, y) = (x, y + y^{m-1}\gamma x^n)$ *if* $m > 1$ *and* $f(x, y) = (x + y\gamma x^{n-1}, y)$ *if* $m = 1$(Note that $x \neq y + y^{m-1}\gamma x^n$ since otherwise $x^2 = x\gamma(y + y^{m-1}\gamma x^n) = 0$. Similarly $x + y\gamma x^{n-1} \neq y$). It is routine to verify in either case that $f\big(f(x, y)\big) = (x, y)$ and therefore S has an even number of elements. Finally consider $T = P - S = \{(x, y) \in P / y^m\gamma x^n \neq 0 \text{ for all } m, n\}$. Let $(x, y) \in T$. Since M is a finite Γ - near – ring, there exists a minimal integer N and an integer t such that $x^N = x^t$ with $1 \leq t < N$, and a minimal integer M and an integer s such that $y^M = y^s$ with $1 \leq s < M$. Given any positive integers a and b, $(x + y^a\gamma x^b)^i = x^i + y^a\gamma x^{b+i-1}$ *and* $(y + y^a\gamma x^b)^i = y^i + y^{a+i-1}\gamma x^b$ for any positive integer i.

Now we show that N is the minimal integer such that $(x + y^{M-1}\gamma x^t)^N = (x + y^{M-1}\gamma x^t)^w$ *with* $1 \leq w \leq N$ (and that w = t) , Let r and u be integers such that $1 \leq u < r \leq N$ *and* $(x + y^{M-1}\gamma x^t)^r = (x + y^{M-1}\gamma x^t)^u$ (the equation holds if u = t and r = N). From this equation we obtain $x^r + y^{M-1}\gamma x^{t+r-1} = x^u + y^{M-1}\gamma x^{t+u-1}$ and thus, $x^r + x^u = y^{M-1}\gamma x^{t+r-1}$ -----→ (1)

Note that $x^r + x^u \neq 0$ *if* $r < n$, *or if* $r = N$ *and* $u < t$. Multiplying equation (1) on the left by x yields $x^{r+1} + x^{u+1} = 0$. This equation can only be true if r = N and u = t, or if r = N – 1 and u = t – 1. But in this later case, equation (1) can be rewritten as $x^r + x^u = y^{M-1}\gamma x^{t+N-1-1} + y^{M-1}\gamma x^{t+t-1-1} = 2y^{M-1}\gamma x^{2t-2} = 0$, and this yields a contradiction. Hence , we must have r = N and u = t. A similar statement is true for M and s. for any positive integers m and n,$(y + y^s\gamma x^{N-1})^m\gamma(x + y^{M-1}\gamma x^t)^n$

$= (y^M + y^{m+s-1}\gamma x^{N-1})\gamma(x^n + y^{M-1}\gamma x^{t+n-1})$

$= y^m\gamma x^n + y^{m+s-1}\gamma x^{N+n-1} + y^{m+M-1}\gamma x^{t+n-1} = y^m\gamma x^n$

Thus, $(x + y^{M-1}\gamma x^t, y + y^s\gamma x^{N-1}) \in T$.

Define a map $g: T \to T$ by $g(x, y) = (x + y^{m-1}\gamma x^t, y + y^s\gamma x^{N-1})$(Note that $x + y^{M-1}\gamma x^t \neq y + y^s\gamma x^{N-1}$ since otherwise $x^2 = x\gamma(x + y^{M-1}\gamma x^t) = x\gamma(y + y^s\gamma x^{N-1}) = 0$). Then $g(x + y^{M-1}\gamma x^t, y + y^s\gamma x^{N-1})$

$= (x + y^{M-1}\gamma x^t + (y + y^s\gamma x^{N-1})^{M-1}\gamma(x + y^{M-1}\gamma x^t)^t, y + y^s\gamma x^{N-1} + (y + y^s\gamma x^{N-1})^s\gamma(x + y^{M-1}\gamma x^t)^{N-1})$

$=(x + y^{M-1}\gamma x^t + y^{M-1}\gamma x^t, y + y^s\gamma x^{N-1} + y^s\gamma x^{N-1})$

$=(x, y)$

Hence T contains an even number of elements. Therefore , since $|E| = |A| + |B| - |A \cap B| + |S| + |T|$, E must have an even number of elements.

References

[1] D.DAnderson, M.Naseer, Beck's coloring of a commutative ring. J. Algebra.159 (1993), 500-514.

[2] D.F.Anderson, P.S.Livingston., The zero-divisor graph of a commutative ring. J. Algebra.217 (1999), 434-447.

[3] I.Beck., Coloring of commutative rings, J. Algebra,.116(1988), 208-226.

[4] J.A.Bondy and U.S.R.Murty., Graph Theory with Applications, North-Holland. Amsterdam (1976).

[5]G.Pilz., Near-Rings, North-Holland, Amsterdam,1983.

[6]R.Rajeswari,N.Meenakumari,T.Tamizh Chelvam, An ideal based zero divisor graph of a gamma near rings,International journal of Mathematical Archieves,Vol(4),No 2,2013 .

[7]Shane Pedmond,structure in the zero – divisor graph of a non commutative ring,Houston J.Mathematics.30(2004)2.

[8] Bh.Satyanarayana, Contributions to Near-ring Theory., Doctoral Thesis, Nagarjuna University, India, 1984.