ABSTRACT:

Let $G = (V, E)$ be a simple graph. A Near Mean Cordial Labeling of $G$ is a function in $f : V(G) \rightarrow \{1, 2, 3, \ldots, p-1, p+1\}$ such that the induced map $f^*$ defined by $f^*(uv) = \begin{cases} 1 & \text{if } (f(u) + f(v)) \equiv 0 \text{ (mod}_2) \\ 0 & \text{else} \end{cases}$ and it satisfies the condition $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ represent the number of edges labeled with 0 and 1 respectively. A graph is called a Near Mean Cordial Graph if it admits a near mean cordial labeling.

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I. INTRODUCTION

By a graph, it means a finite undirected graph without loops or multiple edges. For graph theoretic terminology, we referred Harary [4]. For labeling of graphs, we referred Gallian [1]. A vertex labeling of a graph $G$ is an assignment of labels to the vertices of $G$ that induces for each edge $uv$ a label depending on the vertex labels of $u$ and $v$.

A graph $G$ is said to be labeled if the $n$ vertices are distinguished from one another by symbols such as $v_1, v_2, \ldots, v_n$. In labeling of a particular type, the vertices are assigned distinct values from a given set, which induces distinguish edge values satisfying certain conditions. The concept of graceful labeling was introduced by Rosa [3] in 1967 and subsequently by Golomb [2]. In this paper, it is to be proved that Double Fan (DF$_n$), Triangular Snake (TS$_n$) (When $n \equiv 0,1,3 \text{ (mod } 4)$) and Jelly fish (J(m,n)) are Near Mean Cordial graphs. And also Triangular Snake (TS$_n$) (When $n \equiv 2 \text{ (mod } 4)$) is not Near Mean Cordial Graph.

IL PRELIMINARIES

Definition 2.1: Let $G = (V,E)$ be a simple graph. Let $E(V(G) \rightarrow \{0,1\})$ and the induced edge label assigning $|f(u) - f(v)|$ is called a Cordial Labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by atmost 1 and also the number of edges labeled 0 and the number of edges labeled 1 differ by atmost 1. A graph is called Cordial if it has a cordial labeling.

Definition 2.2: Let $G = (V,E)$ be a simple graph. $G$ is said to be a Near Mean Cordial Graph if $f : V(G) \rightarrow \{0,1,2\}$ such that for each edge $uv$ the induced map $f^*$ defined by $f^*(uv) = \begin{cases} 1 & \text{if } (f(u) + f(v)) \equiv 0 \text{ (mod}_2) \\ 0 & \text{else} \end{cases}$ and it satisfies the condition $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ represent the number of edges labeled with 0 and 1 respectively. A graph is called a Near Mean Cordial Graph if it admits a near mean cordial labeling.

Definition 2.3: Let $G = (V,E)$ be a simple graph. A Near Mean Cordial Labeling of $G$ is a function in $f : V(G) \rightarrow \{1, 2, 3, \ldots, p-1, p+1\}$ such that the induced map $f^*$ defined by $f^*(uv) = \begin{cases} 1 & \text{if } (f(u) + f(v)) \equiv 0 \text{ (mod}_2) \\ 0 & \text{else} \end{cases}$ and it satisfies the condition $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ and $e_f(1)$ represent the number of edges labeled with 0 and 1 respectively. A graph is called a Near Mean Cordial Graph if it admits a near mean cordial labeling.

Definition 2.4: The join $G_1 + G_2$ of $G_1$ and $G_2$ consists of $G_1 \cup G_2$ and all lines joining $V_1$ with $V_2$ as vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ $\cup \{uv : u \in V(G_1)$ and $v \in V(G_2)\}$. The graph $P_n + K_1$ is called a Fan and $P_n + 2K_1$ is called the Double fan (DF$_n$).

Definition 2.5: A Triangular Snake is obtained from the path $(v_1, v_2, \ldots, v_p)$ by replacing every edge by a triangle $C_3$.

Definition 2.6: For integers $m, n \geq 0$. We consider the graph $J(m,n)$ with vertex and edge set $V(J(m,n)) = \{u,v,x,y\} \cup \{x_1,x_2,\ldots, x_m\} \cup \{y_1,y_2,\ldots, y_n\}$ and $E(J(m,n)) = \{(uv),(ux),(uy),(vx),(vy)\} \cup \{(xx) : 1 \leq i \leq m\} \cup \{(yy) : 1 \leq i \leq n\}$. $J(m,n)$ is called a jelly fish.
Definition 2.7: \( K_{1,n} \oplus P_a @ K_{1,m} \) is a graph which is obtained by joining the root of the star \( K_{1,n} \) at one end of the path \( P_a \) and joining the another root of the star \( K_{1,m} \) at the other end of the path \( P_n \).

III. MAIN RESULTS

Theorem 3.1: Double Fan (DF\(_n\)) is a Near Mean Cordial Graph.

Proof: Let \( V(G) = \{u, u_i : 1 \leq i \leq n, v_i : 1 \leq i \leq n \} \)

Let \( E(G) = \{(uu_i) : 1 \leq i \leq n\} U \{(uv_i) : 1 \leq i \leq n\} U \{(v_i v_{i+1}) : 1 \leq i \leq n-1\} \) U \( \{(u_{i}, u_{i+1}) : 1 \leq i \leq n-1\} \)

Define \( f : V(G) \rightarrow \{1, 2, 3, \ldots, 2n, 2n+2\} \) by

Case (i): When \( n \equiv 0 (\text{mod} 4) \):

Let \( f(u) = n+1 \)

\[ f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n}{2} \]

\[ f(u_{2i}) = \frac{n}{2} + i, \quad 1 \leq i \leq \frac{n}{2} \]

\[ f(v_1) = 2n + 2 \]

\[ f(v_{2i+1}) = 2n + (i-1), \quad 1 \leq i \leq \frac{n}{2} - 1 \]

\[ f(v_{2i}) = n + i + 1, \quad 1 \leq i \leq \frac{n}{2} \]

Case (ii): When \( n \equiv 1 (\text{mod} 4) \):

Let \( f(u) = 2n + 2 \)

\[ f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2} \]

\[ f(u_{2i}) = \frac{3(n+1)}{2} + (i-1), \quad 1 \leq i \leq \frac{n-1}{2} \]

\[ f(v_1) = \frac{n+1}{2} + i, \quad 1 \leq i \leq \frac{n+1}{2} \]

\[ f(v_{2i+1}) = \frac{3n+1}{2} - (i-1), \quad 1 \leq i \leq \frac{n-1}{2} \]

Case (iii): When \( n \equiv 2 (\text{mod} 4) \):

Let \( f(u) = \frac{n+2}{2} \)

\[ f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n}{2} \]

\[ f(u_{2i}) = 2n + 2 \]

\[ f(v_{2i+1}) = 2n - (i-2), \quad 2 \leq i \leq \frac{n}{2} \]

\[ f(v_{2i}) = \frac{n+2}{2} + i, \quad 1 \leq i \leq \frac{n}{2} \]

\[ f(v_{2i+1}) = \frac{3n+2}{2} - (i-1), \quad 1 \leq i \leq \frac{n}{2} \]

Case (iv): When \( n \equiv 3 (\text{mod} 4) \):

Let \( f(u) = \frac{3(n+1)}{2} \)

\[ f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2} \]

\[ f(u_{2i}) = 2n + 2 \]

The induced edge labelings are,

\[ f^*(uu_i) = \begin{cases} 1 & \text{if } f(u) + f(u_i) \equiv 0 (\text{mod} 2), \quad 1 \leq i \leq n \\ 0 & \text{else} \end{cases} \]

\[ f^*(uv_i) = \begin{cases} 1 & \text{if } f(u) + f(v_i) \equiv 0 (\text{mod} 2), \quad 1 \leq i \leq n \\ 0 & \text{else} \end{cases} \]

\[ f^*(u_i u_{i+1}) = \begin{cases} 1 & \text{if } f(u_i) + f(u_{i+1}) \equiv 0 (\text{mod} 2), \quad 1 \leq i \leq n - 1 \\ 0 & \text{else} \end{cases} \]

\[ f^*(v_i v_{i+1}) = \begin{cases} 1 & \text{if } f(v_i) + f(v_{i+1}) \equiv 0 (\text{mod} 2), \quad 1 \leq i \leq n-1 \\ 0 & \text{else} \end{cases} \]

Edge condition:-

Here, \( e_r(0) = e_r(1) = 2n-1 \)

So, in all the cases, it satisfies the condition

\[ |e_r(0) - e_r(1)| \leq 1 \]

Hence, \( \text{DF}_n \) or \( (P_{n+2}K_1) \) is a Near Mean Cordial graph.

For example, the Near Mean cordial labeling of \( \text{DF}_3, \text{DF}_5, \text{DF}_{10} \) & \( \text{DF}_8 \) are shown in Figures 3.1.1-3.1.4.
When $n \equiv 3 \pmod{4}$:

**Theorem 3.2:** Triangular Snake (TS$_n$) is a Near Mean Cordial graph (when $n \equiv 0, 1, 3 \pmod{4}$)

**Proof:** Let $V(G) = \{v_i : 1 \leq i \leq n+1, w_i : 1 \leq i \leq n\}$

Let $E(G) = \{(v_iv_{i+1}) : 1 \leq i \leq n\} \cup \{(v_iw_i) : 1 \leq i \leq n\} \cup \{(v_{i+1}w_i) : 1 \leq i \leq n\}$

Define $f: V(G) \to \{1, 2, 3, \ldots, 2n, 2n+2\}$ by

Case (i): When $n \equiv 0 \pmod{4}$:

$$f(v_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+2}{2}$$

$$f(v_{2i}) = 2n+2$$

$$f(v_{2i+2}) = 2n - (i-1), \quad 1 \leq i \leq \frac{n-2}{2}$$

$$f(w_i) = \frac{n+3}{2} + i - 1, \quad 1 \leq i \leq n$$

Case (ii): When $n \equiv 1 \pmod{4}$:

$$f(v_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_{2i}) = 2n+2$$

$$f(v_{2i+2}) = 2n - (i-1), \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(w_i) = \frac{n+3}{2} + i - 1, \quad 1 \leq i \leq n$$

Case (iii): When $n \equiv 3 \pmod{4}$:

$$f(v_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_{2i}) = 2n+2$$

$$f(v_{2i+2}) = 2n - (i-1), \quad 1 \leq i \leq \frac{n-1}{2}$$

$$f(w_i) = \frac{n+3}{2} + i - 1, \quad 1 \leq i \leq n$$

The induced edge labelings are,

$$f^e(v_iw_i) = \begin{cases} 1 & \text{if } f(v_i) + f(w_i) \equiv 0 \pmod{2} \quad 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

$$f^e(v_{i+1}v_{i+1}) = \begin{cases} 1 & \text{if } f(v_{i+1}) + f(v_i) \equiv 0 \pmod{2} \quad 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

$$f^e(w_{i+1}v_{i+1}) = \begin{cases} 1 & \text{if } f(w_i) + f(v_{i+1}) \equiv 0 \pmod{2} \quad 1 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

**Edge condition:**

1. Here $e_f(0) = e_f(1) = 3 \frac{n}{2}$ (when $n \equiv 0 \pmod{4}$)

2. Here $e_f(0) = 3 \frac{n+1}{2}, e_f(1) = 3 \frac{n-1}{2}$ (when $n \equiv 1 \pmod{4}$)

3. Here $e_f(0) = 3 \frac{n-1}{2}, e_f(1) = 3 \frac{n+1}{2}$ (when $n \equiv 3 \pmod{4}$)

Hence, it satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Hence, TS$_n$ is a Near Mean Cordial Graph (when $n \equiv 0, 1, 3 \pmod{4}$).

For example, the Near Mean Cordial labeling of TS$_8$, TS$_9$ & TS$_7$ are shown in Figures 3.2.1-3.2.3.

**Theorem 3.3:** Triangular snake (TS$_n$) is not a Near Mean Cordial graph (when $n \equiv 2 \pmod{4}$)

**Proof:** Let $V(G) = \{v_i : 1 \leq i \leq n+1, w_i : 1 \leq i \leq n\}$

Let $E(G) = \{(v_iw_i) : 1 \leq i \leq n\} \cup \{(v_iw_i) : 1 \leq i \leq n\} \cup \{(v_{i+1}w_i) : 1 \leq i \leq n\}$

Define $f: V(G) \to \{1, 2, 3, \ldots, 2n, 2n+2\}$ by

Consider TS$_6$.

Now the vertex labels are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14.$$ 

Out of which 7 are even numbers and 6 are odd numbers.

If a pair consisting of same parity it gives edge labeling 1.

Otherwise the edge labeling is 0.
In the example of TSₖₙ,

The path Pₗ have 3 ones and 3 zeros.
The curved path have 7 ones and 5 zeros.

On the whole, we get 10 ones and 5 zeros. Clearly in this case |e(0) - e(1)| > 1. If we give any type of labeling, they do not satisfy the conditions of Near Mean Cordial labeling.

Clearly we have, |e(0) - e(1)| > 1.

Hence TSₖₙ is not a Near Mean Cordial Graph.

For example, the Near Mean Cordial Labeling of TS₆ is shown in Figure 3.3.1.

When n ≡ 2 (mod 4).

When m and n is even:

\[ e(0) = \frac{m+n+4}{2}, \quad e(1) = \frac{m+n+6}{2} \]

When m is odd, n is even:

\[ e(0) = e(1) = \frac{m+n+5}{2} \]

When m and n is odd:

\[ e(0) = \frac{m+n+5}{2}, \quad e(1) = \frac{m+n+4}{2} \]

When m is even, n is odd:

\[ e(0) = e(1) = \frac{m+n+5}{2} \]

Hence, it satisfies the condition |e(0) - e(1)| ≤ 1.

Hence, J(m,n) is a Near Mean Cordial Graph. For example, the Near Mean Cordial labeling of J(8,12), J(11,8), J(10,7) & J(9,11) are shown in Figures 3.4.1-3.4.4.
Theorem 3.5: \( K_{1,n} \oplus P_n \oplus K_{1,m} \) is a Near Mean Cordial Graph.

Proof:
Let \( V(G) = \{u_i : 1 \leq i \leq n, v_i : 1 \leq i \leq n, w_i : 1 \leq i \leq m\} \)
Define \( f : V(G) \to \{1, 2, 3, ..., 2n-1, 2n\} \) by

Case (i): When \( m \equiv 0 \pmod{2} \), \( n \in \mathbb{N} \)
Let \( f(u_i) = 2i, \quad 1 \leq i \leq n \)
\( f(v_i) = 2i-1, \quad 1 \leq i \leq n \)
\( f(w_i) = 2n+i, \quad 1 \leq i \leq m \)
\( f(w_m) = m+2n+1 \)

Case (ii): When \( m \equiv 1 \pmod{2} \), \( n \in \mathbb{N} \)
Let \( f(u_i) = 2i-1, \quad 1 \leq i \leq n \)
\( f(v_i) = 2i, \quad 1 \leq i \leq n \)
\( f(w_i) = 2n+i, \quad 1 \leq i \leq m \)
\( f(w_m) = m+2n+1 \)

The induced edge labelings are
\[ f^*(u_iv_1) = \begin{cases} 1 & \text{if } f(u_i) + f(v_1) \equiv 0 \pmod{2}, \quad 1 \leq i \leq n \smallskip \text{else} \end{cases} \]
\[ f^*(v_iv_{i+1}) = \begin{cases} 1 & \text{if } f(v_i) + f(v_{i+1}) \equiv 0 \pmod{2}, \quad 1 \leq i \leq n-1 \smallskip \text{else} \end{cases} \]
\[ f^*(v_nv_1) = \begin{cases} 1 & \text{if } f(v_n) + f(v_1) \equiv 0 \pmod{2}, \quad 1 \leq i \leq m \smallskip \text{else} \end{cases} \]

Edge condition:-
Let \( m = 2k+1, (k \in \mathbb{N}) \)
Here \( e_f(0) = e_f(1) = m-k+n-1 \)
Let \( m = 2k, (k \in \mathbb{N}) \)
Here \( e_f(0) = m-k+n-1 \)
\[ e_f(1) = m-k+n \]
So it satisfies the condition \( |e_f(0) - e_f(1)| \leq 1 \). Hence \( K_{1,n} \oplus P_n \oplus K_{1,m} \) is a Near Mean Cordial Graph. For example, the Near Mean Cordial Labeling of \( K_{1,8} \oplus P_8 \oplus K_{1,9} \), \( K_{1,8} \oplus P_8 \oplus K_{1,10} \), \( K_{1,9} \oplus P_9 \oplus K_{1,10} \), \( K_{1,9} \oplus P_9 \oplus K_{1,11} \) are shown in Figures 3.5.1-3.5.4.

V. REFERENCES


