3 – GRAPHOIDAL PATH COVERS

G. Susi Vinnarasi, V. Maheswari, K. Bala Deepa Arasi
1PG Student, PG& Research Department of Mathematics,
A.P.C.Mahalaxmi College For Women, Thoothukudi, TN, India.
susivinnarasi29@gmail.com

2,3 Assistant Professor of mathematics, PG& Research Department of Mathematics,
A.P.C.Mahalaxmi College For Women, Thoothukudi, TN, India.
mahiraj2005@gmail.com
baladeepa85@gmail.com

Abstract

The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1]. The concept of 2-graphoidal path cover was introduced by K. Nagarajan [7] et al. The concept of 3-graphoidal path cover was introduced by T. Gayathri [4] et al. A 3-graphoidal path cover of a graph G is a collection of paths in G such that every path in G has at least two vertices, every vertex of G is an internal vertex of at most three paths and every edge of G is in exactly one path. The minimum cardinality of it is a 3-graphoidal path covering number \( \eta_3a(G) \). In this paper we find \( \eta_3a(G) \) for some special graphs.

Keywords

Graphoidal cover, Acyclic graphoidal, 2-Graphoidal cover and 3-Graphoidal cover.

1. Introduction

A graph is a pair \( G=(V,E) \), where \( V \) is the set of vertices and \( E \) is the set of edges. We consider only non-trivial, finite, connected, undirected graph without loops or multiple edges. The order and size of \( G \) are denoted by \( p \) and \( q \) respectively. For graph theoretic terminology we refer to Harary [5]. The concept of graphoidal cover was introduced by B.D. Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [2]. The concept of 2-graphoidal path cover was introduced by P.K. Das and K. Ratan Singh [3].

Let \( P=(v_1,v_2,v_3,\ldots,v_{n-1},v_n) \) be a path in a graph \( G=(V,E) \). Then the vertices \( (v_2,v_3,\ldots,v_{n-1}) \) are called internal vertices of \( P \) and \( v_1 \) and \( v_n \) are called external vertices of \( P \). If \( P=(v_1,v_2,\ldots,v_{n-1},v_n=v_1) \) is a closed path then \( v_1 \) may be taken as the external vertex. Let \( \Psi \) be the collection of internally edge disjoint paths in \( G \). A vertex of \( G \) is said to be an internal vertex of \( \Psi \) if it is an internal vertex of some path(s) in \( \Psi \) otherwise it is called an external vertex of \( \Psi \). The number of internal vertices of a path \( P \) in \( \Psi \) is denoted by \( i_\Psi(P) \); the number of internal vertices which appear exactly once in a path of \( \Psi \) by \( t_1(\Psi) \) and \( \max t_1(\Psi)=t_1 \); the number of internal vertices which appear exactly twice in two paths of \( \Psi \) by \( t_2(\Psi) \) and \( \max t_2(\Psi)=t_2 \); the number of internal vertices which appear exactly thrice in three paths of \( \Psi \) by \( t_3(\Psi) \) and \( \max t_3(\Psi)=t_3 \) and the number of external vertices by \( t_\Psi \) and \( \min t_\Psi=t \). The reader may refer [5] and [1] for the terms not defined here.

1.1 Definition

Duplication of a vertex of a graph \( G \) produces a new graph \( G' \) by adding a new vertex \( v' \) such that \( N(v)=N(v') \). In other words, a vertex \( v' \) is said to be duplication of \( v \) if all the vertices which are adjacent to \( v \) in \( G \) is also adjacent to \( v' \) in \( G' \). The graph obtained by duplication of all the vertices of \( G \) is denoted by \( D(G) \).

1.2 Definition

An Udukkai graph \( U_n, n \geq 2 \) is a graph constructed by joining two fan graph \( F_n, n \geq 2 \) with two paths \( P_n, n \geq 2 \) by sharing a common vertex at the centre.

1.3 Definition

An Octopus graph \( O_n, n \geq 2 \) can be constructed by joining a fan graph \( F_n, n \geq 2 \) with a star graph \( K_{1,n} \) by sharing a common vertex which is the centre of the star.
1.4 Definition

A Butterfly graph $C_m \circ C_n \circ W_k$ denote the two cycles of order $m$ and $n$ sharing a common vertex with $k$ pendant edges attached at the common vertex.

1.5 Definition

The graph $C_m \circ K_{1,n}$ is obtained from the cycle $C_m$ by attaching the root vertex of the star $K_{1,n}$ to any one vertex of the cycle $C_m$.

1.6 Definition

The graph $W_{n+1} \circ P_m$ is obtained from the wheel $W_{n+1}$ by attaching a terminal vertex of $P_m$.

1.7 Theorem:[3]

If a graph with $\delta \geq 3$, then there exists a graphoidal cover $\Psi$ of $G$ such that every vertex of $G$ is an internal vertex of at least three paths in $\Psi$.

1.8 Remark:[3]

If $\Delta \leq 3$, then $t_2 = 0$ and hence

\[ \eta_{2a}(G) = \eta(G) \], where is the minimum graphoidal covering number.

1.9 Theorem:[4]

For any graph with $\delta = 1$ and $\Delta \geq 6$, $\eta_{3a} \geq |q - p - (t_1 + t_2 + t_3) + t|$.

1.10 Definition

A 3-graphoidal path cover of a graph $G$ is a collection $\Psi$ of paths in $G$ such that

i. Every path in $\Psi$ has at least two vertices.

ii. Every vertex of $G$ is an internal vertex of at most three paths in $\Psi$.

iii. Every edge of $G$ is in exactly one path in $\Psi$.

The minimum cardinality of a 3-graphoidal cover of $G$ is called the 3-graphoidal covering number of $G$ and it is denoted by $\eta_{3a}(G)$ or $\eta_{3a}$.

2. MAIN RESULTS

2.1 Proposition

Let $G$ be a $C_m \circ K_{1,n}$ graph, then

\[ \eta_{3a}(G) = \begin{cases} n & \text{if } n = 4, 5 \\
 n - 1 & \text{if } n \geq 6 \end{cases} \]

Proof

Consider, $C_m \circ K_{1,n}$ as in figure 1.

For $n = 4, 5$, $\eta_{2a} = \eta_{3a} = 5$.

For $n \geq 6$, $q = m + n$, $p = m + n$, $\Delta \geq 6$, $t_1 = t_2 = 0$, $t_3 \leq 1$, $t \geq n$.

Then, $\eta_{3a} \geq q - p - (t_1 + t_2 + t_3) + t$.

\[ \geq m + n - (m + n) - (0 + 0 + 1) + n \]

\[ \eta_{3a} \geq n - 1 \]

Let $P_1 = <v_1, u_1, \ldots, u_m>$, $P_2 = <v_2, u_1, u_m>$, $P_3 = <v_3, u_4, v_4>$.

The paths $P_1$, $P_2$, $P_3$ together with the remaining edges form a 3-graphoidal path cover $\Psi$ of $C_m \circ K_{1,n}$.

| $|\Psi|$ = $3 + |E(C_m \circ K_{1,n})| - |E(P_1| + E(P_2) + E(P_3)) |$
| = $3 + (m + n) - ((m - 2) - 2) |$
| = $3 + |n - 4| = n - 1$

$\eta_{3a} \leq n - 1$.

Hence, $\eta_{3a} = n - 1$.

2.2 Proposition

Let $G$ be a $W_{n+1} \circ P_m$ graph, then
Proof

Consider \( W_{n+1} \oplus P_m \).

Here, \( \Delta \geq 6 \), \( q = m + 2n - 1 \), \( p = m + n + 1 \), \( t_1 = 0 \), \( t_2 \leq 1 \), \( t_3 \leq 1 \), \( t \geq 1 \).

Then, \( \eta_{3a} \geq | q - p - (t_1 + t_2 + t_3) + t | \).

\[ \geq | m + 2n - 1 - (m + n) - (0 + 1 + 1) + 1 | \]

\[ \eta_{3a} \geq n - 2. \]

Figure 2

Let \( P_1 = \langle v_1, v_2, \ldots, v_9, w, v_{10} \rangle \)

\( P_2 = \langle v_9, \ldots, v_n, v_1, w, v_2 \rangle \), \( P_3 = \langle v_3, w, v_4 \rangle \)

The paths \( P_1, P_2, P_3 \) together with the remaining edges form a 3-graphoidal path cover \( \Psi \) of \( G \)

\[ |\Psi| = 3 + |E(G) - \{E(P_1) + E(P_2) + E(P_3)\}| \]

\[ = 3 + |(m + 2n - 1) - (m + n + 4)| \]

\[ = 3 + |n - 5| = n - 2 \Rightarrow \eta_{3a} \geq n - 2. \]

Hence, \( \eta_{3a} = n - 2. \)

2.3 Proposition

Let \( G \) denote the Franklin graph with 12 vertices and 18 edges as given in figure. Then \( \eta_{3a} (G) = 6. \)

Proof

Here, \( \Delta \geq 6 \); \( q = 18 \), \( p = 12 \), \( t_1 = 0 \), \( t_2 = 0 \), \( t_3 \leq 1 \) since the graph forms a cycle, at least one vertex must appear as an external vertex \( t \geq 1 \).

Figure 3

Then, \( \eta_{3a} \geq | q - p - (t_1 + t_2 + t_3) + t | \).

\[ \geq | 18 - 12 - (0 + 0 + 1) + 1 | \]

\[ \eta_{3a} \geq 6. \]

\( P_1 = \langle v_8, v_7, v_1, v_2, v_3, v_4, v_5, v_6, v_{12}, w, v_{10} \rangle \)

\( P_2 = \langle v_{12}, v_{11}, w, v_8, v_2 \rangle \), \( P_3 = \langle v_7, w, v_9, v_{10}, v_4 \rangle \)

\( P_1 = \langle v_1, v_6 \rangle \), \( P_2 = \langle v_3, v_{11} \rangle \), \( P_3 = \langle v_5, v_9 \rangle \)

we have, \( \eta_{3a} \leq 6 \).

Hence, \( \eta_{3a} = 6. \)

2.4 Proposition

Let \( G \) be a \( D(K_{1,n}) \) graph. Then \( \eta_{3a} (G) = 2n - 3. \)

Proof

Let \( u_1, u_2, \ldots, u_n \) be the vertices of the star graph and \( u \) be its centre. Then \( u'_1, u'_2, \ldots, u'_n \) be its duplication and \( u' \) be the duplication of \( u \).

Figure 4

Since, \( d(u) = 2n \), we have \( t_3 \leq 1 \), \( p = 2n + 2 \), \( q = 3n \), \( t \geq n \)

\[ \eta_{3a} \geq | q - p - (t_1 + t_2 + t_3) + t | \]

\[ \geq | 3n - 2n - 2 - (0 + 0 + 1) + n | \]
\[ \geq 2n-3. \]
\[ \eta_{3a} \geq 2n-3 \]

Let \( P_1 = <u_1,u,u_1',u_0> \)
\[ P_2=<u_2,u,u_2',u_{n-1}> \]
\[ P_3=<u_3,u,u_3',u_{n-2}> , P_4=<u,u,u'> \]
\[ P_5=<u,u_5,u'> \ldots P_{n-3}=<u,u_{n-3},u'> \]

The paths \( P_1, P_2, \ldots, P_{n-3} \) together with the remaining edges form a 3-graphoidal path cover \( \Psi \) of \( D(K_{1,n}) \)

\[ |\Psi|=n-3+|E(D(K_{1,n})-\{E(P_1)+E(P_2)+E(P_3)\}| \]
\[ =n-3+(3n)-((12+2(n-6))] \]
\[ =n-3+|n|=2n-3. \]
\[ \eta_{3a} \leq 2n-3. \]

Hence, \( \eta_{3a} = 2n-3. \)

2.5 Proposition

For a graph \( U_n, n \geq 2, \eta_{3a}(U_n)=2n-1 \)

Proof

Here \( \Delta \geq 6, q=6n-4, p=4n-1, t_1=0, t_2=0, t_3 \leq 1 \), since the graph contains cycle, at least one vertex must appear as an external vertex, \( t \geq 3. \)

Then, \( \eta_{3a} \geq |q-p-(t_1+t_2+t_3)+t|. \)
\[ \geq |6n-4-4n+1- (0+0+1)+3| \]
\[ \eta_{3a} \geq 2n-1. \]

Now, \( P_1 = <v_1,v_2,\ldots,v_n,w,t_1t_2,\ldots,t_n> \)
\[ P_2=<s_n,s_{n-1},\ldots,s_1,w,u_1,u_2,\ldots,u_n> \]
\[ P_3=<u_1,w,v_n> \]

The paths \( P_1, P_2, P_3 \) together with the remaining edges form a 3-graphoidal path cover \( \Psi \) of \( U_n. \)

\[ |\Psi|=3+|E(U_n)-\{E(P_1)+E(P_2)+E(P_3)\}| \]
\[ =3+(6n-4)-(4n)] \]
\[ =3+2n-4=2n-1. \]
\[ \eta_{3a} \leq 2n-1. \]

Hence, \( \eta_{3a}(U_n)=2n-1 \)

2.6 Proposition

For a graph \( O_n, n \geq 2, \eta_{3a}(O_n)=2n-3 \)

Proof

Here \( \Delta \geq 6, q=3n-1, p=2n+1, t_1=0, t_2=0, t_3 \leq 1, t \geq n. \)

Then, \( \eta_{3a} \geq |q-p-(t_1+t_2+t_3)+t|. \)
\[ \geq |3n-1-(2n+1)- (0+0+1)+n| \]
\[ \eta_{3a} \geq 2n-3. \]

Let \( P_1=<u_n,u_{n-1},\ldots,u_1,w,v_1> \)
\[ P_2=<u_1,w,v_n> , P_3=<u_n,w,v_1> \]

The paths \( P_1, P_2, P_3 \) together with the remaining edges form a 3-graphoidal path cover \( \Psi \) of \( O_n. \)

\[ |\Psi|=3+|E(O_n)-\{E(P_1)+E(P_2)+E(P_3)\}| \]
\[ \begin{align*}
3 &+ |(3n-1)-(n+5)| \\
= &+ |2n-6| = 2n-3.
\end{align*} \]

\( \eta_{3a} \leq 2n-3. \)

Hence, \( \eta_{3a}(O_n) = 2n-3. \)

### 2.7 Proposition

For a graph \( F_{2,n} \), \( \eta_{3a}(F_{2,n}) = 2n-3 \)

#### Proof

Here \( \Delta \geq 6 \), \( q=4n-2 \), \( p=2n+1 \), \( t_1=0 \) \( t_2=0 \), \( t_3 \leq 1 \), since the graph contains cycles at least one vertex must appear as an external vertex, \( t \geq 1 \).

Then, \( \eta_{3a} \geq | q - p - (t_1 + t_2 + t_3) + t | \).

\[ \geq |4n-2-(2n+1)-(0+0+1)+1| \]

\( \eta_{3a} \geq 2n-3. \)

Now, \( P_1 = \langle u_n, u_{n-1}, \ldots, u_1, u, v_n, v_{n-1}, \ldots, v_1 \rangle \)

\( P_2 = \langle u_2, u_3, v_2 \rangle \), \( P_3 = \langle u_n, u, v_1 \rangle \)

The paths \( P_1, P_2, P_3 \) together with the remaining edges form a 3-graphoidal path cover \( \Psi \) of \( C_m:C_n@W_k \).

\[ |\Psi|=3+|E(G)-\{E(P_1)+E(P_2)+E(P_3)\}| \]

\[ =3+|(m+n+k)-(m+n+2)| \]

\[ =3+k-2 \]

\( \eta_{3a}(C_m:C_n@W_k) \leq k+1. \)

Hence, \( \eta_{3a}(C_m:C_n@W_k) = k+1. \)

### References


[3] P.K. Das and K. Ratan Singh, On 2-Graphoidal Covering Number of a Graph,


