On S - Near Rings and S' - Near Rings with Right Bipotency

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Abstract:-- A right near ring \((N, +, \cdot)\) is an algebraic system with two binary operations such that (i) \((N, +)\) is a group - (not necessarily abelian) with 0 as its identity element, (ii) \((N, \cdot)\) is a semigroup (we write \(xy\) for \(x \cdot y\) for all \(x, y \in N\)) and (iii) \((x + y)z = xz + yz\) for all \(x, y, z \in N\). We say that \(N\) is zero symmetric if \(n0 = 0\) for all \(n \in N\). \(N\) is called a S - near ring or an S' - near ring according as \(x \in Nx\) or \(x \in xN\) for all \(x \in N\). A subgroup \(M\) of \(N\) is called an N-subgroup if \(NM \subseteq M\) and an invariant N-subgroup if, in addition, \(MN \subseteq M\). An element \(a \in N\) is said to be distributive, if \(a(b + c) = ab + ac\) for all \(b\) and \(c\) in \(N\); \(N\) is called distributively generated (d.g.), if the additive group of \(N\) is generated by the multiplicative semigroup of distributive elements of \(N\).

A near ring \(N\) is defined to be right bipotent if \(aN = a^2N\) for each \(a \in N\). In this paper, we have proved some more results on right bipotent near rings by using the concepts of S' - near ring; subcommutativity; regularity; reduced property etc. It is proved that every right bipotent near ring is an S' - near ring and it is also S - near ring if it is also subcommutative. Every regular near ring is central and reduced if it is right bipotent. Some special characterizations are obtained in such a way that, a reduced right bipotent near ring is a near field if \(N = N_d\) and it is a division ring if it is dgnr.

Keywords:-- S near ring, S’- near ring, near field, right bipotent near ring, subcommutative, nilpotent, right N - subgroup, zero divisors, regular near ring, division ring, distributively generated near ring.

1. Introduction

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring. Taussky [24] in 1936 and B.H.Neumann [13] in 1940 considered near rings in which addition need not be commutative. Since then the theory of near rings has been developed much. Later Frolich [6], Beidleman [2], Oswald [14] and many other researchers had done and have been doing extensive work on different aspects of near rings. Gunter Pilz [5] "Near rings" is an extensive collection of the work done in the area of near rings.

A near ring \(N\) is defined to be left bipotent if \(Na = Na^2\) for each \(a \in N\). The definitions for S - Near ring and S’ - Near ring...
are dealt in P(r,m) Near rings by R. Balakrishnan and S. Suryanarayanan in [1].

2. Preliminaries

Definition 2.1 [9]

N is said to be subcommutative, if aN = Na for all a ∈ N.

Definition 2.2 [5]

An element n ∈ N is called nilpotent if n^k = 0 for some positive integer k.

Definition 2.3 [8]

A near ring N is regular if for each a in N, there exists x in N such that a = axa.

Definition 2.4

An element e in N is called idempotent if e^2 = e.

Definition 2.5 [5]

An idempotent a in N is called a central if ax = xa for all x in N.

Definition 2.6 [5]

Let (P, +) be a group with 0 and let N be a near ring. Let µ: N × P → P; (P, µ) is called an N-group if for all p ∈ P and for all n,n_1 ∈ N we have (n + n_1)p = np + n_1p and (nn_1)p = n(n_1p). N^P stands for N-groups.

Definition 2.7 [5]

A subgroup S of N^P with NS ⊂ S is a N-subgroup of P.

Definition 2.8 [8]

An additive group A of N is called a left N-subgroup if NA ⊆ A where NA = {ra/r ∈ N, a ∈ A}.

Definition 2.9 [8]

An additive group A of N is called a right N-subgroup if AN ⊆ A where AN = {ar/r ∈ N, a ∈ A}.

Definition 2.10 [8]

For any subset A of a near ring N, Define √A = {x ∈ N/x^n ∈ A, for some n}.

Definition 2.11 [12]

An element 0 ≠ x ∈ N is called a right zero divisor if ∃ 0 ≠ a ∈ N such that ax = 0

Definition 2.12 [12]

An element 0 ≠ x ∈ N is called a left zero divisor if ∃ 0 ≠ a ∈ N such that xa = 0.

Definition 2.13 [5]

If all non zero elements of N are left (right) cancelable, we say that N fulfills the left (right) cancellation law.

Definition 2.14 [8]

N is called a near-field if it contains an identity and each non zero element has a multiplicative inverse.

Notation 2.15 [5]

Let N_d = {d ∈ N|d is distributive}

Definition 2.16 [5]

If N = N_d, N is said to be distributive.

Definition 2.17 [1]

N is called an S - near ring according as x ∈ Nx for all x ∈ N.

Definition 2.18 [1]

N is called an S’ - near ring according as x ∈ xN for all x ∈ N.

Definition 2.19 [25]

A near ring N is defined to be right bipotent if aN = a^2N for each a in N.
3. Main Results

Theorem 3.1
Every Right Bipotent near ring is an $S'$- near ring.

Proof:
Let $N$ be right bipotent. This implies $a^2N = aN$. Therefore $a \in a^2N = aN$. This implies $a \in aN$. Hence $N$ is $S'$- near ring.

Corollary 3.2
Every $S$- near ring is $S'$ - near ring if it is subcommutative with vice versa.

Proof:
Let $N$ be $S$ - near ring.
Then, $x \in Nx = xN$ for all $x \in N$. This implies $x \in xN$. Hence $N$ is $S'$- near ring.
Converse follows.

Result 3.3
Any right bipotent subcommutative near ring is an $S$ - near ring.

Theorem 3.4
Homomorphic images of right bipotent $S'$ - near rings are also such.

Proof:
Let $f: N \to N'$ be a homomorphism of near rings $N$ onto $N'$, and let $N$ be a right bipotent $S'$- near ring. If $a \in N'$, there exists $b \in N$ such that $f(b) = a$. By assumption, we have $bN = b^2N$. Then $f(bN) = f(b)f(N) = aN'$ and $f(b^2N) = f(b^2)f(N) = [f(b)]^2f(N) = a^2N'$. Thus $bN' = b^2N'$. Now since $b \in bN$, we have $a = f(b) \in f(bN) = aN'$.

Theorem 3.5
A regular near ring $N$ is right bipotent if each idempotent in $N$ is central.

Proof:
$N$ is regular, so far $a$ in $N$, there exists $x$ in $N$ such that $a = axa$. Let $ax = e$. Now, $(ax)^2 = (ax)(ax) = (axa)x = ax$. Therefore $ax$ is an idempotent. Now $a = axa = aax$ (since idempotents are central) $= a^2x$. Hence $aN = a^2N$ and $N$ is right bipotent.

Theorem 3.6
Let $N$ be an $S'$ - near ring, then $N$ is regular iff for each $a (\neq 0)$ in $N$, there exists an idempotent $e$ such that $aN = eN$.

Proof:
If $N$ is a regular near ring, then for every $a$ in $N$, there exists $x$ in $N$ such that $a = axa$. Let $ax = e$. Now, $(ax)^2 = (ax)(ax) = (axa)a = ax = e$. (i.e) $e^2 = e$. Therefore $e$ is an idempotent and $aN = axaN \subseteq axN = eN \subseteq eN$). Conversely, Let $N$ be an $S'$ - near ring satisfying the given condition. For any $d \in N$, there exists an idempotent $b$ such that $d \in dN = bN$. This implies $d = bu$ for some $u$ in $N$. Also $b \in bN = dN$. This implies $b = dy$ for some $y$ in $N$. Now $dyd = dybu = bbu = b^2u = bu = d$. Therefore $dyd = d$. Hence $N$ is a regular near ring.

Theorem 3.7
A right bipotent near ring $N$ is regular iff it is an $S'$ - near ring.

Proof:
Let $N$ be regular near ring. This implies for each $a$ in $N$, there exists $x$ in $N$ such that $a = axa$. Let $ax = e$. Now $(ax)^2 = (ax)(ax) = (axa)x = ax$. Therefore $ax$ is an idempotent.
Now $a = axa = aax$. This implies $a \in eN$. Therefore every regular near ring is an $S'$ - near ring. Conversely, Let $N$ be a right bipotent $S'$ - near ring. Then for each
Clearly $A \subseteq \sqrt{A}$. Now let $a \in \sqrt{A}$, then $a^n \in A$ for some $n$. Also we have $aN = a^2N = \cdots = a^nN$ in a right bipotent near ring. Since $N$ is an $S'$ - near ring, $a \in aN = a^nN$. This gives $a = a^nb$ for some $b$ in $N$. Thus $a \in A$, (since $a^n \in A$ and $A$ is a right $N$-subgroup of $N$). Hence $\sqrt{A} \subseteq A$. Conversely, we have to prove that if $N$ is an $S'$ - near ring with the condition $A = \sqrt{A}$ for every right $N$-subgroup $A$ of $N$ then $N$ is right bipotent. For $a \in N$, $a^3 \in a^2N$ and $a \in \sqrt{a^2N} = a^2N$. Then $aN \subseteq a^2N \subseteq aN$ and $N$ is right bipotent.

**Theorem 3.11**

Let $N$ be a right bipotent near ring with no zero divisors. If $N$ has a non zero distributive element, then $N$ is a near field.

**Proof:**

$N$ is regular. Let $d$ be a non zero distributive element in $N$, then there exists $x$ in $N$ such that $d = dx$. Let $dx = e$. Now, $(dx) = (dx)(dx) = (dx)x = dx$. Therefore $dx$ is an idempotent. If $r$ is any element in $N$, then $r(d - edx) = 0$. This implies $r(d - ed) = 0$. This gives $r = re = 0$ (since $d$ is a distributive element). From this, we get $r = re$. That is, $e$ is a right identity in $N$. If $a \in N$ with $a \neq 0$ then $aN = a^2N$. Therefore, $ae = a^2y$ for some $y$ in $N$. This implies $a(e - ay) = 0$. This gives $e - ay = 0$ (since $a \neq 0$). From this, we get $e = ay$. That is, $y$ is a right inverse of $a$. Hence $N$ is a near field.

**Corollary 3.12**

Let $N$ be a right bipotent distributively generated (d.g.) near ring with no zero divisors then $N$ is a division ring.
Proof:
By Theorem 3.11, $N$ is a near field and so $(N, +)$ is abelian (see(6)). Moreover, a d.g. near ring with $(N, +)$ abelian is a ring (13). Therefore, $N$ is a division ring.

Bibliography

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