INTRODUCTION TO MEAN LABELING

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Abstract

Graph labeling is an essential and interesting topic in graph theory. There are nearly 200 graph labeling techniques. Graph theory was introduced in the year 1960. In this paper, the graph is taken as simple, finite and undirected. V(G) represents vertex set and E(G) represents Edge set. A graph labeling is the assignment of labels, that is represented by integers, to the edges or vertices, or both, of a graph. We shall study some of the types of labelling with illustration which gives a clear idea about it. The topics are explained with examples and supporting results and diagrams for clear idea about the concept. The study is limited with certain types.

Keywords: Mean Graph – Even Mean labelling – Odd Mean Labelling – Strongly Multiplicative – Harmonious Multiplicative – Geometric mean labelling – Skolem difference

Introduction

In this paper, the graph are taken as simple, finite and undirected. V(G) represents vertex set and E(G) represents Edge set. A graph labeling is the assignment of labels, that is represented by integers, to the edges or vertices, or both, of a graph. A vertex labelling is a function of V to a set of labels. A graph with such a vertex labelling function is defined as Vertex – labeled graph.

An edge labeling is a function of E to a set of labels and a graph with such a function is called as an edge-labeled graph. We shall discuss about labelling concept in this paper.

Definition – 1

A graph is said to be mean graph if a graph with p vertices and q edges has an injective function f from the vertices of G to {0, 1, 2 ….., q} such that when each edge uv is labeled with \((f(u) + f(v))/2\) if \(f(u) + f(v)\) is even and \((f(u) + f(v) + 1)/2\) if \(f(u) + f(v)\) is odd, then the resulting edge labels are distinct.

Definition – 2

A graph with p vertices and q edges is said to be a relaxed mean graph if there exists a function f from the vertex set of G to {0, 1, 2, …., q – 1, q + 1} such that the induced map \(f^*\) from the edge set of G to {1, 2, …., q} defined by

\[
f^*(e = uv) = \begin{cases} 
\frac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even} \\
\frac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd}
\end{cases}
\]
then the edges get distinct from 1,2,3,…q.

Definition – 3

A function f is called an *Even Mean Labeling* of a graph G with p vertices and q edges. If f is an injection from the vertices of G to the set \{2,4,6,…2q\} such that when each edge uv is assigned the label \([f(u)+f(v)/2]\), then the resulting edge labels are distinct. A graph, which admits an Even Mean labeling, is said to be Even Mean Graph.

Definition – 4

A function f is called an *Odd mean labeling* of a graph G with p vertices and q edges. If f is an injection from the vertices of G to the set \{1,3,5,….,2q-1\} such that when each edge uv is assigned the label \([f(u)+f(v)/2]\), then the resulting edge labels are distinct. A graph, which admits an odd mean labeling, is said to be odd mean graph.

Definition – 5

A graph with p vertices is *Strongly Multiplicative* if the vertices of G can be labeled with distinct integers 1,2,3,…p such that the labels induced on the edges by the product of the end vertices are distinct.

Definition – 6

A graph of order n is said to be a *Strongly graph* if its vertices can be assigned the values 1,2,3,4…n in such a way that, when an edge whose vertices are labeled i and j is labeled with the value i+j+ij, all edges have different labels.

Definition – 7

A *harmonious labeling* on a graph G is an injection from the vertices of G to the group of integers modulo k, where k is the number of edges of G, that induces a bijection between the edges of G and the numbers modulo k by taking the edge label for an edge \((x, y)\) to be the sum of the labels of the two vertices \(x, y \pmod k\).

Definition – 8

A *harmonious graph* is one that has a harmonious labeling. Odd cycles are harmonious, as is the Petersen graph. It is conjectured that trees are all harmonious if one vertex label is allowed to be reused.\(^8\) The seven-page book graph \(K_{1,7} \times K_2\) provides an example of a graph that is not harmonious.

Definition – 9

A Graph \(G=(V,E)\) with p vertices and q edges is said to be a *Geometric mean* if it is possible to label the vertices \(x \in V\) with distinct labels \(f(x)\) from 1,2,…q+1 in such a way that when each edge \(e=uv\) is labeled
with \( f(e=uv) = \lceil \sqrt{f(u)f(v)} \rceil \) or \( \lfloor \sqrt{f(u)f(v)} \rfloor \), then the resulting edge labels are distinct. In this case \( f \) is called Geometric mean labeling of \( G \).

**Definition – 10**

Let \( G = (V,E) \) be a skolem difference mean graph with \( p \) vertices and \( q \) edges. If one of the *Skolem difference mean labeling* of \( G \) satisfies the condition that all the labels of the vertices are odd, then we call this skolem difference mean labeling an extra skolem difference mean labeling and call the graph \( G \) an extra skolem difference mean graph.

**Results**

In this section, it is proved that comb \( P_n \Theta K_1, \forall \ n \ C_{2n+1}, \forall \ n \ K_{1,n}, \forall \ n \) and \( P_n \forall \ n \) are even and odd mean graphs.

**Result – 1** Every Comb graph is odd and even mean graph.

Proof: Let \( G = P_n \Theta K_1, \forall \ n \) be a comb graph with \( 2n \) vertices and \( 2n-1 \) edges. The even mean labeling for vertices of comb \( P_n \Theta K_1 \) is defined by

\[
u_i = 4i - 2, \ i = 1,2,\ldots \ n \]

\[
v_i = 4i, \ i = 1,2,\ldots \ n \]

Edge labelings are defined by \( e_i = 4i, \ i = 1,2,\ldots \ n - 1 \ e_i' = 4i - 1, \ i = 1,2,\ldots \ n \).

The odd Mean labeling for vertices of comb \( P_n \Theta K_1 \) is defined by

\[
u_i = 4i - 3, \ i = 1,2,\ldots \ n \]

\[
v_i = 4i - 1, \ i = 1,2,\ldots \ n \]
Edge labelings are defined by

\[ e_i = 4i - 1, \ i = 1, 2, \ldots, n - 1 \]

\[ e'_i = 4i - 2, \ i = 1, 2, \ldots, n \]

Therefore, the labeling of vertices and edges are distinct. Hence the graph is even and odd mean graph.

**Result – 2**

Every cycle of odd length is even and odd mean graph

*Proof:* Let \( G = C_n \), \( \forall \ n \geq 3 \) and \( n \) is odd

The even mean labeling for vertices of \( C_n \) is defined by

\[ u_i = 2i, \ i = 1, 2, \ldots, n \]

Edge labelings are defined by

\[ e_i = 2i + 1, \ i = 1, 2, \ldots, n-1, \]

\[ e_n = n+1. \]

The odd mean labeling for vertices of cycle \( C_n \) is defined by \( u_i = 2i-1, i = 1, 2, 3, \ldots, n \).

Edge labelings are defined by \( e_i = 2i, i = 1, 2, 3, \ldots, n-1, \) \( e_n = n \).

From above \( t \), the labeling of vertices and edges are distinct.

Hence the graph is even and odd mean graph.

Example Every cycle of odd length is an even and odd mean graph.

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**Result – 3**

Every star graph is odd and even mean graph

*Proof:* Let \( G = K_{1,n} \) \( \forall \ n \) be a star graph with \( n + 1 \) vertices and \( n \) edges.

The odd mean labeling for vertices of star graph \( K_{1,n} \) is defined by
v_0 = 1, u_i = 2i+1, i = 1,2,3……n

Edge labelings are defined by \( e_i = i + 1 \), \( i = 1,2,3\ldots n \).

The even mean labeling for vertices of star graph \( K_{1,n} \), \( \forall n \) is defined by

\[ v_0 = 2, u_i = 2i+2, i = 1,2,3\ldots n \]

Edge labeling are defined by \( e_i = i + 2 \), \( i = 1,2,3\ldots n \).

From the above, the vertex and edge labeling are distinct.

Hence the star graph is odd and even mean graph.

Example - Star graph \( K_{1,12} \) is odd and even mean graph

The Graph \( K_{1,12} \) Figure 6.9 The Graph \( K_{1,12} \) Theorem 6.4: Every path graph \( P_n \), \( \forall n \) is odd and even mean graph. Proof: Let \( G = P_n \), \( \forall n \) be a path graph with \( n \) vertices and \( n-1 \) edges.

The odd mean labeling for vertices of path graph \( P_n \) is defined by

\[ u_i = 2i-1, i = 1,2,3\ldots n \]

Edge labelings are defined by

\[ e_i = 2i, i = 1,2,\ldots n - 1 \]

The even mean labeling for vertices of path graph \( P_n \) is defined by

\[ u_i = 2i, i = 1,2,\ldots n. \]

\[ e_i = 2i+1, i = 1,2,3\ldots n - 1 \]

From the above assignment, the vertex and edge labelings are distinct.

Hence the graph \( G = P_n \), \( \forall n \) is odd and even mean graph.

Results

Theorem 2.1 \( Ln A K_{2} \) is a Geometric mean graph.

**Proof:** Let \( u_1 u_2 \ldots \ldots u_n \) and \( v_1 v_2 \ldots \ldots v_n \) be two paths of length \( n \). Join \( u_i \) and \( v_i \), \( 1 \leq i \leq n \).

The resultant graph is \( Ln \). For \( 1 \leq i \leq n \), join \( u_i \) with two vertices \( s_i, t_i \) and \( v_i \) with two vertices \( x_i, y_i \).

The resultant graph is \( LnA_{K_{2}} \) whose edge set is \( E = \{ u_i u_{i+1}, v_i v_{i+1}, | 1 \leq i \leq n - 1 \} \cup \{ s_i u_i , t_i u_i , u_i v_i , v_i x_i , v_i y_i , 1 \leq i \leq n \} \)

Define a function \( f: V(LnA_{K_{2}}) \) on geometric mean graphs
\[ f(s_1) = 1, \]
\[ f(s_i) = 7(i-1), \quad 2 \leq i \leq n \]
\[ f(t_i) = 7i - 5, \quad 1 \leq i \leq n \]
\[ f(u_i) = 7i - 4, \quad 1 \leq i \leq n \]
\[ f(x_i) = 7i - 2, \quad 1 \leq i \leq n \]
\[ f(y_i) = 7i - 1, \quad 1 \leq i \leq n \]

Edges are labeled with \( f(s_iu_i) = 7i - 6, \quad 1 \leq i \leq n \)
\[ f(t_iu_i) = 7i - 5, \quad 1 \leq i \leq n \]
\[ f(u_iu_i+1) = 7i - 4, \quad 1 \leq i \leq n - 1 \]
\[ f(v_iu_i+1) = 7i - 3, \quad 1 \leq i \leq n - 1 \]
\[ f(v_iu_i+1) = 7i - 2, \quad 1 \leq i \leq n - 1 \]
\[ f(u_iu_i+1) = 7i - 1, \quad 1 \leq i \leq n - 1 \]
\[ f(v_iu_i+1) = 7i, \quad 1 \leq i \leq n - 1 \]

Then we get distinct edge labels. Thus \( f \) provides a Geometric mean labeling for \( L_n A K_2 \).

Example:

Theorem 2.3: Triangular ladder \( TL_n \) is a Geometric mean graph.

Proof: Let \( u_1, u_2, \ldots, u_n \) and \( v_1, v_2, \ldots, v_n \) be the two path of length \( n \).

Join \( u_i \) and \( v_i \), \( 1 \leq i \leq n \).

The resultant graph is \( L_n \). For \( 1 \leq i \leq n - 1 \), Join \( u_i \) and \( v_i + 1 \), and the resultant graph is \( TL_n \) and whose edge set is \( E = \{ u_iu_{i+1}, vivi+1, uivi+1 / 1 \leq i \leq n - 1 \} \cup \{ uivi / 1 \leq i \leq n - 1 \} \)

Triangular Ladder \( TL_n \) has \( 4n - 3 \) edges.

Define a function \( f: V(TL_n) \rightarrow \{1,2,\ldots,q+1\} \) by

\[ f(u_1) = 1, \]
\[ f(u_i) = 4(i-1), \quad 2 \leq i \leq n - 1. \]
\[ f(v_1) = 3, \]
f(vi) = 4i - 2, 2 ≤ i ≤ n

Edges are labeled with f(uivi) = 4i - 3, 1 ≤ i ≤ n

f(uiui+1) = 4i - 2, 1 ≤ i ≤ n - 1

f(uivi+1) = 4i - 1, 1 ≤ i ≤ n - 1

f(vivi+1) = 4i, 1 ≤ i ≤ n - 1

Hence TL_n is a Geometric mean graph.

Theorem: TL_nAK1 is a Geometric mean graph.

Proof: Let u_1, u_2, ..., u_n and v_1, v_2, ..., v_n two paths of length n.

Join u_i and v_i, (1 ≤ i ≤ n).

The resultant graph is L_n. For 1 ≤ i ≤ n - 1, join u_i and v_i+1 the resultant graph is TL_n. For 1 ≤ i ≤ n, add two new vertices x_i and y_i and join u_i with x_i and v_i with y_i.

The resultant graph is TL_nAK1, Whose edge set is E = {u_i u_i+1, v_i v_i+1, u_i v_i+1 | 1 ≤ i ≤ n - 1} ∪ {u_i v_i, x_i y_i, y_i v_i | 1 ≤ i ≤ n} On geometric mean graphs.

Define a function f: V (L_nAK1) → {1, 2, ..., q+1} by

f(x_1) = 1,

f(x_i) = 6i - 2, 2 ≤ i ≤ n

f(y_1) = 2,

f(y_i) = 6i - 5, 2 ≤ i ≤ n

f(u_1) = 3,

f(u_i) = 6i - 3, 2 ≤ i ≤ n

f(v_1) = 4,

f(v_i) = 6i - 4, 2 ≤ i ≤ n
Edges are labeled with

\[ f(u_1v_1) = 3, \]
\[ f(u_{i}v_{i}) = 6i - 4, \quad 2 \leq i \leq n \]
\[ f(x_1u_1) = 1, \]
\[ f(x_{i}u_{i}) = 6i - 3, \quad 2 \leq i \leq n \]
\[ f(y_1v_1) = 2, \quad f(y_{i}v_{i}) = 6i - 5, \quad 2 \leq i \leq n \]
\[ f(u_1v_2) = 4, \]
\[ f(u_{i}v_{i+1}) = 6i - 2, \quad 2 \leq i \leq n - 1 \]
\[ f(u_{i}u_{i+1}) = 6i, \quad 1 \leq i \leq n - 1 \]
\[ f(v_1v_2) = 5, \quad f(v_{i}v_{i+1}) = 6i - 1, \quad 2 \leq i \leq n - 1 \]

Hence \( f \) provides a Geometric mean labeling of \( TL_{n}AK_1 \).

Conclusion: We have discussed about various types of Graph labelling and its application with certain results. There are many graph labelling types apart from this study, Only selective study has been made.

References:


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