Error Analysis of Solution Of Time Fractional Convection Diffusion Equation

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Abstract: The aim of this paper is to develop the Crank-Nicolson finite difference scheme and Adomian Decomposition Method to solve time fractional convection diffusion equation. Furthermore, we give a detailed analysis of the Crank-Nicolson scheme and generate the discrete model. Also, we prove solution of the scheme is unconditionally stable and convergent. As an application of the scheme we solve test problem and their solution represented graphically by a powerful software Mathematica. We also develop the Adomian Decomposition Scheme for time fractional convection diffusion equation. We analyze the solutions of convection diffusion equation by both methods and estimate the error.

IndexTerms: Convection diffusion equation, Crank-Nicolson Finite difference scheme, Adomian Decomposition Method, Mathematica.

I. INTRODUCTION

Convection diffusion equation (CDE) describes the transport occurring in fluid by combination of convection and diffusion. This equation is useful to model many natural processes such as causes of environmental degradation due to pollution of airand ground water, the spreading of harmful chemical discharge etc. The convection diffusion equation has many applications in environmental engineering, heat transfer, chemical engineering, biology etc. Therefore many researchers are studying this equation. [1]

In recent years, fractional calculus is playing very important role to study convection diffusion equation. Feng et al [2] has given the solution of the space fractional diffusion equation with variable coefficients on a finite domain. Obidat [3] has solved the space-time fractional advection diffusion equation (ADE). Zhuang et al. [4] made a numerical study to the variable order fractional ADE with a nonlinear source term.

Yang et al. [5] studied the local fractional variational iteration method for diffusion and wave equations. Rocca et al. [6] gave the solution of fractional ADE and study the diffusion process of solar rays. Zhuang and Liu [7] also solved time fractional diffusion equation by an implicit finite difference scheme. Liu et al. [8] used fractional method of lines to solve the space fractional Fokker Plank equation. Above review shows that CDE has a wide range of practical and industrial applications. Due to the importance of CDE the present paper solves and analyzes time fractional CDE using Crank-Nicolson method and Adomian decomposition method. Also we will analyze the solutions graphically.

In the year 1967, Caputo, introduced new and useful definition of fractionalderivative popularly known as Caputo fractional derivative. we define it as [12, 14]follows

Definition 1.1*The Caputo time fractional derivative of order α, (0 < α ⩽ 1) is defined as follows*

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{\partial U(x,t)}{\partial \xi}\frac{d\xi}{(t-\xi)^\alpha}, & 0<\alpha<1 \\ \dfrac{\partial U(x,t)}{\partial t}, & \alpha = 1 \end{cases}$$

Where, Γ (.) is a Gamma function

.

We organize the paper as follows: In section 2, we develop the Crank-Nicolson fractional order finite difference scheme for time fractional convection diffusion equation. The stability of the solution is proved in section 3 and the concept of convergence is discussed in section 4. In section 5, the numerical solution of time fractional convection diffusion equation is obtained and it is represented graphically by Mathematica software. In the last section we develop the method of Adomian Decomposition Method for time fractional convection diffusion equation and estimate error by comparing the results.

II.FINITE DIFFERENCE SCHEME

We consider the following time fractional convection diffusion equation (TFCDE) with initial and boundary conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \lambda\frac{\partial u(x,t)}{\partial x} - C\frac{\partial^2 u(x,t)}{\partial x^2} = 0, 0<x<L, 0<\alpha\le 1, t>0 \quad (2.1)$$

$$initial\ condition: u(x,0) = f(x), 0\le x\le L \quad (2.2)$$

$$Boundary\ \ conditions: u(0,t) = g_1(t), u(L,t) = g_2(t), t\ge 0 \quad (2.3)$$

For the time fractional order Crank-Nicolson implicit numerical approximation scheme, we define $h = \frac{(x_R - x_L)}{N} = \frac{L}{M}$ and $\tau = \frac{T}{N}$ the space and time steps respectively, such that $t_k = k\tau$; k =0,1….,N be the integration time $0 \le t_k \le T$ and $x_i = x_L + ih$ for i = 0,1……,M. Define $u_i^k = u(x_i, t_k)$ and let u_i^k denote the numerical approximation to the exact solution $u(x_i, t_k)$.

In the differential equation (2.1), the time fractional derivative term is approximatedby the following scheme

$$\frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{k+1}} \frac{1}{(t_{k+1} - \xi)^\alpha} \frac{\partial u(x_i, \xi)}{\partial \xi} d\xi$$

$$= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [u_i^{k+1} - u_i^k] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j [u_i^{k+1-j} - u_i^{k-j}]$$

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 1,2,\dots,k$.

For $\frac{\partial^2 u(x,t)}{\partial x^2}$, we adopt the second order central difference scheme and for $\frac{\partial u(x,t)}{\partial x}$, we adopt the backward difference scheme in space for each interior grid points x_i, $0 < i < M$, as follows [13]

$$\frac{\partial^2 u(x,t)}{\partial x^2} = [\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{h^2} + \frac{u_{i-1}^k - u_i^k + u_{i+1}^k}{h^2}]$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{u_i^{k+1} - u_{i-1}^{k+1}}{h}$$

Using time fractional approximation, the Crank-Nicolson implicit type numerical approximationto equation (2.1) − (2.3) is given as follows

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} [u_i^{k+1} - u_i^k] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j [u_i^{k-j+1} - u_i^{k-j}]$$

$$= \frac{C}{2} [\frac{u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}}{h^2} + \frac{u_{i-1}^k - u_i^k + u_{i+1}^k}{h^2}] - \lambda \frac{u_i^{k+1} - u_{i-1}^{k+1}}{h}$$

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, j = 1,2,\dots,k$.
The initial condition is approximated as $u_i^0 = f(x_i), i = 1,2,\dots,M$.
For the two boundary points x_0 and x_M the corresponding discretisation schemes are $u_0^k = g_(t)$ and $u_M^k = g_2(t), k = 0,1,2,\dots,N$
After simplification, we get

$$u_i^{k+1} - u_i^k + \sum_{j=1}^{k} b_j [u_i^{k-j+1} - u_i^{k-j}] = r[u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1} + u_{i-1}^k - 2u_i^k + u_{i+1}^k] - \mu[u_i^{k+1} - u_{i-1}^{k+1}]$$

Where, $r = \frac{C\tau^\alpha \Gamma(2-\alpha)}{2h^2}$, $\mu = \frac{\lambda \tau^\alpha \Gamma(2-\alpha)}{h}$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, for $i = 1,\dots,M-1$, $k = 0,1,2,\dots$ and $j = 1,2,\dots,k$

$$-(r+\mu)u_{i-1}^{k+1} + (1+2r+\mu)u_i^{k+1} - ru_{i+1}^{k+1} = ru_{i-1}^k + (1-2r)u_i^k + ru_{i+1}^k - \sum_{j=1}^{k} b_j [u_i^{k-j+1} - u_i^{k-j}]$$

Therefore, the fractional approximated IBVP is

$$-(r+\mu)u_{i-1}^1 + (1+2r+\mu)u_i^1 - ru_{i+1}^1 = ru_{i-1}^0 + (1-2r)u_i^0 + ru_{i+1}^0 \quad (2.4)$$

$$-(r+\mu)u_{i-1}^{k+1} + (1+2r+\mu)u_i^{k+1} - ru_{i+1}^{k+1} = ru_{i-1}^k + (1-b_1-2r)u_i^k + ru_{i+1}^k - \sum_{j=1}^{k-1} (b_j - b_{j+1}) U_i^{k-j} + b_k u_i^0 \quad (2.5)$$

initial condition: $u_i^0 = f(x_i), i = 1,2,\dots,M-1$ (2.6)

boundary conditions: $u_0^k = g_1(t)$ and $u_M^k = g_2(t), k = 0,1,2,\dots$ (2.7)

Where, $r = \frac{C\tau^\alpha \Gamma(2-\alpha)}{2h^2}$, $\mu = \frac{\lambda \tau^\alpha \Gamma(2-\alpha)}{h}$ and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, for $i = 1,\dots,M-1, k = 0,1,2,\dots$ and $j = 1,2,\dots,k$.

Therefore, the fractional approximated IBVP (2.1) − (2.3) can be written in the following matrix equation form

Where $U^k = (u_1^k, u_2^k, \dots, u_{M-1}^k)^T, k = 0,1,2,..$ and

$$A = \begin{pmatrix} (1+2r+\mu) & -r & \cdots & \cdots & \cdots & 0 & \\ -(r+\mu) & (1+2r+\mu) & -r & \cdots & & & \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots & \\ 0 & & & & & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots & \\ 0 & \cdots & & & -(r+\mu) & (1+2r+\mu) & \end{pmatrix}$$

and

$$B = \begin{pmatrix} (1-2r) & r & \cdots & \cdots & 0 \\ r & (1-2r) & r & \cdots & \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & \vdots \\ 0 & & & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & \vdots \\ 0 & \cdots & & r & (1-2r) \end{pmatrix}$$

and

$$X = \begin{pmatrix} (1-2r-b_1) & r & \cdots & \cdots & \cdots & 0 & \\ r & (1-2r-b_1) & r & \cdots & & & \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots & \\ 0 & & & & & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots & \\ 0 & \cdots & & & r & (1-2r-b_1) & \end{pmatrix}$$

$$S = \begin{pmatrix} 2r+\mu \\ \vdots \\ 0 \\ \vdots \\ 2r \end{pmatrix}$$

III. STABILITY

Theorem 3.1

The solution of the discretised scheme (2.4) − (2.7) for the time fractional convection diffusion equation (2.1) − (2.3) is unconditionally stable.

Proof: We assume that$\| E^k \|_\infty = |\epsilon_l^k| = \max_{1\le i\le M-1} |\epsilon_i^k|$, $\alpha, r > 0, 1 = b_1 > b_2 >, \dots > 0$ for $i = 1,2,\dots, M-1, k = 1,2,\dots, N$.
Therefore, from equation (2.4) $k = 0$, we get

$$\begin{aligned} |\epsilon_l^1| &= |-(r+\mu)\epsilon_{i-1}^1 + (1+2r+\mu)\epsilon_i^1 - r\epsilon_{i+1}^1| \\ &= |r\epsilon_{i-1}^0 + (1-2r)\epsilon_i^0 + r\epsilon_{i+1}^0| \\ &\le |\epsilon_l^0| \\ \therefore ||E^1||_\infty &\le |\epsilon_l^0| \\ \therefore ||E^1||_\infty &\le ||E^0||_\infty \end{aligned}$$

Suppose that

$$\| E^k \|_\infty \le \| E^0 \|_\infty.$$

From equation (2.5), we get

$$\begin{aligned}|\epsilon_l^{k+1}| &= \left|-(r+\mu)u_{i-1}^{k+1} + (1+2r+\mu)u_i^{k+1} - ru_{i+1}^{k+1}\right| \\ &= |r\epsilon_{i-1}^k + (1-b_1-2r)\epsilon_i^k + r\epsilon_{i+1}^k \\ &\quad + \sum_{j=1}^{k-1}(b_j - b_{j+1})\epsilon_i^{k-j} + b_k\epsilon_i^0| \\ &\le |\epsilon_l^0| \\ \| E^{k+1} \|_\infty &\le \| E^0 \|_\infty.\end{aligned}$$

Hence, by induction we prove the scheme is unconditionally stable.

IV.CONVERGENCE

In this section, we discuss the convergence of the approximate scheme (2.4) − (2.7). Let $U(x_i,t_k)$ be the exact solution of the time fractional diffusion equation (2.1) − (2.3) and U_i^k be the exact solution of the discrete equation (2.4) − (2.7) at mesh points (x_i,t_k), where i = 1, 2, ...,M − 1 and k = 1, 2, ...,N. Define $e_i^k = U(x_i,t_k) - U_i^k,\ i = 1,2,\dots,M-1$ and $k = 1,2,\dots,N$ and $E^k = (e_1^k, e_2^k, \dots, e_{M-1}^k)$. Clearly $E^0 = 0, E_0^k = 0$ and $E_M^k = 0$. Now substituting $U(x_i,t_k)$ into equation (2.1) and U_i^k into (2.4) then subtracting from (2.4), we get

$$-(r+\mu)e_{i-1}^1 + (1+2r+\mu)e_i^1 - re_{i+1}^1 = re_{i-1}^0 + (1-2r)e_i^0 + re_{i+1}^0 + R_i^1$$

$$-(r+\mu)e_{i-1}^{k+1} + (1+2r+\mu)e_i^{k+1} - re_{i+1}^{k+1} = re_{i-1}^k + (1-b_1-2r)e_i^k + re_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_kU_i^0 + R_i^1$$

Where, $r = \frac{C\tau^\alpha\Gamma(2-\alpha)}{2h^2}$, $\mu = \frac{\lambda\tau^\alpha\Gamma(2-\alpha)}{h}$, $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, for $i = 1,\dots,M-1$, $k = 0,1,2,..$ and $j = 1,2,\dots,k$

Theorem 4.1 The fractional order Crank-Nicolson finite difference scheme (2.4) − (2.7) for TFADE (2.1) − (2.3) is convergent and the solution U_i^k of the discrete scheme (2.4)-(2.7) and the solution $U(x_i,t_k)$ of the equation (2.1) – (2.3) satisfy

$\| U(x_i,t_k) - U_i^k \| \le \| E \|_\infty + O(\tau^{1-\alpha} + h^2), i = 1,\dots,M-1, k = 1,2,\dots,N.$

Proof: Let us assume that

$$|e_l^k| = \max_{1\le i\le M-1}|\epsilon_i^k| = \| E^k \|_\infty, for\, l = 1,2,\dots$$

And $T_l^k = \max_{1\le i\le N}|T_i^k|$, $T_j^n = h^2[O(\tau^{1-\alpha}) + O(h^2)]$ then from equation (2.4), we get

$$\begin{aligned}|e_l^1| &= |-(r+\mu)e_{i-1}^1 + (1+2r+\mu)e_i^1 - re_{i+1}^1| \\ &= |re_{i-1}^0 + (1-2r)e_i^0 + re_{i+1}^0| \\ \therefore ||E^1||_\infty &\le |e_l^0| + h^2[O(\tau^{1-\alpha}) + O(h^2)] \\ \therefore ||E^1||_\infty &\le ||E^0||_\infty + h^2[O(\tau^{1-\alpha}) + O(h^2)]\end{aligned}$$

Suppose that

$$\| E^k \|_\infty \le \| E^0 \|_\infty + h^2[O(\tau^{1-\alpha}) + O(h^2)].$$

From equation (2.5), we get

$$\begin{aligned}|e_l^{k+1}| &= |-(r+\mu)_{i-1}^{k+1} + (1+2r+\mu)e_i^{k+1} - re_{i+1}^{k+1}| + |T_i^{k+1}| \\ &= |re_{i-1}^k + (1-b_1-2r)e_i^k + re_{i+1}^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_ke_i^0| + |T_i^{k+1}| \\ &= |e_l^k| + h^2[O(\tau^{1-\alpha}) + O(h^2)] \\ \| E^{k+1} \|_\infty &\le \| E^0 \|_\infty + h^2[O(\tau^{1-\alpha}) + O(h^2)].\end{aligned}$$

Hence, by induction, we prove

$$\| E^{k+1} \|_\infty \le \| E^0 \|_\infty + h^2[O(\tau^{1-\alpha}) + O(h^2)].$$

$$\| U(x_i,t_k) - U_i^k \| \le \| E \|_\infty + O(\tau^{1-\alpha} + h^2), i = 1,\dots,M-1, k = 1,2,\dots,N.$$

Hence, the proof of the theorem is completed.

V.NUMERICAL SOLUTIONS

In this section, we obtain the approximated solution of time fractional convection diffusion equation with initial and boundary conditions by Crank Nicolson Finite Difference Method and Adomian Decomposition Method. Afterwards, the comparative analysis and error estimation are presented. We consider the following time fractional convection diffusion equation with suitable initial and boundary conditions

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\lambda\frac{\partial u(x,t)}{\partial x}-C\frac{\partial^2 u(x,t)}{\partial x^2}=0, 0<x<L, 0<\alpha\leq 1, t>0$$

$initialcondition: u(x,0)=e^x, 0\leq x\leq L$
$Boundaryconditions: u(0,t)=e^t, u(L,t)=e^{1+t}, t\geq 0.$

The same model is used to obtain the approximate solution by Adomian decompositionmethod. Using Adomian decomposition method, we have

$$u_0(x,t) = u(x,0) = e^x$$
$$u_1(x,t) = J^\alpha[-\lambda D_x u_0(x,t)+CD_x^2 u_0(x,t)] = (C-\lambda)e^x\frac{t^\alpha}{\Gamma(\alpha+1)}$$
$$u_2(x,t) = (C-\lambda)^2 e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$\vdots$$

Therefore, the series solution for the IBVP is given by

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots.$$

Substituting values of components in above equation, we get the solution as follow

$$u(x,t) = e^x+(C-\lambda)e^x\frac{t^\alpha}{\Gamma(\alpha+1)}+(C-\lambda)^2 e^x\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots$$

$$u_{EXACT}(x,t) = e^x+e^(C-\lambda)t, when\alpha=1$$

$$u_{ADM}(x,t) = e^x.\sum_{n=0}^{\infty}\frac{(C-\lambda)^n t^{n\alpha}}{\Gamma(n\alpha+1)}$$

Obtained Numerical solutions At T = 0.3, T = 0.003, H = 0.1, Simulated In The Following Figure

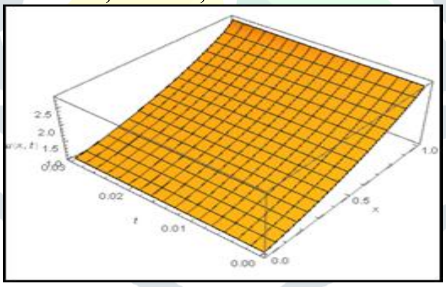

Fig.5.1 : The exact solution of convection diffusion equation

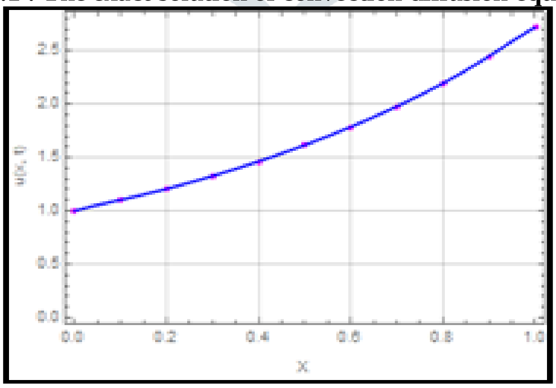

Fig.5.4 : Comparison of solutions by both methods

Following table shows the error estimation of both Methods

x	EXACT	CN	ADM	EXACT - CN	EXACT - ADM
0.1	1.08546	1.10287	1.08211	0.01741	0.00335
0.2	1.19961	1.20563	1.19592	0.00602	0.00369
0.3	1.32578	1.32543	1.32169	0.00035	0.00409
0.4	1.46521	1.46158	1.4607	0.00363	0.00451
0.5	1.61931	1.61422	1.61432	0.00509	0.00499
0.6	1.78961	1.78468	1.7841	0.00493	0.00551
0.7	1.97783	1.97594	1.97173	0.00189	0.0061
0.8	2.18584	2.19382	2.1791	0.00798	0.00674
0.9	2.41573	2.44901	2.40828	0.03328	0.00745

Table 1 Error Estimation of the methods used

VI. CONCLUSION

1. We obtain the approximate solution of time fractional convection diffusion equation by Crank-Nicolson finite difference method and Adomian decomposition method.
2. We simulate these solutions graphically by mathematica and compared them.
3. We observe that the numerical solution by Crank-Nicolson finite difference method and Adomian decomposition method are very close to exact solution.
4. We demonstrated that the solutions obtained by both methods have second order Convergence. From error estimation, we observed that Adomian decomposition method has faster convergence than Crank-Nicolson finite difference method.

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